Part (a):

A general EM wave in the space between the two plates is a superposition of two plane waves reflecting back and forth from each plate:

$$\begin{aligned} F_{\mu\nu}(x) &= C^{(1)}_{\mu\nu} e^{ik_1 x} + C^{(2)}_{\mu\nu} e^{ik_2 x} \\ \text{for} \quad k^\mu_{1,2} &= (\omega, k_x, k_y, \pm k_z) \\ \text{where } \omega &= c\sqrt{k_x^2 + k_y^2 + k_z^2} \end{aligned} \tag{S.1}$$

and the polarization tensors $C^{(1)}_{\mu\nu}$ and $C^{(2)}_{\mu\nu}$ obey the Maxwell equations for the corresponding wave vectors k^μ_1 and k^μ_2. In addition, the net wave (S.1) must obey the boundary conditions at each conducting plane:

$$\text{at } z = 0 \text{ or } z = b \text{ and any } x, y: \quad E_x = E_y = B_z = 0. \tag{S.2}$$

For the sake of notational simplicity, let's focus on the wave propagating in the x direction in the (x, y) plane, thus let $k_y = 0$. In this case, the two independent polarizations of the two reflecting waves (S.1) are as follows:

(A) The electric field **E** of each plane wave points in the y direction while the magnetic field vector **B** lies in the (x, z) plane.

(B) The magnetic field **B** of each plane wave points in the y direction while the electric field vector lies in the (x, y) plane.

For the (A) polarization, the E_y field must vanish at both $z = 0$ and $z = b$, so its z-dependence is a standing wave with nodes at both ends. Thus,

$$E_y(x, y, z, t) = e^{ik_x x - i\omega t} \times \sin\frac{n\pi z}{b} \quad \text{for } n = 1, 2, 3, 4, \ldots, \tag{S.3}$$

or in other words,

$$k_z = \frac{\pi}{b} \times \text{a positive integer}. \tag{S.4}$$

For the (B) polarization, the magnetic field in y direction is related by the electric field in x

direction by the Maxwell–Ampere Law

$$\nabla \times \mathbf{B} \;=\; \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \quad\Longrightarrow\quad \frac{\partial B_y}{\partial z} \;=\; \frac{i\omega}{c}E_x\,, \tag{S.5}$$

so the Dirichlet boundary conditions for the E_x at $z = 0$ and $z = b$ translate to the Neumann boundary conditions for the B_y. Consequently, the z-dependence of the B_y is a standing wave with anti-nodes at both ends, thus

$$B_y(x, y, z, t) \;=\; e^{ik_x x - i\omega t} \times \cos\frac{n\pi z}{b} \quad \text{for } n \;=\; 0, 1, 2, 3, 4, \ldots. \tag{S.6}$$

Or in other words,

$$k_z \;=\; \frac{\pi}{b} \times \text{a non-negative integer.} \tag{S.7}$$

Altogether, the allowed values of k_z have form $k_z = \pi n/b$ for non-negative integers n. For $n = 0$ there is only one polarization (B), while for the positive $n = 1, 2, 3, \ldots$ there are two polarizations (A) and (B).

Finally, for the 2D wave vector (k_x, k_y) pointing in a general direction, we may repeat the above analysis almost verbatim but in more complicated notations for the directions of the electric and magnetic fields for the (A) and (B) polarization. Thus, for any k_x and k_y, the spectrum of k_z comprises $n \times \pi/b$ for non-negative integers n, and there are two polarizations for $n > 0$ but only one for $n = 0$. *Quod erat demonstrandum.*

Part (b):

For the square plates with periodic boundary conditions the 2D wave vectors (k_x, k_y) have discrete spectrum

$$k_x \;=\; \frac{2\pi n_x}{L}\,, \quad k_y \;=\; \frac{2\pi n_y}{L} \quad \text{for integer} n_x \text{and} n_y\,, \tag{S.8}$$

while according to part (a)

$$k_z \;=\; \frac{\pi n_z}{b} \quad \text{for integer } n_z \;\geq\; 0. \tag{S.9}$$

Consequently, eq. (2) for the regularized vacuum energy of the cavity becomes

$$E(\tau) \;=\; \sum_{n_x=-\infty}^{+\infty}\sum_{n_y=-\infty}^{+\infty}\sum_{n_z=0}^{+\infty}\frac{\hbar\omega(\mathbf{k})}{2}\times e^{-\tau\omega(\mathbf{k})}\times\begin{cases}1 & \text{for } n_z=0,\\ 2 & \text{for } n_z>0.\end{cases} \tag{S.10}$$

For large plate size $L\to\infty$, the spectrum of k_x and k_y becomes near-continuous with density

$$d\#(k_x,k_y) \;=\; \frac{L^2}{(2\pi)^2}\times dk_x\,dk_y \tag{S.11}$$

so we may replace the sum over n_x and n_y with integrals over k_x and k_y,

$$E(\tau) \;=\; \frac{L^2}{(2\pi)^2}\iint dk_x\,dk_y\sum_{n_z=0}^{+\infty}\frac{\hbar\omega(\mathbf{k})}{2}\times e^{-\tau\omega(\mathbf{k})}\times\begin{cases}1 & \text{for } n_z=0,\\ 2 & \text{for } n_z>0,\end{cases} \tag{S.12}$$

where the integration range is the whole 2D momentum space.

Next, since

$$\omega(\mathbf{k}) \;=\; c\sqrt{k_x^2+k_y^2+k_z^2} \tag{S.13}$$

does not care about the sign of the k_z, we may replace the sum over non-negative n_z with the sum over all integer n_z — positive, zero, or negative — but count each such n_z only once. This way, we would get two similar terms for $\pm n_z\neq 0$ but only one term for $n_z=0$, which is precisely agrees with the polarization count in eq. (S.12). Thus,

$$E(\tau) \;=\; \frac{L^2}{(2\pi)^2}\iint dk_x\,dk_y\sum_{n_z=-\infty}^{+\infty}\frac{\hbar\omega(\mathbf{k})}{2}\times e^{-\tau\omega(\mathbf{k})}. \tag{S.14}$$

Now, for each n_z we can get $k_z=\pi n_z/b$ by integrating (over the whole real axis)

$$\int dk_z\,\delta\left(k_z-\frac{\pi n_z}{b}\right) \;=\; \frac{b}{\pi}\int dk_z\,\delta\left(\frac{bk_z}{\pi}-n_z\right) \tag{S.15}$$

and hence

$$\sum_{n_z} f(k_z) \;=\; \frac{b}{\pi}\int dk_z\,f(k_z)\times\sum_{n_z}\delta\left(\frac{bk_z}{\pi}-n_z\right).$$

In this way, eq. (S.14) becomes a 3D integral over the wave vector $\mathbf{k} = (k_x, k_y, k_z)$,

$$E(\tau) \;=\; \frac{L^2}{(2\pi)^2} \times \frac{b}{\pi} \iiint d^3\mathbf{k}\, \frac{\hbar\omega(\mathbf{k})}{2} \times e^{-\tau\omega(\mathbf{k})} \times \sum_{n_z} \delta\left(\frac{bk_z}{\pi} - n_z\right). \tag{S.16}$$

Finally, dividing both sides of this equation by the cavity's volume $V = L^2 b$, we get

$$\frac{E(\tau)}{V} \;=\; \frac{\hbar}{(2\pi)^3} \int d^3\mathbf{k}\, \omega(\mathbf{k}) \times e^{-\tau\omega(\mathbf{k})} \times \sum_{n_z} \delta\left(\frac{bk_z}{\pi} - n_z\right). \tag{S.17}$$

Eq. (4) follows from this formula via $\omega(\mathbf{k}) = c|\mathbf{k}|$.

Part (c):

Eq. (5) — or rather its corollary

$$\sum_{n=-\infty}^{+\infty} F(n) \;=\; \int\limits_{-\infty}^{+\infty} dx\, F(x) \times \sum_{n=-\infty}^{+\infty} \delta(n-x) \;=\; \sum_{m=-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} dx\, F(x) \times e^{2\pi mix} \tag{S.18}$$

is known as the Poisson's resummation formula. To prove it, let's calculate the sum on the LHS of eq. (5), or rather the regulated version of this sum,

$$\begin{aligned} S(x,\epsilon) \;&\stackrel{\text{def}}{=}\; \sum_{m=-\infty}^{+\infty} e^{2\pi mix} \times e^{-\epsilon|m|} \\ &\langle\!\langle \text{ using } m' = -m \text{ for } m < 0 \,\rangle\!\rangle \\ &=\; 1 \;+\; \sum_{m=1}^{\infty} e^{+2\pi imx} \times e^{-\epsilon m} \;+\; \sum_{m'=1}^{\infty} e^{-2\pi im'x} \times e^{-\epsilon m'} \\ &=\; 1 \;+\; \sum_{m=1}^{\infty} \left(e^{+2\pi ix-\epsilon}\right)^m \;+\; \sum_{m'=1}^{\infty} \left(e^{-2\pi ix-\epsilon}\right)^{m'} \\ &=\; 1 \;+\; \frac{e^{+2\pi ix-\epsilon}}{1-e^{+2\pi ix-\epsilon}} \;+\; \frac{e^{-2\pi ix-\epsilon}}{1-e^{+2\pi ix-\epsilon}} \\ &=\; \frac{\sinh(\epsilon)}{\cosh(\epsilon)-\cos(2\pi x)}\,. \end{aligned} \tag{S.19}$$

In the small ϵ limit, this expression becomes

$$S(x,\epsilon) \;\xrightarrow[\epsilon\to+0]{}\; \frac{2\epsilon}{\epsilon^2 + 4\sin^2(\pi x)} \;\longrightarrow\; \begin{cases} O(\epsilon) & \text{for } \sin(\pi x) \neq 0, \\ \frac{2}{\epsilon} & \text{for } \sin(\pi x) = 0. \end{cases} \tag{S.20}$$

Since $\sin(\pi x)$ vanishes at integer x — and only at integer x, — the sum (S.19) becomes very large at integer values of x and very small everywhere else. Moreover, for any integer n and any $c \gg \epsilon$ but $c \ll 1$, we have

$$\begin{aligned}
\int\limits_{n-c}^{n+c} dx\, \frac{2\epsilon}{\epsilon^2 + 4\sin^2(\pi x)} &= \int\limits_{-c}^{+c} dx'\, \frac{2\epsilon}{\epsilon^2 + 4\sin^2(\pi x')} \qquad \langle\!\langle\, \text{where } x' = x - n \,\rangle\!\rangle \\
&\approx \int\limits_{-c}^{+c} dx'\, \frac{2\epsilon}{\epsilon^2 + (2\pi x')^2} \\
&= \frac{4\epsilon}{2\pi\epsilon} \times \arctan\frac{2\pi c}{\epsilon} \\
&\approx 1,
\end{aligned} \tag{S.21}$$

and since the integrand is very small at all non-integer x, it follows that

$$\int\limits_{\substack{\text{any range containing}\\ \text{only one integer}}} \lim_{\epsilon\to 0} S(x,\epsilon)\, dx \;=\; 1. \tag{S.22}$$

In a similar way, for any smooth function $f(x)$,

$$\int\limits_{\substack{\text{any range containing}\\ \text{only one integer}}} f(x) \times \lim_{\epsilon\to 0} S(x,\epsilon)\, dx \;=\; f(\text{that integer}), \tag{S.23}$$

and therefore

$$\int\limits_{\text{anyrange}} f(x) \times \lim_{\epsilon\to 0} S(x,\epsilon) \;=\; \sum_n f(n) \tag{S.24}$$

where the sum is over all integers which happen to lie withing the integration range. In other words,

$$\lim_{\epsilon\to 0} S(x,\epsilon) \;=\; \sum_{n=-\infty}^{+\infty} \delta(x-n), \tag{S.25}$$

and hence

$$\sum_{m=-\infty}^{+\infty} e^{2\pi i m x} \;=\; \lim_{\epsilon\to +0} \sum_{m=-\infty}^{+\infty} e^{2\pi i m x} \times e^{-\epsilon|m|} \;=\; \sum_{n=-\infty}^{+\infty} \delta(x-n). \tag{S.26}$$

Quod erat demonstrandum.

Part (d):

Let's approximate the infinite empty space by a cube of large volume $L^3 \to \infty$ with periodic boundary conditions. The EM wave modes in this cube are plane waves with

$$\mathbf{k} = \frac{2\pi}{L}(n_x, n_y, n_z) \quad \text{for integer } n_x, n_y, n_z \tag{S.27}$$

and two polarizations for each $\mathbf{k}$. For large L, the spectrum of $\mathbf{k}$ becomes near-continuous with density

$$d\#\mathbf{k} = \frac{L^3}{(2\pi)^3}\, d^3\mathbf{k}, \tag{S.28}$$

hence the regularized vacuum energy of the cube becomes

$$\begin{aligned} E(\tau) &= \sum_{\mathbf{k}} 2_{\text{polarizations}} \times \hbar\omega(\mathbf{k}) \times e^{-\tau\omega(\mathbf{k})} \\ &= \frac{L^3}{(2\pi)^3} \int d^3\mathbf{k}\, \hbar\omega(\mathbf{k}) \times e^{-\tau\omega(\mathbf{k})} \\ &= \frac{L^3\hbar c}{(2\pi)^3} \int d^3\mathbf{k}\, |\mathbf{k}| \times e^{-\tau c|\mathbf{k}|}. \end{aligned} \tag{S.29}$$

This energy is proportional to the cube's volume L^3, so the L-independent *energy density*

$$\frac{E(\tau)}{\text{volume} = L^3} = \frac{\hbar c}{(2\pi)^3} \int d^3\mathbf{k}\, |\mathbf{k}| \times e^{-\tau c|\mathbf{k}|} \tag{S.30}$$

of a large cube should be the same as the energy density of infinite space without any walls.

By inspection, eq. (S.30) for the vacuum energy density of the infinite space looks precisely like the $m = 0$ term in eq. (6) for the vacuum energy density of the cavity between the plates. Hence, according to eq. (3), the Casimir energy density of the cavity obtains from eq. (7) where the sum skips the $m = 0$ term.

Part (e):

To evaluate the RHS of eq. (7), we start by integrating $\int d^3\mathbf{k}$ in spherical coordinates for the wave vector $\mathbf{k}$:

$$\begin{aligned}
\int d^3\mathbf{k}\,|\mathbf{k}| \times e^{-\tau c|\mathbf{k}|} \times e^{2mibk_z} &= \\
&= \int_0^\infty dk\,k^2 \int_0^\pi d\theta\,\sin\theta \int_0^{2\pi} d\phi\,k \times e^{-\tau ck} \times e^{2imbk\cos\theta} \\
&= 2\pi \int_0^\pi d\theta\,\sin\theta \times \int_0^\infty dk\,k^3 \times \exp\bigl(-k(\tau c - 2imb\cos\theta)\bigr) \\
&= 2\pi \int_0^\pi d\theta\,\sin\theta \times \frac{\Gamma(4) = 6}{(\tau c - 2imb\cos\theta)^4} \\
&= 12\pi \int_{\tau c - 2imb}^{\tau c + 2imb} \frac{d(\tau c - 2imb\cos\theta)}{2imb}\,\frac{1}{(\tau c - 2imb\cos\theta)^4} \\
&= 12\pi \times \frac{-1/3}{2imb}\left[\frac{1}{(\tau c + 2imb)^3} - \frac{1}{(\tau c - 2imb)^2}\right] \\
&= -4\pi \times \frac{2(2mb)^2 - 6(\tau c)^2}{[(2mb)^2 + (\tau c)^2]^3}\,.
\end{aligned} \tag{S.31}$$

Next, we remove the regularization by taking the $\tau \to +0$ limit, which yields

$$\lim_{\tau\to+0} \int d^3\mathbf{k}\,(\cdots) = -4\pi \times \frac{2(2mb)^2}{[(2mb)^2]^3} = -\frac{\pi}{2m^4b^4}\,. \tag{S.32}$$

Consequently,

$$\frac{E_{\text{Casimir}}}{\text{volume}} = -\frac{\hbar c}{16\pi^2 b^4} \sum_{m\neq 0} \frac{1}{m^4} \tag{S.33}$$

where the sum evaluates to

$$\sum_{m\neq 0} \frac{1}{m^4} = 2\sum_{m=1}^{\infty} \frac{1}{m^4} = 2\zeta(4) = \frac{\pi^4}{45}\,. \tag{S.34}$$

Thus altogether,

$$\frac{E_{\text{Casimir}}}{\text{volume}} = -\frac{\pi^2}{720}\frac{\hbar c}{b^4}. \tag{S.35}$$

Part (f):

The two reflective plates divide the infinite space into 3 regions — two semi-infinite regions outside the plates, and one cavity between them — and the Casimir energy is the difference between the net vacuum energy of the three regions and the vacuum energy of the empty space without any plates. But since the vacuum energy — or even the UV-regularized vacuum energy — or a large region is proportional to its volume, the $E(\tau)$ of the infinite space or of a semi-infinite region is infinite, and one must be very careful subtracting such infinities.

To avoid this trouble, we were careful calculating the vacuum energy *densities* rather than the net vacuum energies, and now we must be careful subtracting the net vacuum energies of exactly same net volumes of space.

Thus, let's put the whole system — the plates, and the spaces between and outside the plates — into a huge box of size $L \times L \times L'$ with periodic boundary conditions. In effect, such boundary conditions in z direction merge the two outside regions into a single cavity with two reflective walls of width $L' - b$, so altogether we have two cavities, one narrow and one wide. Consequently, the net Casimir energy is

$$E_{\text{Casimir}} = E_{\text{vac}}[\text{narrow cavity}] + E_{\text{vac}}[\text{wide cavity}] - E_{\text{vac}}[\text{whole box without the plates}], \tag{S.36}$$

or rather

$$E_{\text{Casimir}} = \lim_{\tau\to 0}\Big(E(\tau)[\text{narrow cavity}] + E(\tau)[\text{wide cavity}] - E(\tau)[\text{whole box}]\Big). \tag{S.37}$$

Or in terms of energy densities $\mathcal{E} = E/\text{volume}$

$$\begin{aligned}
E_{\text{Casimir}} &= \lim_{\tau\to 0}\left(\begin{array}{r} L^2 b \times \mathcal{E}(\tau)[\text{narrow cavity}] \;+\; L^2(L'-b)\times\mathcal{E}(\tau)[\text{wide cavity}] \\ -\; L^2 L' \times \mathcal{E}(\tau)[\text{whole box}] \end{array}\right) \\
&= L^2 b \times \lim_{\tau\to 0}\Big(\mathcal{E}(\tau)[\text{narrow cavity}] \;-\; \mathcal{E}(\tau)[\text{whole box}]\Big) \\
&\quad +\; L^2(L'-b)\times \lim_{\tau\to 0}\Big(\mathcal{E}(\tau)[\text{wide cavity}] \;-\; \mathcal{E}(\tau)[\text{whole box}]\Big) \\
&= L^2 b \times \mathcal{E}_{\text{Casimir}}[\text{narrow cavity}] \;+\; L^2(L'-b)\times\mathcal{E}_{\text{Casimir}}[\text{wide cavity}].
\end{aligned} \tag{S.38}$$

At this point, we may use eq. (S.35) for the Casimir energy densities of the two cavities to obtain

$$\begin{aligned}
E_{\text{Casimir}} &= L^2 b \times \mathcal{E}_{\text{Casimir}}(b) \;+\; L^2(L'-b)\times\mathcal{E}_{\text{Casimir}}(L'-b) \\
&= -\frac{\pi^2\hbar c}{720 b^4}\times L^2 b \;-\; \frac{\pi^2\hbar c}{720(L'-b)^4}\times L^2(L'-b) \\
&= -\frac{\pi^2\hbar c L^2}{720 b^3} \;-\; \frac{\pi^2\hbar c L^2}{720(L'-b)^3} \\
&\xrightarrow[L'\to\infty]{} \;-\frac{\pi^2\hbar c L^2}{720 b^3}\,.
\end{aligned} \tag{S.39}$$

And this is the net Casimir energy of the cavity, including the (vanishing) effect of the outside regions of space.

Two noteworthy features of the Casimir energy (S.39): (1) it's negative, and (2) its magnitude decreases with the cavity's width b. Consequently, when we allow the walls to move, the Casimir energy acts as a potential energy for such motion, which results in the mechanical force on the walls attracting them towards each other. Specifically,

$$F \;=\; -\frac{dE_{\text{Casimir}}}{db} \;=\; -\frac{\pi^2\hbar c L^2}{240 b^4} \tag{S.40}$$

where the overall $-$ sign indicates that the force is attractive. And since this force is proportional to the plates' area L^2, it means the negative Casimir pressure

$$P \;=\; \frac{F}{L^2} \;=\; -\frac{\pi^2\hbar c}{240 b^4}\,. \tag{S.41}$$

Numerically, for $b = 1.000$ micron the Casimir pressure is $1.300\cdot 10^{-3}$ Pa.