Part (a):

Classically,

$$\frac{d}{dt}\mathbf{L}_{\text{mech}} = \mathbf{v}\times\vec{\pi} + \mathbf{x}\times\mathbf{F} = 0 + \mathbf{x}\times\mathbf{F} \tag{S.1}$$

where $\mathbf{F}$ is the net force on the charged particle. In presence of the EM fields (1), this force is

$$\mathbf{F} = q\mathbf{E} + \frac{q}{c}\mathbf{v}\times\mathbf{B} = \frac{qQ}{r^2}\,\mathbf{n} + \frac{qM}{cr^2}\,\mathbf{v}\times\mathbf{n}, \tag{S.2}$$

hence

$$\frac{d}{dt}\mathbf{L}_{\text{mech}} = (\mathbf{x}=r\mathbf{n})\times\mathbf{F} = 0 + \frac{qM}{cr}\,\mathbf{n}\times(\mathbf{v}\times\mathbf{n}) = \frac{qM}{cr}\big(\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\big). \tag{S.3}$$

At the same time,

$$\frac{d}{dt}\mathbf{J}_{\text{EM}} = -\frac{qM}{c}\,\frac{d\mathbf{n}}{dt} = -\frac{qM}{c}\,\frac{\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}}{r}\,. \tag{S.4}$$

By inspection of the last two formulae, the separate angular momenta $\mathbf{L}_{\text{mech}}$ and $\mathbf{J}_{\text{EM}}$ are not conserved, but the net angular momentum (2) is conserved,

$$\frac{d}{dt}\mathbf{J}_{\text{net}} = \frac{d}{dt}\mathbf{L}_{\text{mech}} + \frac{d}{dt}\mathbf{J}_{\text{EM}} = 0. \tag{S.5}$$

Quod erat demonstrandum.

Part (b):

Let's start by verifying eq. (7). Since the 3 coordinate operators $\hat{x}_i$ commute with each other, we have

$$[\hat{x}_i, \hat{J}^{\rm EM}_j] \;=\; 0 \tag{S.6}$$

and therefore

$$\begin{aligned} [\hat{x}_i, \hat{J}_j] \;&=\; [\hat{x}_i, \hat{L}_j] \\ &=\; [\hat{x}_i, \epsilon_{jk\ell}\hat{x}_k\hat{\pi}_\ell] \;=\; \epsilon_{jk\ell}\hat{x}_k[\hat{x}_i, \hat{\pi}_\ell] \\ &=\; \epsilon_{jk\ell}\hat{x}_k \times i\hbar\delta_{i\ell} \;=\; i\hbar\epsilon_{jki}\hat{x}_k \\ &=\; i\hbar\epsilon_{ijk}\hat{x}_k\,. \end{aligned} \tag{S.7}$$

Verifying eq. (8) takes more work. First,

$$\begin{aligned} [\hat{\pi}_i, \hat{L}_j] \;&=\; [\hat{\pi}_i, \epsilon_{jk\ell}\hat{x}_k\hat{\pi}_\ell] \\ &=\; \epsilon_{jk\ell}\hat{x}_k \times [\hat{\pi}_i, \hat{\pi}_\ell] + \; \epsilon_{jk\ell}[\hat{\pi}_i, \hat{x}_k] \times \hat{\pi}_\ell \\ &=\; \epsilon_{jk\ell}\hat{x}_k \times \frac{iqM\hbar}{c}\,\epsilon_{i\ell m}\frac{\hat{x}_m}{\hat{r}^3} \;+\; \epsilon_{jk\ell} \times -i\hbar\delta_{ik} \times \hat{\pi}_\ell \\ &=\; -i\hbar\epsilon_{ji\ell}\hat{\pi}_\ell \;+\; \frac{iqM\hbar}{c}\,(\delta_{jm}\delta_{ki} - \delta_{ji}\delta_{mk})\,\frac{\hat{x}_k\hat{x}_m}{\hat{r}^3} \\ &=\; +i\hbar\epsilon_{ij\ell}\hat{\pi}_\ell \;+\; \frac{iqM\hbar}{c}\,\frac{\hat{n}_i\hat{n}_j - \delta_{ij}}{\hat{r}} \end{aligned} \tag{S.8}$$

where $\hat{n}_i \stackrel{\rm def}{=} \hat{x}_i/\hat{r}$. On the bottom line of this formula, the first term is precisely what we want in eq. (8), but the second term is something we do not want. Fortunately, this second term is canceled by the commutator of $\hat{\pi}_i$ with the other part of the net angular momentum,

$$\begin{aligned} [\hat{\pi}_i, \hat{J}^{\rm EM}_j] \;&=\; -\frac{qM}{c}\,[\hat{\pi}_i, \hat{n}_j] \\ &=\; -\frac{qM}{c} \times -i\hbar\widehat{\frac{\partial n_j}{\partial x_i}} \\ &=\; +\frac{i\hbar qM}{c} \times \frac{\delta_{ij} - \hat{n}_i\hat{n}_j}{\hat{r}}\,. \end{aligned} \tag{S.9}$$

Thus altogether,

$$[\hat{\pi}_i, \hat{J}^{\rm net}_j] \;=\; [\hat{\pi}_i, \hat{L}_j] \;+\; [\hat{\pi}_i, \hat{J}^{\rm EM}_j] \;=\; +i\hbar\epsilon_{ij\ell}\hat{\pi}_\ell \;+\; 0, \tag{S.10}$$

precisely as in eq. (8).

Finally, eq. (9) follows from eqs. (7) and (8). Indeed, eq. (7) implies that $\hat{r}^2 = \hat{x}_i\hat{x}_i$ and hence $\hat{r}$ commute with the $\hat{j}_j$, and therefore

$$[\hat{J}_i^{\rm EM},\hat{J}_j] \;=\; -\frac{qM}{c}\left[\frac{\hat{x}_i}{\hat{r}},\hat{j}_j\right] \;=\; -\frac{qM}{c}\,\frac{1}{\hat{r}}\,[\hat{x}_i,\hat{j}_j] \;=\; -\frac{qM}{c}\,\frac{1}{\hat{r}}\times i\hbar\epsilon_{ijk}\hat{x}_k \;=\; i\hbar\epsilon_{ijk}\hat{J}_k^{\rm EM}. \tag{S.11}$$

At the same time, eqs. (7) and (8) together lead to

$$\begin{aligned} [\hat{L}_i,\hat{j}_j] \;&=\; [\epsilon_{ik\ell}\hat{x}_k\hat{\pi}_\ell,\hat{j}_j] \\ &=\; \epsilon_{ik\ell}\hat{x}_k\times\Big([\hat{\pi}_\ell,\hat{J}_j] \;=\; i\hbar\epsilon_{\ell jm}\hat{\pi}_m\Big) \;+\; \epsilon_{ik\ell}\Big([\hat{x}_k,\hat{j}_j] \;=\; i\hbar\epsilon_{kjn}\hat{x}_n\Big)\times\hat{\pi}_\ell \\ &=\; i\hbar\hat{x}_n\hat{\pi}_m\times\Big(\delta_{kn}\epsilon_{ik\ell}\epsilon_{\ell jm} \;+\; \delta_{\ell m}\epsilon_{ik\ell}\epsilon_{kjn}\Big) \end{aligned} \tag{S.12}$$

where

$$\begin{aligned} \delta_{kn}\epsilon_{ik\ell}\epsilon_{\ell jm} \;+\; \delta_{\ell m}\epsilon_{ik\ell}\epsilon_{kjn} \;&=\; \big(\delta_{ij}\delta_{nm}-\delta_{im}\delta_{nj}\big) \;+\; \big(\delta_{mj}\delta_{ni}-\delta_{ji}\delta_{mn}\big) \\ &=\; \delta_{mj}\delta_{ni} \;-\; \delta_{im}\delta_{nj} \\ &=\; \epsilon_{ijk}\epsilon_{knm}\,, \end{aligned} \tag{S.13}$$

hence

$$\begin{aligned} [\hat{L}_i,\hat{j}_j] \;&=\; i\hbar\hat{x}_n\hat{\pi}_m\times\epsilon_{ijk}\epsilon_{knm} \\ &=\; i\hbar\epsilon_{ijk}\times\epsilon_{knm}\hat{x}_n\hat{\pi}_m \\ &=\; i\hbar\epsilon_{ijk}\times\hat{L}_k\,. \end{aligned} \tag{S.14}$$

Finally, combining eqs. (S.11) and (S.14), we arrive at

$$[\hat{J}_i,\hat{J}_j] \;=\; [\hat{L}_j,\hat{J}_j] \;+\; [\hat{J}_i^{\rm EM},\hat{J}_j] \;=\; i\hbar\epsilon_{ijk}\hat{L}_k \;+\; i\hbar\epsilon_{ijk}\hat{J}_k^{\rm EM} \;=\; i\hbar\epsilon_{ijk}\hat{J}_k\,, \tag{S.15}$$

precisely as in eq. (9). *Quod erat demonstrandum.*

Part (c):

Eqs. (7–9) imply that the $\hat{\mathbf{x}}$, $\hat{\vec{\pi}}$, and $\hat{\mathbf{J}}$ operators act as vectors under the space rotations jenerated by the angular momenta $\hat{J}_j$. Consequently, all the scalar combinations made from these operators act as scalars under such rotations and therefore commute with the $\hat{j}_j$. In particular, eq. (7) implies that $\hat{r}^2 = \hat{x}_i\hat{x}_i$ commutes with all the $\hat{j}_j$ and hence the $\hat{r}$ and the $1/\hat{r}$ operators also commute with all the $\hat{j}_j$. In the same way, eq. (8) implies that the $\hat{\vec{\pi}}^2 = \hat{\pi}_i\hat{\pi}_i$ operator also commutes with all the $\hat{j}_j$. Thus, both terms in eq. (10) for the Hamiltonian commute with the angular momenta $\hat{j}_j$, so the whole Hamiltonian also commutes with them,

$$[\hat{H}, \hat{j}_j] \;=\; 0. \tag{S.16}$$

Therefore, in the Heisenberg picture of QM, the angular momentum operators $\hat{j}_j$ are time-independent. In other words, the $\hat{j}_j$ are conserved operators.

Part (d):

By definition

$$\hat{\mathbf{J}} \;=\; \hat{\mathbf{x}}\times\hat{\vec{\pi}} \;+\; \hat{\mathbf{J}}^{\rm EM} \;=\; \hat{\mathbf{x}}\times\hat{\mathbf{p}} \;-\; \frac{q}{c}\,\hat{\mathbf{x}}\times\mathbf{A}(\hat{\mathbf{x}}) \;-\; \frac{qM}{c}\,\hat{\mathbf{n}}\,. \tag{S.17}$$

In this formula

$$\mathbf{x}\times\mathbf{A} \;=\; r\mathbf{n}\times M\,\frac{\pm 1-\cos\theta}{r\sin\theta}\,\mathbf{e}_\phi \;=\; M\,\frac{\pm 1-\cos\theta}{\sin\theta}\,\big(\mathbf{n}\times\mathbf{e}_\phi \;=\; -\mathbf{e}_\theta\big) \tag{S.18}$$

where

$$\mathbf{e}_\theta \;=\; (+\cos\theta\cos\phi, +\cos\theta\sin\phi, -\sin\theta) \tag{S.19}$$

is the unit vector in the θ direction. Focusing on the z component $\hat{J}_z$ of the angular momentum, we have

$$[\mathbf{x}\times\mathbf{A}]_z \;=\; M\,\frac{\pm 1-\cos\theta}{\sin\theta}\,(+\sin\theta) \;=\; M\,(\pm 1-\cos\theta), \tag{S.20}$$

hence

$$\left[-\frac{q}{c}\,\mathbf{x}\times\mathbf{A}(\mathbf{x})\;-\;\frac{qM}{c}\,\mathbf{n}\right]_z\;=\;-\frac{Mq}{c}\,(\pm1-\cos\theta)\;-\;\frac{qM}{c}\,\cos\theta\;=\;\mp\frac{Mq}{c}\tag{S.21}$$

and therefore

$$\hat{J}_z\;=\;[\hat{\mathbf{x}}\times\hat{\mathbf{p}}]_z\;\mp\frac{Mq}{c}\,.\tag{S.22}$$

Finally, in the polar coordinate basis, the $[\hat{\mathbf{x}}\times\hat{\mathbf{p}}]_z$ operator acts as $-i\hbar\partial/\partial\phi$, thus altogether

$$\hat{J}_z\psi(r,\theta,\phi)\;=\;-i\hbar\frac{\partial\psi}{\partial\phi}\;\mp\;\frac{Mq}{c}\times\psi,\tag{S.23}$$

precisely as in eq. (12).

<u>Part (e)</u>:
According to eqs. (S.18) and (S.19),

$$[\mathbf{x}\times\mathbf{A}]_x\;\pm'\;i[\mathbf{x}\times\mathbf{A}]_y=\;-M\,\frac{\pm1-\cos\theta}{\sin\theta}\,\cos\theta\exp(\pm' i\phi)\tag{S.24}$$

where $\pm$ denotes the gauge choice (Northern vs. Southern hemisphere) while $\pm'$ is a separate sign choice, same on both sides of this equation. Likewise,

$$n_x\;\pm'\;in_y\;=\;\sin\theta\exp(\pm' i\phi),\tag{S.25}$$

hence

$$\begin{aligned}\left[-\frac{q}{c}(\mathbf{x}\times\mathbf{A})\;-\;\frac{qM}{c}\mathbf{n}\right]_x\;\pm'\;i\left[-\frac{q}{c}(\mathbf{x}\times\mathbf{A})\;-\;\frac{qM}{c}\mathbf{n}\right]_y\;&=\\ =\;\frac{qM}{c}\,\exp(\pm' i\phi)\times\left(\frac{(\pm1-\cos\theta)\cos\theta}{\sin\theta}\;-\;\sin\theta\right)&\\ =\;\frac{qM}{c}\,\exp(\pm' i\phi)\times\frac{\pm\cos\theta-1}{\sin\theta}\,.&\end{aligned}\tag{S.26}$$

Also, in the spherical coordinates

$$[\hat{\mathbf{x}}\times\hat{\mathbf{p}}]_x\;\pm'\;i[\hat{\mathbf{x}}\times\hat{\mathbf{p}}]_y\;=\;\hbar\exp(\pm' i\phi)\left(\pm'\frac{\partial}{\partial\theta}\;+\;i\coth\theta\,\frac{\partial}{\partial\phi}\right);\tag{S.27}$$

you can find this formula in any undergraduate QM textbook. Thus altogether, plugging the last two formulae into eq. (S.17) for the net angular momentum, we get

$$\hat{J}_x \ \pm' \ i\hat{J}_y \ = \ \hbar \exp(\pm' i\phi) \left(\pm' \frac{\partial}{\partial\theta} + i \coth\theta \, \frac{\partial}{\partial\phi} + \frac{qM}{\hbar c} \, \frac{\pm\cos\theta - 1}{\sin\theta} \right), \tag{S.28}$$

in perfect agreement with eqs. (13). *Quod erat demonstrandum.*

Part (f):

Because of the spherical symmetry of the quantum system in question, we expect all the eigenstates to have wavefucntions of the form

$$\psi(r,\theta,\phi) \ = \ f(r) \times g(\theta) \times h(\phi). \tag{S.29}$$

Moreover, in light of eq. (12), the states of definite m should have

$$h(\phi) \ = \ \exp(im'\phi) \quad \text{for} \quad m' \ = \ m \ \pm \frac{qM}{\hbar c} \, . \tag{S.30}$$

Or rather, in the Northern hemisphere gauge

$$h_N(\phi) \ = \ \exp(im_N\phi) \quad \text{for} \quad m_N \ = \ m \ + \ \frac{qM}{\hbar c} \, , \tag{S.31}$$

while in the Southern hemisphere gauge

$$h_S(\phi) \ = \ \exp(im_S\phi) \quad \text{for} \quad m_S \ = \ m \ - \ \frac{qM}{\hbar c} \, . \tag{S.32}$$

Both h_N and h_s must be single-valued functions of the angle ϕ, so both m_N and m_S must be integer. Consequently:

1. $qM/\hbar c$ must be integer or half-integer — this is the Dirac's charge quantiation condition.

2. For integer $qM/\hbar c$, the eigenvalue m of the $\hat{J}_z$ must be integer, and hence j must also be integer. But for a half-integer $qM/\hbar c$, the eigenvalue m mst be half-integer, and hence j must also be half-integer.

Now consider a multiplet of states $|j,m\rangle$ of definite j and all possible m ranging from $-j$ to $+j$ by 1. In this multiplet, the state with maximal $m=+j$ must be annihilated by the $\hat{J}_+$ operator,

$$(\hat{J}_x + i\hat{J}_y)\,|j,m=j\rangle \;=\; 0. \tag{S.33}$$

In polar coordinates, this operator acts as in the top eq. (13), so for a wave function of the form (S.29) with $h(\phi)$ as in eq. (S.30), we have

$$\begin{aligned}
\hat{J}_+\psi(r,\theta,\phi) \;&=\; \hbar\exp(+i\phi)f(r)h(\phi)\times\left(\begin{aligned} &+\frac{dg}{d\theta} - \left(m' \,=\, m\pm\frac{qM}{\hbar c}\right)\coth\theta\times g(\theta)\\ &\qquad - \frac{qM}{\hbar c}\,\frac{1\mp\cos\theta}{\sin\theta}\times g(\theta)\end{aligned}\right)\\
&=\; \hbar\exp(+i\phi)f(r)h(\phi)\times\left(+\frac{dg}{d\theta} - \left(m\coth\theta + \frac{qM}{\hbar c}\,\frac{1}{\sin\theta}\right)\times g(\theta)\right).
\end{aligned} \tag{S.34}$$

For the state with $m=+j$ the LHS here must vanish, so the $g(\theta)$ function must obey the differential equation

$$\frac{dg}{d\theta} \;=\; \left(m\coth\theta + \frac{qM}{\hbar c}\,\frac{1}{\sin\theta}\right)\times g. \tag{S.35}$$

Consequently,

$$\begin{aligned}
d\log g(\theta) \;&=\; \frac{dg}{g} \;=\; \left(m\coth\theta + \frac{qM}{\hbar c}\,\frac{1}{\sin\theta}\right)d\theta\\
&=\; \left(m - \frac{qM}{\hbar c}\right)\times\frac{\cos\theta-1}{2\sin\theta}\,d\theta \;+\; \left(m + \frac{qM}{\hbar c}\right)\times\frac{\cos\theta+1}{2\sin\theta}\,d\theta\\
&=\; \left(m - \frac{qM}{\hbar c}\right)\times\left(\frac{-\sin(\theta/2)d\theta}{2\cos(\theta/2)} \;=\; d\log\cos(\theta/2)\right)\\
&\qquad + \left(m + \frac{qM}{\hbar c}\right)\times\left(\frac{\cos(\theta/2)d\theta}{2\sin(\theta/2)} \;=\; d\log\sin(\theta/2)\right)
\end{aligned} \tag{S.36}$$

and therefore

$$g(\theta) \;=\; \text{const}\times\bigl(\cos(\theta/2)\bigr)^{n_1}\times\bigl(\sin(\theta/2)\bigr)^{n_2} \quad\text{for}\quad n_{1,2} \;=\; (m=j)\;\mp\;\frac{qM}{\hbar c}\,. \tag{S.37}$$

To make this solution regular at both $\theta=0$ and $\theta=\pi$, both n_1 and n_2 must be non-negative integers. Consequently, we need

$$j \;=\; \left|\frac{qM}{\hbar c}\right| \;+\; \text{a non-negative integer}, \tag{S.38}$$

precisely as in eq. (14). *Quod erat demonstrandum.*

Part (g):
First,

$$\hat{\mathbf{J}}^2 \;=\; \left(\hat{\mathbf{L}} + \frac{qM}{c}\hat{\mathbf{n}}\right)^2 \;=\; \hat{\mathbf{L}}^2 \;+\; \left(\frac{qM}{c}\right)^2 \;+\; \frac{qM}{c}\big(\hat{\mathbf{n}}\cdot\hat{\mathbf{L}} + \hat{\mathbf{L}}\cdot\hat{\mathbf{n}}\big) \;=\; \hat{\mathbf{L}}^2 \;+\; \left(\frac{qM}{c}\right)^2 \tag{S.39}$$

because

$$\hat{\mathbf{n}}\cdot\hat{\mathbf{L}} \;=\; \hat{\mathbf{L}}\cdot\hat{\mathbf{n}} \;=\; 0. \tag{S.40}$$

Second,

$$\hat{\mathbf{L}}^2 \;=\; \big(\hat{\mathbf{x}}\times\hat{\vec{\pi}}\big)^2 \;=\; \hat{r}^2\big(\hat{\vec{\pi}}^2 \;-\; \hat{\pi}_r^2\big), \tag{S.41}$$

which obtains exactly as in QM of a particle in a central potential without the magnetic field. To be safe, I'll derive this formula for the present case in a momemt. But once we have this formula, eq. (15) follows immediately from eqs. (S.39) and (S.41).

Now let's derive eq. (S.41). Classically, it follows from the basic vector algebra:

$$\mathbf{L}^2 \;=\; (\mathbf{x}\times\vec{\pi})^2 \;=\; \mathbf{x}^2\,\vec{\pi}^2 \;-\; (\mathbf{x}\cdot\vec{\pi})^2 \;=\; r^2\big(\vec{\pi}^2 \;-\; (\mathbf{n}\cdot\vec{\pi})^2\big) \;=\; r^2\big(\vec{\pi}^2 \;-\; \pi_r^2\big). \tag{S.42}$$

But in the quantum mechanics, we have to watch out for the commutators, thus

$$\begin{aligned}
\hat{\mathbf{L}}^2 \;&=\; \big(\epsilon_{ijk}\hat{x}_j\hat{\pi}_k\big)\,\big(\epsilon_{i\ell m}\hat{x}_\ell\hat{\pi}_m\big) \\
&=\; \hat{x}_j\hat{\pi}_k\hat{x}_\ell\hat{p}_m \times \big(\epsilon_{ijk}\epsilon_{i\ell m} \;=\; \delta_{j\ell}\delta_{km} \;-\; \delta_{jm}\delta_{k\ell}\big) \\
&=\; \hat{x}_j\hat{\pi}_k\hat{x}_j\hat{\pi}_k \;-\; \hat{x}_j\hat{\pi}_k\hat{x}_k\hat{\pi}_j\,,
\end{aligned} \tag{S.43}$$

where

$$\hat{x}_j\hat{\pi}_k\hat{x}_j\hat{\pi}_k \;=\; \hat{x}_j\hat{x}_j\hat{\pi}_k\hat{\pi}_k \;+\; \hat{x}_j\big([\hat{\pi}_k,\hat{x}_j] = -i\hbar\delta_{jk}\big)\hat{\pi}_k \;=\; \hat{r}^2\,\hat{\vec{\pi}}^2 \;-\; i\hbar\hat{x}_j\hat{\pi}_j\,, \tag{S.44}$$

while

$$\begin{aligned}
\hat{x}_j\hat{\pi}_k\hat{x}_k\hat{\pi}_j \;&=\; [\hat{x}_j,\hat{\pi}_k]\hat{x}_k\hat{\pi}_j \;+\; \hat{\pi}_k\hat{x}_k\hat{x}_j\hat{\pi}_j\\
&=\; [\hat{x}_j,\hat{\pi}_k]\hat{x}_k\hat{\pi}_j \;+\; [\hat{\pi}_k,\hat{x}_k]\hat{x}_j\hat{\pi}_j \;+\; (\hat{x}_k\hat{\pi}_k)(\hat{x}_j\hat{\pi}_j)\\
&=\; i\hbar\delta_{jk}\hat{x}_k\hat{\pi}_j \;-\; i\hbar\delta_{kk}\hat{x}_j\hat{\pi}_j + \;(\hat{x}_k\hat{\pi}_k)^2\\
&=\; i\hbar(\hat{\mathbf{x}}\cdot\hat{\vec{\pi}}) \;-\; 3i\hbar(\hat{\mathbf{x}}\cdot\hat{\vec{\pi}}) \;+\; (\hat{\mathbf{x}}\cdot\hat{\vec{\pi}})^2 \;=\; (\hat{\mathbf{x}}\cdot\hat{\vec{\pi}})^2 \;-\; 2i\hbar(\hat{\mathbf{x}}\cdot\hat{\vec{\pi}}).
\end{aligned} \tag{S.45}$$

Altogether, this gives us

$$\hat{\mathbf{L}}^2 \;=\; \hat{r}^2\,\hat{\vec{\pi}}^2 \;-\; (\hat{\mathbf{x}}\cdot\hat{\vec{\pi}})^2 \;+\; i\hbar(\hat{\mathbf{x}}\cdot\hat{\vec{\pi}}). \tag{S.46}$$

Now let's compare the second and the third terms here to $\hat{r}^2\hat{\pi}_r^2$. First,

$$\hat{\pi}_r \;\stackrel{\text{def}}{=}\; \tfrac{1}{2}\big(\hat{n}_i\hat{\pi}_i + \hat{\pi}_i\hat{n}_i\big) \;=\; \hat{n}_i\hat{\pi}_i \;+\; \tfrac{1}{2}[\hat{\pi}_i,\hat{n}_i] \;=\; \hat{n}_i\hat{\pi}_i \;+\; \tfrac{1}{2}(-i\hbar)\left(\frac{\partial n_i}{\partial x_i} \;=\; \frac{2}{r}\right) \;=\; \hat{n}_i\hat{\pi}_i \;-\; \frac{i\hbar}{\hat{r}}\,. \tag{S.47}$$

Second,

$$\hat{r}^2\hat{\pi}_r^2 \;=\; \hat{r}\hat{\pi}_r\hat{r}\hat{\pi}_r \;+\; \hat{r}[\hat{r},\hat{\pi}_r]\hat{\pi}_r \tag{S.48}$$

where

$$[\hat{r},\hat{\pi}_r] \;=\; [\hat{r},\hat{n}_i\hat{\pi}_i] \;=\; \hat{n}_i[\hat{r},\hat{\pi}_i] = \;\hat{n}_i\left(i\hbar\widehat{\frac{\partial r}{\partial x_i}} \;=\; i\hbar\hat{n}_i\right) \;=\; i\hbar. \tag{S.49}$$

Consequently,

$$\hat{r}^2\hat{\pi}_r^2 \;=\; (\hat{r}\hat{\pi}_r)^2 \;+\; i\hbar(\hat{r}\hat{\pi}_r) \;=\; (\hat{x}_i\hat{\pi}_i - i\hbar)^2 \;+\; i\hbar(\hat{x}_i\hat{\pi}_i - i\hbar) \;=\; (\hat{x}_i\hat{\pi}_i)^2 \;-\; i\hbar(\hat{x}_i\hat{\pi}_i), \tag{S.50}$$

and comparing this formula to the RHS of eq. (S.46), we immediately see that

$$\hat{\mathbf{L}}^2 \;=\; \hat{r}^2\,\hat{\vec{\pi}}^2 \;-\; \hat{r}^2\hat{\pi}_r^2, \tag{S.51}$$

precisely as in eq. (S.41).

This completes our derivation of eq. (S.41) and hence eq. (15)

Part (h):

In light of eq. (15), the Hamiltonian of the charged particle orbiting a dyon can be written as

$$\hat{H} = \frac{\hat{\pi}_r^2}{2m} + \frac{\hat{\mathbf{J}}^2 - (qM/c)^2}{2m\hat{r}^2} - \frac{qQ}{\hat{r}}. \tag{S.52}$$

In the coordinate basis

$$\hat{\pi}_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right), \qquad \hat{\pi}_r^2 = -\hbar^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right), \tag{S.53}$$

so the radial wave function $f(r)$ (*cf.* eq. (S.29)) of a bound state $|n_r, j, m\rangle$ of energy $E < 0$ and anjular momentum j obeys the radial Schrödinger equation

$$\frac{\hbar^2}{2m}\left(-f''(r) - \frac{2}{r}f'(r) + \frac{j(j+1) - (qM/\hbar c)^2}{r^2}f(r)\right) - \frac{qQ}{r}f(r) = Ef(r). \tag{S.54}$$

This equation looks exactly like the radial Schrödinger equation for the hydrogen atom — except for having

$$\lambda(\lambda+1) \stackrel{\rm def}{=} j(j+1) - (qM/\hbar c)^2 \tag{18}$$

instead of $\ell(\ell+1)$ — and it can be solved in exactly the same way. You can find a solution — and there are many different way to solve eq. (S.54) — in any undergraduate QM textbook; but since ν is generally non-integral while many textbook solutions make use of ℓ being an integer, let me write down a solution of my own.

First, let me introduce a couple of parameters:

$$\kappa = \frac{1}{\hbar}\sqrt{-2mE} \tag{S.55}$$

for a bound state of negative energy $E < 0$, and

$$\nu = \frac{qQm}{\hbar^2\kappa}. \tag{S.56}$$

In terms of these parameters (as well as λ), eq. (S.54) becomes

$$f'' + \frac{2}{r}\times f' - \frac{\lambda(\lambda+1)}{r^2}\times f + \frac{2\nu\kappa}{r}\times f = \kappa^2\times f. \tag{S.57}$$

Now let's take the asymptotic limits $r \to \infty$ and $r \to 0$. For $r \to \infty$, we may crudely approximate eq. (S.57) as

$$f'' \;\approx\; \kappa^2 f, \tag{S.58}$$

so the normalizable solution behaves as $f(r) \sim \exp(-\kappa r)$. In the opposite limit of $r \to 0$, we approximate eq. (S.57) as

$$f'' \;+\; \frac{2}{r} \times f' \;-\; \frac{\lambda(\lambda+1)}{r^2} \times f \;\approx\; 0, \tag{S.59}$$

with the normalizable solution being $f \sim r^\lambda$. In light of these asymptotic limits, we let

$$f(r) \;=\; r^\lambda \times \exp(-\kappa r) \times \Phi(r) \tag{S.60}$$

for some (hopefully) regular function $\Phi(r)$. Following eq. (S.60), we have

$$f'(r) \;=\; r^\lambda \exp(-\kappa r) \times \left(\Phi' \;+\; \frac{\lambda}{r}\Phi \;-\; \kappa\Phi\right), \tag{S.61}$$

$$f''(r) \;=\; r^\lambda \exp(-\kappa r) \times \left(\Phi'' \;+\; \left(\frac{2\lambda}{r} - 2\kappa\right)\Phi' \;+\; \left(\frac{\lambda(\lambda-1)}{r^2} - \frac{2\lambda\kappa}{r} + \kappa^2\right)\Phi\right), \tag{S.62}$$

and consequently eq. (S.57) becomes

$$\Phi'' \;+\; 2\left(\frac{\lambda+1}{r} - \kappa\right) \times \Phi' \;+\; 2\frac{(\nu-\lambda-1)\kappa}{r} \times \Phi. \tag{S.63}$$

To solve this equation, we rewrite it as

$$r \times \Big(\Phi'' \;-\; 2\kappa\Phi'\Big) \;+\; 2\Big((\lambda+1)\Phi' \;+\; (\nu-\lambda-1)\kappa\Phi\Big) \;=\; 0 \tag{S.64}$$

and then Laplace transform it to a first-order differential equation. Thus, we look for $\Phi(r)$ in the form of a contour integral in the complex plane,

$$\Phi(r) \;=\; \int\limits_\Gamma dt\, e^{tr} \times F(t) \tag{S.65}$$

for some analytic function $F(t)$ and some contour Γ. To allow integration by parts, Γ should be either a closed contour, or else both ends should extend to ∞ in directions along which the integrand dies off rapidly enough.

Given the Laplace transform (S.65) of the Φ function itself, we have

$$\frac{d\Phi}{dr} = \int\limits_\gamma dt\, e^{tr} \times t \times F(t), \tag{S.66}$$

$$\frac{d^2\Phi}{dr^2} = \int\limits_\gamma dt\, e^{tr} \times t^2 \times F(t), \tag{S.67}$$

while

$$\begin{aligned} r \times \Phi(r) &= \int\limits_\Gamma dt\, \left(re^{tr} = \frac{\partial e^{tr}}{\partial t} \right) \times F(t) \\ \langle\!\langle \text{ by parts } \rangle\!\rangle &= -\int\limits_\Gamma dt\, e^{tr} \times \frac{dF}{dt}\,, \end{aligned} \tag{S.68}$$

and likewise

$$r \times \Big(\Phi'' - 2\kappa\Phi' \Big) = -\int\limits_\Gamma dt\, e^{tr} \times \frac{d}{dt}\Big(t^2 F(t) - 2\kappa t F(t) \Big). \tag{S.69}$$

Plugging all these formulae into eq. (S.64), we may recast it as an equation for the $F(t)$, namely

$$-\frac{d}{dt}\Big((t^2 - 2\kappa t) F(t) \Big) + 2\big((\lambda+1)t + (\nu-\lambda-1)\kappa \big) F(t) = 0. \tag{S.70}$$

This is a fairly easy first-order differential equation. To solve, we rewrite it as

$$t(t-2\kappa) \times \frac{dF}{dt} = 2(\lambda t + \nu\kappa - \lambda\kappa) \times F(t), \tag{S.71}$$

hence

$$\frac{dF/dt}{F} = \frac{2\lambda t + 2\kappa\nu - 2\kappa\lambda}{t(t-2\kappa)} = \frac{\lambda-\nu}{t} + \frac{\lambda+\nu}{t-2\kappa}\,, \tag{S.72}$$

$$\begin{aligned} d\log F &= \frac{dF}{F} = (\lambda-\nu)\frac{dt}{t} + (\lambda+\nu)\frac{dt}{t-2\kappa} \\ &= (\lambda-\nu)d\log(t) + (\lambda+\nu)d\log(t-2\kappa), \end{aligned} \tag{S.73}$$

$$F(t) \;=\; \text{const} \times t^{\lambda-\nu} \times (t-2\kappa)^{\lambda+\nu}, \tag{S.74}$$

and therefore

$$\psi_{\text{radial}} \;=\; f(r) \;= \text{const} \times r^{\lambda} e^{-\kappa r} \times \int\limits_{\Gamma} dt\, e^{tr} \times t^{\lambda-\nu} \times (t-2\kappa)^{\lambda+\nu}. \tag{S.75}$$

It remain to determine the integration contour Γ in this formula. For generic λ and ν, the integrand in eq. (S.75) has two branch cuts, one from $t=0$ to $t=2\kappa$ and the other from $t=0$ to $t=\infty$; let's lay the combined branch cut along the real axis, from $t=-\infty$ to $t=+2\kappa$. Since there are no other singularities, the integration contour must therefore surround this cut, with both running to $-\infty$ on two sides of the cut. Consequently, the integral in eq. (S.75) becomes

$$2 \times \int\limits_{-\infty}^{+2\kappa} dt\, e^{rt} \times \text{disc}\Big[t^{\lambda-\nu} \times (t-2\kappa)^{\lambda+\nu}\Big] \tag{S.76}$$

where 'disc' stands for the discontinuity of $[\cdots]$ across the real axis.

For large r, the exponential e^{rt} grows rapidly with t, so the integral (S.76) is dominated by its right end at $t=2\kappa$, so asymptotically

$$\text{for } r\to\infty: \quad \text{theintegral} \;\sim\; e^{+2\kappa r} \times r^{\text{some power}} \tag{S.77}$$

and therefore

$$\psi_{\text{rad}}(r) \;\sim\; e^{+\kappa r} \times r^{\text{some power}}. \tag{S.78}$$

Such a radial wave function is un-normalizable, so there are no good solutions for generic λ and ν.

To get a good, normalizable solution of the radial wave equation, we need the integrand of eq. (S.75) to have a different geometry of singularities that would allow a different kind of an integration contour. Such geometry obtains when $\lambda - \nu$ is a negative integer, *i.e.*

$$\nu \;=\; \lambda \;+\; n_r, \quad n_r \;=\; 1,2,3,4,\ldots, \tag{S.79}$$

hence

$$\kappa \;=\; \frac{mqQ}{\nu\hbar^2} \;=\; \frac{mqQ}{(\lambda+n_r)\hbar^2} \tag{S.80}$$

and the bound state energy

$$E \;=\; -\frac{\hbar^2\kappa^2}{2m} \;=\; -\frac{m(qQ)^2}{2\hbar^2(\lambda+n_r)^2} \tag{S.81}$$

precisely as in eq. (17). Indeed, for $\lambda - \nu = -n_r$, the integrand

$$e^{tr} \times t^{\lambda-\nu} \times (t-2\kappa)^{\lambda+\nu} \tag{S.82}$$

has an isolated pole at $t = 0$ in addition to a branch cut from $t = -\infty$ to $t = +2\kappa$. Let's reroute the branch cut so it lies away from the pole at $t = 0$. Then in addition to the integration contour surrounding the branch cut, we have another option for the contour — a small circle around the pole around $t = 0$. For such a contour, the integral extracts the residue of this pole, hence

$$\psi_{\rm rad}(r) \;=\; \text{const} \times r^\lambda e^{-\kappa r} \times \underset{@t=0}{\text{Residue}} \left[\frac{e^{tr} \times (t-2\kappa)^{2\lambda+n_r}}{t^{n_r}}\right]. \tag{S.83}$$

As a function of r, the residue here is a polynomial of degree $n_r - 1$, so

$$\text{for } r\to\infty \quad \psi_{\rm rad} \;\sim\; e^{-\kappa r} \times r^{\text{some power}} \tag{S.84}$$

which makes for a normalizable radial wavefunction.