Problem **1**(a):

Classically,

$$\mathcal{L} \;=\; \mathcal{L}_{\rm YM} \;+\; D^\mu\Phi^\dagger D_\mu\Phi \;-\; V(\Phi^\dagger,\Phi) \tag{S.1}$$

where

$$D_\mu\Phi^i \;=\; \partial_\mu\Phi_i \;+\; igA^a_\mu(T^a_{(r)})^i{}_j\Phi^j\,, \qquad D^\mu\Phi^*_i \;=\; \partial^\mu\Phi^*_i \;-\; igA^{a\mu}\Phi^*_j(T^a_{(r)})^j{}_i, \tag{S.2}$$

and $V(\Phi^\dagger,\Phi)$ is some kind of a G–invariant potential. For renormalizability's sake, V should be a polynomial of degree 4 (or less), and to keep my notations simple I assume that V has only the quadratic mass term and the quartic interaction term, thus

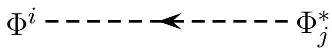
$$V \;=\; m^2\times\Phi^*_i\Phi^i \;+\; \tfrac14\lambda^{ij}_{k\ell}\times\Phi^*_i\Phi^*_j\Phi^k\Phi^\ell \tag{S.3}$$

for a suitable G-invariant coupling array $\lambda^{ij}_{k\ell}$. For example, for scalars Φ^i in the fundamental $\mathbf{N}$ multiplet of the $SU(N)$ gauge group, $\lambda^{ij}_{k\ell}=\lambda\times\left(\delta^i_k\delta^j_\ell+\delta^i_\ell\delta^j_k\right)$. But the details of the $\lambda^{ij}_{k\ell}$ coupling are not germane for the present problem, so I'll keep them generic as long as they are G-invariant.

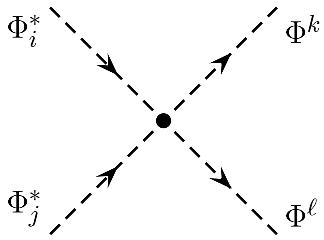
In the quantum field theory, the net bare Lagrangian comprises the classical terms (S.1) plus the ghost Lagrangian, the gauge fixing terms, and the whole slew of counterterms. Altogether, we have

$$\begin{aligned}
\mathcal{L} \;=\;& -\frac14(F^a_{\mu\nu})^2 \;-\; \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \;+\; \partial_\mu\bar c^a D^\mu c^a\\
&\quad+\; D_\mu\Phi^\dagger D^\mu\Phi \;-\; m^2\Phi^\dagger\Phi \;-\; \tfrac14\lambda^{ij}_{k\ell}\,\Phi^*_i\Phi^*_j\Phi^k\Phi^\ell\\
&-\;\frac{\delta_3}{4}(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu)^2 \;+\; g\delta_1^{(3g)}f^{abc}A^b_\mu A^c_\nu\partial_\mu A^{a\nu} \;-\; \frac{g^2\delta_1^{(4g)}}{4}(f^{abc}A^b_\mu A^c_\nu)^2\\
&+\;\delta_2^{(\rm gh)}\partial_\mu\bar c^a\partial^\mu c^a \;-\; g\delta_1^{(\rm gh)}f^{abc}\partial_\mu\bar c^a A^{b\mu}c^c\\
&+\;\delta_2^{(\phi)}\partial_\mu\Phi^\dagger\partial^\mu\Phi \;+\; ig\delta_1^{(\phi1g)}\,A^a_\mu\times\left(\partial^\mu\Phi^\dagger T^a_{(r)}\Phi \;-\; \Phi^\dagger T^a_{(r)}\partial^\mu\Phi\right)\\
&\quad+\; g^2\delta_1^{(\phi2g)}A^a_\mu A^b\mu\times\Phi^\dagger T^a_{(r)}T^b_{(r)}\Phi\\
&-\;\delta_m^{(\phi)}\Phi^\dagger\Phi \;-\; \tfrac14(\delta_\lambda)^{ij}_{k\ell}\,\Phi^*_i\Phi^*_j\Phi^k\Phi^\ell.
\end{aligned} \tag{S.4}$$

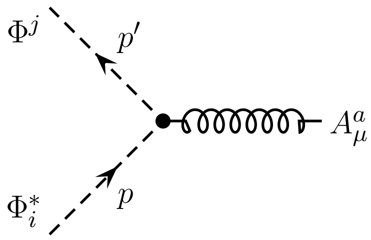
Note that all the terms in this bare Lagrangian which pertain only to the vector and ghost fields are exactly the same as in the fermionic QCD, *cf.* my notes on QCD Feynman rules. Consequently, in the Feynman rules of the present theory, the gluon propagator, the three-gluon and the four-gluon vertices, the ghost propagator and the ghost-gluon vertex are exactly as in my notes, and I don't need to repeat them here. But let me write down the explicit Feynman rules pertaining to the scalar fields, in particular, the scalar propagator and the scalar vertices:



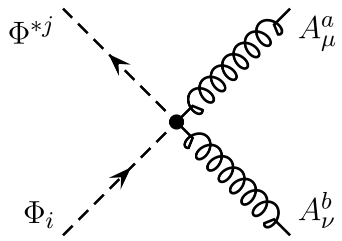
$$= \frac{i\delta^i_j}{p^2 - m^2 + i0}\,, \tag{S.5}$$



$$= -i\lambda^{ij}_{k\ell}\,, \tag{S.6}$$



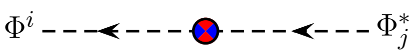
$$= ig(p+p')_\mu (T^a_{(r)})^j{}_i, \tag{S.7}$$



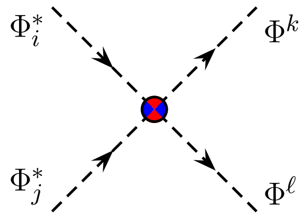
$$= ig^2 g_{\mu\nu}\{T^a_{(r)}, T^b_{(r)}\}^j{}_i. \tag{S.8}$$

Note the anticommutator of the group generators in the two-scalar two-gluon vertex: It follows from permutations of the two gluon lines.

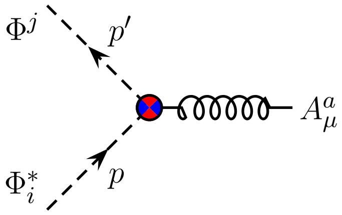
In addition, there are several counterterm vertices involving the scalar fields. Although such vertices are not germane to the present exercise, let me list them here for the completeness sake:



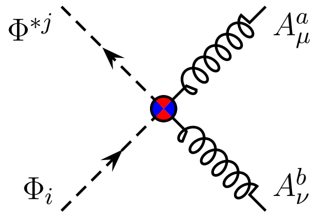
$$= \delta^i_j\big(i\delta^{(\phi)}_m - i\delta^{(\phi)}_2 p^2\big), \tag{S.9}$$



$$= \; -i\big(\delta_\lambda\big)^{ij}_{k\ell}\,, \tag{S.10}$$



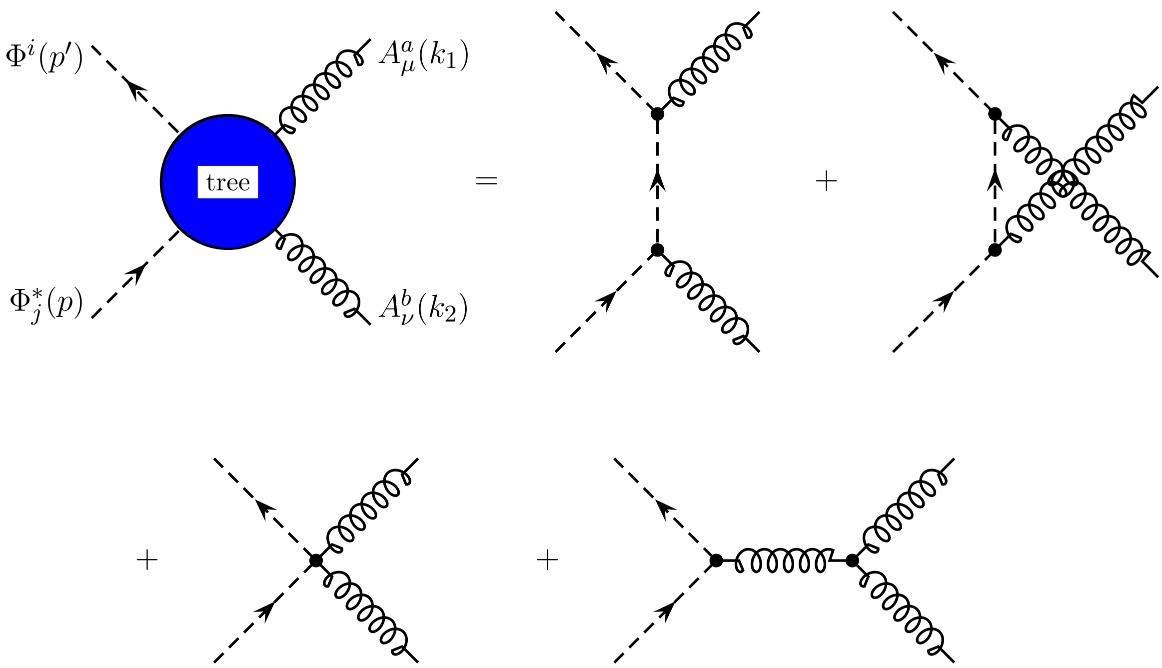
$$= \; ig\delta_1^{(\phi 1g)} \times (p+p')_\mu (T^a_{(r)})^j{}_i, \tag{S.11}$$



$$= \; ig^2\delta_1^{(\phi 2g)} \times g_{\mu\nu}\{T^a_{(r)}, T^b_{(r)}\}^j{}_i. \tag{S.12}$$

Problem **1**(b):

At the tree level there are four diagrams for the $\Phi + \Phi^* \to g + g$ annihilation process, namely



(S.13)

The amplitude stemming from each of these 4 diagrams has form

$$\mathcal{M}[\text{diagram}\#n] \;=\; e^*_{1\mu}\, e^*_{2\nu} \times \mathcal{M}^{\mu\nu}_n \tag{S.14}$$

where

$$\mathcal{M}^{\mu\nu}_1 \;=\; \frac{-g^2}{(p-k_1)^2 - m^2}\,(2p-k_1)^\mu (k_2 - 2p')^\nu \times (T^b T^a)^i{}_j\,, \tag{S.15}$$

$$\mathcal{M}^{\mu\nu}_2 \;=\; \frac{-g^2}{(p'-k_1)^2 - m^2}\,(k_1-2p')^\mu (2p - k_2)^\nu \times (T^a T^b)^i{}_j\,, \tag{S.16}$$

$$\mathcal{M}^{\mu\nu}_3 \;=\; +g^2 g^{\mu\nu} \times \{T^a, T^b\}^i{}_j\,, \tag{S.17}$$

$$\begin{aligned}\mathcal{M}^{\mu\nu}_4 \;=\; &-\frac{ig^2}{(k_1+k_2)^2}\,(p-p')_\lambda\,(T^c)^i{}_j \\ &\times f^{abc}\big(g^{\mu\nu}(k_1-k_2)^\lambda \,+\, g^{\nu\lambda}(2k_2+k_1)^\mu \,+\, g^{\lambda\mu}(-2k_1-k_2)^\nu\big),\end{aligned} \tag{S.18}$$

so the net tree-level amplitude is

$$\mathcal{M}_{\text{net}} \;=\; e^*_{1\mu}\, e^*_{2\nu} \times \mathcal{M}^{\mu\nu}_{\text{net}} \;=\; e^*_{1\mu}\, e^*_{2\nu} \times \big(\mathcal{M}^{\mu\nu}_1 + \mathcal{M}^{\mu\nu}_2 + \mathcal{M}^{\mu\nu}_3 + \mathcal{M}^{\mu\nu}_4\big) \tag{S.19}$$

Problem **1**(c):

Our task is to verify that the net amplitude (S.19) satisfies

$$k_1^\mu e_2^\nu \mathcal{M}_{\mu\nu} \;=\; 0 \tag{S.20}$$

provided $e_2^\nu k_{2\nu} = 0$ and all external momenta are on shell. Let's start by calculating the $k_{1\mu}\mathcal{M}^{\mu\nu}_n$ for each of the 4 diagrams. For the first diagram's amplitude (S.15),

$$k_{1\mu}\mathcal{M}^{\mu\nu}_1 \;=\; -g^2 (k_2 - 2p')^\nu \times \frac{2(pk_1) - k_1^2}{(p-k_1)^2 - m^2} \times (T^b T^a)^j{}_i \tag{S.21}$$

where for the on-shell momenta $p^2 = p'^2 = m^2$, $k_1^2 = k_2^2 = 0$,

$$\frac{2(pk_1) - k_1^2}{(p-k_1)^2 - m^2} \;=\; -1 \tag{S.22}$$

and hence

$$k_{1\mu}\mathcal{M}_1^{\mu\nu} \;=\; +g^2(k_2-2p')^\nu\times(T^bT^a)^j{}_i\,. \tag{S.23}$$

Likewise, for the second diagram's amplitude (S.16),

$$\begin{aligned} k_{1\mu}\mathcal{M}_2^{\mu\nu} \;&=\; -g^2(2p-k_2)^\nu\times\frac{k_1^2-2(p'k_1)}{(p'-k_1)^2-m^2}\times(T^aT^b)^j{}_i \\ &\qquad\langle\!\langle\,\text{for the on-shell momenta}\,\rangle\!\rangle \\ &=\; +g^2(k_2-2p)^\nu\times 1\times(T^aT^b)^i{}_j\,. \end{aligned} \tag{S.24}$$

For the third diagram's amplitude (S.17) we have

$$k_{1\mu}\mathcal{M}_3^{\mu\nu} \;=\; +g^2k_1^\nu\times\{T^a,T^b\}^i{}_j \;=\; g^2k_1^\nu\times(T^aT^b)^i{}_j \;+\; g^2k_1^\nu\times(T^bT^a)^i{}_j\,, \tag{S.25}$$

so adding the first three diagrams together, we obtain

$$\begin{aligned} k_{1\mu}\times\mathcal{M}_{1+2+3}^{\mu\nu} \;&=\; g^2(T^aT^b)^i{}_j\times\bigl((k_2-2p')\,+\,k_1\bigr)^\nu \;+\; g^2(T^bT^a)^i{}_j\times\bigl((k_2-2p)\,+\,k_1\bigr)^\nu \\ &\qquad\langle\!\langle\,\text{using momentum conservation } k_1+k_2=p+p'\,\rangle\!\rangle \\ &=\; g^2(T^aT^b)^i{}_j\times(p-p')^\nu \;+\; g^2(T^bT^a)^i{}_j\times(p'-p)^\nu \\ &=\; g^2(p-p')^\nu\times(T^aT^b\;-\;T^bT^a)^i{}_j \\ &=\; g^2(p-p')^\nu\times if^{abc}(T^c)^i{}_j\,. \end{aligned} \tag{S.26}$$

As to the fourth diagram's amplitude (S.18),

$$\begin{aligned} k_{1\mu}\mathcal{M}_4^{\mu\nu} \;&=\; g^2(p'-p)_\lambda\times if^{abc}(T^c)^i{}_j\times\frac{1}{(k_1+k_2)^2}\times \\ &\qquad\times k_{1\mu}\bigl[g^{\mu\nu}(k_1-k_2)^\lambda\;+\;g^{\nu\lambda}(2k_2+k_1)^\mu\;+\;g^{\lambda\mu}(-2k_1-k_2)^\nu\bigr], \end{aligned} \tag{S.27}$$

where the expression on the second line is exactly similar to its analogue in the fermionic QCD, *cf.* eqs. (37–38) on page 9 of my notes on QCD Ward identities. Just as in my notes, for the on-shell photon momenta

$$k_{1\mu} \times [\cdots] \;=\; (k_1+k_2)^2 g^{\nu\lambda} \;-\; (k_1+k_2)^\nu (k_1+k_2)^\lambda \;+\; k_2^\nu k_2^\lambda\,. \tag{S.28}$$

hence plugging each of the 3 terms here into eq. (S.27), we obtain

$$k_{1\mu}\mathcal{M}_4^{\mu\nu} \;=\; k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu} \;+\; k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu} \;+\; k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu}\,, \tag{S.29}$$

where

$$k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu} \;=\; g^2(p'-p)^\nu \times if^{abc}(T^c)^i{}_j\,, \tag{S.30}$$

$$k_{1\mu}\mathcal{M}_{4,b}^{\mu\nu} \;=\; -g^2(k_1-k_2)^\nu \times \frac{(p'-p)_\lambda(k_1+k_2)^\lambda}{(k_1+k_2)^2} \times if^{abc}(T^c)^i{}_j\,, \tag{S.31}$$

$$k_{1\mu}\mathcal{M}_{4,c}^{\mu\nu} \;=\; g^2 k_2^\nu \times \frac{(p'-p)_\lambda k_2^\lambda}{(k_1+k_2)^2} \times if^{abc}(T^c)^i{}_j\,. \tag{S.32}$$

By inspection of eqs. (S.30) and (S.26), the first term's contribution precisely cancels the combined contributions of the diagrams 1, 2, and 3,

$$k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu} \;+\; k_{1\mu}\mathcal{M}_{1+2+3}^{\mu\nu} \;=\; 0. \tag{S.33}$$

As to the second term's contribution (S.31), it vanishes for the on-shell scalars' momenta $p^2 = p'^2 = m^2$; indeed,

$$(p'-p)_\lambda(k_1+k_2)^\lambda \;=\; (p'-p)_\lambda(p+p')^\lambda \;=\; p'^2-p^2 \;=\; m^2-m^2 \;=\; 0 \quad\Longrightarrow\quad k_{1\mu}\mathcal{M}_{4,b}^{\mu\nu} \;=\; 0. \tag{S.34}$$

Finally, the third term's contribution (S.32) does not vanish but its ν index belongs to the k_2^ν factor, thus

$$k_{1\mu}\mathcal{M}_{\rm net}^{\mu\nu} \;=\; k_{1\mu}\mathcal{M}_{4,c}^{\mu\nu} \;=\; [\text{stuff}] \times k_2^\nu\,. \tag{S.35}$$

Consequently, when the net amplitude is contracted with the polarization vector $e^*_{2\nu}$ of the second gluon, it vanishes when the second gluon is transversely polarized, $k_2 e_2^* = 0$, but not if the other gluon's polarization is longitudinal. And this is in accordance to the weak form of Ward Identity: *On-shell amplitudes involving one longitudinal gluon vanish, but only if all the other gluons are transverse.*

Problem **1**(d):

For the first longitudinal gluon $e_1^\mu \propto k_1^\mu$, hence in light of eqs. (S.35) and (S.32),

$$\begin{aligned}\mathcal{M}(\Phi+\Phi^*\to g_L+g_L) &= e^*_{1\mu}\mathcal{M}^{\mu\nu}e^*_{2\nu} = \frac{e_1^*}{k_1}\times k_{1\mu}\mathcal{M}^{\mu\nu}e^*_{2\nu} = \frac{e_1^*}{k_1}\times k_{1\mu}\mathcal{M}^{\mu\nu}_{4,c}e^*_{2\nu} \\ &= \frac{e_1^*}{k_1}\times g^2(k_2e_2^*)\times\frac{(p'-p)_\lambda k_2^\lambda}{(k_1+k_2)^2}\times if^{abc}(T^c)^i{}_j\,.\end{aligned} \tag{S.36}$$

For the specific longitudinal polarizations in question,

$$\frac{e_1^*}{k_1} = \frac{1}{\sqrt{2}\omega_1}\,,\quad (e_2^*k_2) = \sqrt{2}\omega_2\,, \tag{S.37}$$

where in the center-of-mass system

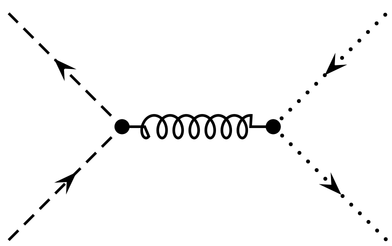
$$\omega_1 = \omega_2 = \tfrac{1}{2}E_{\rm cm}\quad\text{while}\quad (k_1+k_2)^2 = s = E^2_{\rm cm}, \tag{S.38}$$

hence

$$\mathcal{M}(\Phi+\Phi^*\to g_L+g_L) = \frac{g^2}{s}\times((p'-p)k_2)\times if^{abc}(T^c)^i{}_j\,. \tag{S.39}$$

Problem **1**(e):

There is only one tree diagram for the $\mathcal{M}(\Phi+\Phi^*\to \mathrm{gh}+\overline{\mathrm{gh}})$ amplitude, namely



Evaluating this diagram, we get

$$i\mathcal{M} = ig(p-p')^\mu(T^c)^i{}_j\times\frac{-ig_{\mu\nu}}{(k_1+k_2)^2}\times gf^{cab}k_2^\nu \tag{S.40}$$

and hence

$$\mathcal{M}(\Phi+\Phi^*\to \mathrm{gh}+\overline{\mathrm{gh}}) = \frac{g^2}{s}\times((p'-p)k_2)\times if^{abc}(T^c)^i{}_j\,. \tag{S.41}$$

Problem **1**(f):

By inspection, the tree amplitudes (S.39) and (S.41) for the scalars annihilating into a pair of longitudinal gluons and into a ghost-antighost pair are exactly equal to each other,

$$\mathcal{M}(\Phi+\Phi^*\to \mathrm{gh}+\overline{\mathrm{gh}}) \;=\; \mathcal{M}(\Phi+\Phi^*\to g_L+g_L). \tag{S.42}$$

Naively, this means that the partial cross-section of these two processes are also equal to each other. However, In the Hilbert space of both physical and unphysical particles, the longitudinal gluon states have positive norm while the (anti)ghost states have negative norm. Consequently,

$$\frac{d\sigma(\Phi+\Phi^*\to g_L+g_L)}{d\Omega} \;=\; +\frac{|\mathcal{M}|^2}{64\pi^2 s} \quad\text{while}\quad \frac{d\sigma(\Phi+\Phi^*\to \mathrm{gh}+\overline{\mathrm{gh}})}{d\Omega} \;=\; -\frac{|\mathcal{M}|^2}{64\pi^2 s}, \tag{S.43}$$

so the two partial cross-sections actually have opposite signs. Therefore, in the net annihilation cross-section into two gluons of any polarizations or into a ghost-antighost pair,

$$\begin{aligned}\frac{d\sigma(\Phi^*+\Phi\to g+g\text{ or }\mathrm{gh}+\overline{\mathrm{gh}})}{d\Omega} \;=\;& \frac{d\sigma(\Phi+\Phi^*\to g_T+g_T)}{d\Omega}\\ &+\;\frac{d\sigma(\Phi+\Phi^*\to g_L+g_L)}{d\Omega}\;+\;\frac{d\sigma(\Phi+\Phi^*\to \mathrm{gh}+\overline{\mathrm{gh}})}{d\Omega},\end{aligned} \tag{S.44}$$

the last two terms cancel each other, so the net cross-section is the same as cross-section for annihilation into physical transverse gluons only,

$$\frac{d\sigma(\Phi^*+\Phi\to g+g\text{ or }\mathrm{gh}+\overline{\mathrm{gh}})}{d\Omega} \;=\; \frac{d\sigma(\Phi+\Phi^*\to g_T+g_T)}{d\Omega}\,. \tag{S.45}$$

Problem **2**(a):

The tree-level amplitude for the $\Phi+\Phi^*\to g+g$ annihilation process, was worked out in problem 1(b), *cf.* eqs. (S.14) through (S.18). When both final-state 'gluons' have transverse polarizations, $(k_1e_1^*)=(k_2e_2^*)=0$, the net amplitude simplifies to

$$\mathcal{M}_{\text{net}} \;=\; \mathcal{M}_1\;+\;\mathcal{M}_2\;+\;\mathcal{M}_3\;+\;\mathcal{M}_4\,, \tag{S.46}$$

$$\mathcal{M}_1 \;=\; \frac{4g^2}{t-m^2} \times (pe_1^*)(p'e_2^*) \times (T^bT^a)^i{}_j\,, \tag{S.47}$$

$$\mathcal{M}_2 \;=\; \frac{4g^2}{u-m^2} \times (p'e_1^*)(pe_2^*) \times (T^aT^b)^i{}_j\,, \tag{S.48}$$

$$\mathcal{M}_3 \;=\; g^2 \times (e_1^*e_2^*) \times \{T^bT^a\}^i{}_j\,, \tag{S.49}$$

$$\mathcal{M}_4 \;=\; -\frac{g^2}{s} \times X \times [T^a,T^b]^i{}_j\,, \tag{S.50}$$

where s, t, and u are the Mandelstam variables, and

$$\begin{aligned} X \;&=\; (p-p')_\lambda e^*_{1\mu} e^*_{2\nu} \times \left[g^{\mu\nu}(k_1-k_2)^\lambda \;+\; g^{\nu\lambda}(2k_2+k_1)^\mu \;+\; g^{\lambda\mu}(-2k_1-k_2)^\nu\right] \\ &=\; (u-t)\times(e_1^*e_2^*) \;+\; 2(e_1^*k_2)\times\bigl(e_2^*(p-p')\bigr) \;-\; 2(e_2^*k_1)\times\bigl(e_1^*(p-p')\bigr). \end{aligned} \tag{S.51}$$

The color dependencies of amplitudes (S.49) and (S.50) are already in the form (2), while the amplitudes (S.47) and (S.48) can be brought into this form using

$$\begin{aligned} (T^aT^b) \;&=\; \tfrac{1}{2}\{T^a,T^b\} \;+\; \tfrac{1}{2}[T^a,T^b]\,, \\ (T^bT^a) \;&=\; \tfrac{1}{2}\{T^a,T^b\} \;-\; \tfrac{1}{2}[T^a,T^b]\,, \end{aligned} \tag{S.52}$$

Thus altogether,

$$\mathcal{M}_{\rm net}(\Phi^i+\Phi^*_j\to g^a+g^b) \;=\; F\times\{T^bT^a\}^i{}_j \;+\; iG\times[T^a,T^b]^i{}_j \tag{1}$$

for

$$F \;=\; \frac{2g^2}{t-m^2}\times(pe_1^*)(p'e_2^*) \;+\; \frac{2g^2}{u-m^2}\times(p'e_1^*)(pe_2^*) \;+\; g^2\times(e_1^*e_2^*) \tag{S.53}$$

and

$$G \;=\; \frac{2ig^2}{t-m^2}\times(pe_1^*)(p'e_2^*) \;-\; \frac{2ig^2}{u-m^2}\times(p'e_1^*)(pe_2^*) \;+\; \frac{ig^2}{s}\times X. \tag{S.54}$$

Problem **2**(b):

First, let us average over the scalar particles' color indices $i, j = 1, 2, \ldots, \dim(r)$. For fixed adjoint colors a and b of the gauge bosons , let

$$M \;=\; F\{T^a_{(r)}, T^b_{(r)}\} \;+\; iG\,[T^a_{(r)}, T^b_{(r)}] \tag{S.55}$$

be a matrix (in the representation (r) of the gauge group) whose elements $M^i{}_j$ are annihilation amplitudes (1) for the scalar particles Φ^i and Φ^*_j of specific colors i, j. Then averaging over those colors gives

$$\frac{1}{\dim^2(r)} \sum_{i,j} \left|M^i{}_j\right|^2 \;=\; \frac{1}{\dim^2(r)} \sum_{i,j} M^i{}_j (M^\dagger)^j{}_i = \;\frac{1}{\dim^2(r)}\; \mathrm{tr}(MM^\dagger). \tag{S.56}$$

For the specific form (S.55) of the matrix M, we write

$$\begin{aligned} M \;&=\; (F+iG)\, T^a_{(r)} T^b_{(r)} \;+\; (F-iG)\, T^b_{(r)} T^a_{(r)}\,, \\ M^\dagger \;&=\; (F+iG)^*\, T^b_{(r)} T^a_{(r)} \;+\; (F-iG)^*\, T^a_{(r)} T^b_{(r)}\,, \end{aligned} \tag{S.57}$$

and therefore

$$\begin{aligned} \mathrm{tr}(MM^\dagger) \;&=\; |F+iG|^2\ \mathrm{tr}_{(r)}(T^aT^bT^bT^a) \;+\; (F-iG)(F+iG)^*\ \mathrm{tr}_{(r)}(T^bT^aT^bT^a) \\ &\qquad +\; (F+iG)(F-iG)^*\ \mathrm{tr}_{(r)}(T^aT^bT^aT^b) \;+\; |F-iG|^2\ \mathrm{tr}_{(r)}(T^bT^aT^aT^b) \\ &\qquad\langle\!\langle \text{ using the cyclic symmetry of the trace } \rangle\!\rangle \\ &=\; \mathrm{tr}_{(r)}(T^aT^aT^bT^b) \times \Big(|F+iG|^2 \;+\; |F-iG|^2 \;=\; 2|F|^2 + 2|G|^2\Big) \\ &\qquad +\; \mathrm{tr}_{(r)}(T^aT^bT^aT^b) \times \begin{pmatrix} (F-iG)(F+iG)^* \;+\; (F+iG)(F-iG)^* \\ \qquad\qquad = 2|F|^2 \;-\; 2|G|^2 \end{pmatrix} \\ &=\; 2(|F|^2 + |G|^2) \times \mathrm{tr}_{(r)}(T^aT^aT^bT^b) \\ &\qquad + 2(|F|^2 - |G|^2) \times \begin{pmatrix} \mathrm{tr}_{(r)}(T^aT^bT^aT^b) \;=\; \mathrm{tr}_{(r)}(T^aT^aT^bT^b) \\ \qquad\qquad -\; \mathrm{tr}_{(r)}(T^a[T^a,T^b]T^b) \end{pmatrix} \\ &=\; 4|F|^2 \times \mathrm{tr}_{(r)}(T^aT^aT^bT^b) \;+\; 2(|G|^2 - |F|^2) \times \mathrm{tr}_{(r)}(T^a[T^a,T^b]T^b). \end{aligned} \tag{S.58}$$

Our next step is to sum over the color indices a and b of the gauge bosons. In the context of eq. (S.58), we have

$$\sum_{a,b} \mathrm{tr}_{(r)}(T^aT^aT^bT^b) \;=\; \mathrm{tr}_{(r)}\left(\Big(\sum_a T^aT^a\Big)\Big(\sum_b T^bT^b\Big)\right) \;=\; \mathrm{tr}_{(r)}\,(C_2\,C_2) \;=\; C^2(r)\times\dim(r) \tag{S.59}$$

and

$$\begin{aligned}
\sum_{a,b} \mathrm{tr}_{(r)}(T^a[T^a,T^b]T^b) &= \sum_{a,b}\sum_c if^{abc}\,\mathrm{tr}_{(r)}(T^aT^cT^b) \qquad\qquad (\mathrm{S.60})\\
&= \tfrac{1}{2}\sum_{a,b,c} if^{abc}\,\mathrm{tr}_{(r)}(T^aT^cT^b - T^aT^bT^c)\\
&= \tfrac{1}{2}\sum_{a,b,c} if^{abc}\sum_d if^{cbd}\,\mathrm{tr}_{(r)}(T^aT^d)\\
&= \tfrac{1}{2}\sum_{a,b,c,d} (if^{abc})(if^{dcb})\times R(r)\delta^{ad}\\
&= \tfrac{1}{2}R(r)\sum_a\sum_{b,c}\Big(if^{abc}=(T^a_{(\mathrm{adj})})^{bc}\Big)\Big(if^{acb}=(T^a_{(\mathrm{adj})})^{cb}\Big)\\
&= \tfrac{1}{2}R(r)\sum_a \mathrm{tr}\Big(T^a_{(\mathrm{adj})}T^a_{(\mathrm{adj})}\Big)\\
&= \tfrac{1}{2}R(r)\times C(G)\dim(G)\\
&= \tfrac{1}{2}C(G)C(r)\dim(r). \qquad\qquad (\mathrm{S.60})
\end{aligned}$$

Therefore,

$$\sum_{a,b}\mathrm{tr}(MM^\dagger) \;=\; C(r)\dim(r)\times\big[4C(r)|F|^2 \;+\; C(G)\big(|G|^2-|F|^2\big)\big] \tag{S.61}$$

and hence in light of eq. (S.56),

$$\frac{1}{\dim^2(r)}\sum_{ij}\sum_{ab}|\mathcal{M}|^2 \;=\; \frac{C(r)}{\dim(r)}\times\big(4C(r)\,|F|^2 \,+\, C(\mathrm{adj})\,(|G|^2-|F|^2)\big)\,. \tag{3}$$

Eq. (4) follows from this as a special case.

Problem $\mathbf{2}$(c):

Let us take a closer look at eqs. (S.53) and (S.54). In the center of mass frame, $\mathbf{p}' = -\mathbf{p}$, $\mathbf{k}_2 = -\mathbf{k}_1$, and the gluons' polarizations $e^{\mu}_{1,2}$ are purely spatial and transverse, $e^0_{1,2} = 0$ and $\mathbf{k}_{1,2} \cdot \mathbf{e}_{1,2} = 0$. Consequently, eqs. (S.51), (S.53) and (S.54) simplify to

$$\begin{aligned} X &= (t-u)\,(\mathbf{e}_1^* \cdot \mathbf{e}_2^*) \;+\; 0 \;+\; 0, \\ F &= -2g^2(\mathbf{e}_1^* \cdot \mathbf{p})(\mathbf{e}_2^* \cdot \mathbf{p}) \left(\frac{1}{t-m^2} + \frac{1}{u-m^2}\right) \;-\; g^2(\mathbf{e}_1^* \cdot \mathbf{e}_2^*), \\ G &= -2ig^2(\mathbf{e}_1^* \cdot \mathbf{p})(\mathbf{e}_2^* \cdot \mathbf{p}) \left(\frac{1}{t-m^2} - \frac{1}{u-m^2}\right) \;+\; ig^2\,\frac{t-u}{s}\,(\mathbf{e}_1^* \cdot \mathbf{e}_2^*). \end{aligned} \tag{S.62}$$

Furthermore, in the center of mass frame $E = E' = \omega_1 = \omega_2$, $|\mathbf{k}| = \omega = E$, $|\mathbf{p}| = \beta E$,

$$s \;=\; 4E^2, \qquad t - m^2 \;=\; -2E^2(1-\beta\cos\theta), \qquad u - m^2 \;=\; -2E^2(1+\beta\cos\theta),$$

hence

$$\begin{aligned} \frac{t-u}{s} &= \beta\cos\theta, \\ \frac{1}{t-m^2} + \frac{1}{u-m^2} &= \frac{-1}{m^2 + \mathbf{p}^2\sin^2\theta}\,, \\ \frac{1}{t-m^2} - \frac{1}{u-m^2} &= \frac{-\beta\cos\theta}{m^2 + \mathbf{p}^2\sin^2\theta}\,, \end{aligned} \tag{S.63}$$

and therefore

$$\begin{aligned} F &= -g^2 \left((\mathbf{e}_1^*\mathbf{e}_2^*) \;-\; \frac{2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p})}{m^2 + \mathbf{p}^2\sin^2\theta} \right), \\ G &= ig^2 \left((\mathbf{e}_1^*\mathbf{e}_2^*) \;+\; \frac{2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p})}{m^2 + \mathbf{p}^2\sin^2\theta} \right) \times \beta\cos\theta. \end{aligned} \tag{S.64}$$

Now consider the gluons' polarization vectors. For the problem at hand it is easier to use linear polarizations for which the $\mathbf{e}_1$ and $\mathbf{e}_2$ are real unit vectors. Specifically, for each gluon there is a choice of two transverse $\mathbf{e}$, one parallel to the $(\mathbf{p}, \mathbf{k})$ plane and one perpendicular to it. In the coordinate system where

$$\mathbf{p} = \beta E(0,0,1) \qquad \text{and} \quad \mathbf{k} = E(\sin\theta, 0, \cos\theta), \tag{S.65}$$

the two polarization vectors are

$$\mathbf{e}_{\parallel} = (-\cos\theta, 0, +\sin\theta) \quad \text{and} \quad \mathbf{e}_{\perp} = (0,1,0). \tag{S.66}$$

For these vectors

$$(\mathbf{p}\mathbf{e}_{\parallel}) = \beta E\sin\theta, \qquad (\mathbf{p}\mathbf{e}_{\perp}) = 0, \tag{S.67}$$

so according to eqs. (S.64),

$$\text{for } \mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_{\perp}\,, \quad F = -g^2 \quad \text{and} \quad G = ig^2\beta\cos\theta, \tag{S.68}$$

$$\text{for } \mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_{\parallel}\,, \quad F = -g^2(1-2A) \quad \text{and} \quad G = ig^2(1+2A)\beta\cos\theta \tag{S.69}$$

where

$$A = \frac{\mathbf{p}^2\sin^2\theta}{m^2 + \mathbf{p}^2\sin^2\theta}\,, \tag{S.70}$$

and finally

$$\text{for } \mathbf{e}_1 = \mathbf{e}_{\perp},\, \mathbf{e}_2 = \mathbf{e}_{\parallel} \text{ or } \textit{vice verse}, \quad F = G = 0. \tag{S.71}$$

Problem **2**(d):

According to eq. (S.71), the two gluons produced in the $\Phi\Phi^*$ annihilation must have similar polarizations: either both are polarized $\parallel$ to the $(\mathbf{p},\mathbf{k})$ plane of scattering or both are polarized $\perp$ to the plane. Consequently, there are only two polarized partial cross sections to consider, namely

$$\left(\frac{d\sigma(\perp)}{d\Omega}\right)_{\text{c.m.}} = \frac{g^4}{64\pi^2 E_{\text{c.m.}}^2\beta}\,\frac{C(r)}{\dim(r)} \times \Big(4C(r)\,-\,(1-\beta^2\cos^2\theta)C(G)\Big),$$
$$\left(\frac{d\sigma(\|)}{d\Omega}\right)_{\text{c.m.}} = \frac{g^4}{64\pi^2 E_{\text{c.m.}}^2\beta}\,\frac{C(r)}{\dim(r)} \times \begin{pmatrix} 4C(r)\times(1-2A) \\ -\,C(G)\times\Big((1-2A)^2-\beta^2(1+2A)^2\cos^2\theta\Big)\end{pmatrix}. \tag{S.72}$$

Note that the angular dependence of the $\|$ polarized partial cross section is more complicated than it looks because A is θ–dependent according to eq. (S.70).

In the limit of non-relativistic scalar particles, $\beta \ll 1$ leads to $A \ll 1$ and hence to the expected isotropy and polarization independence of the annihilation cross-section,

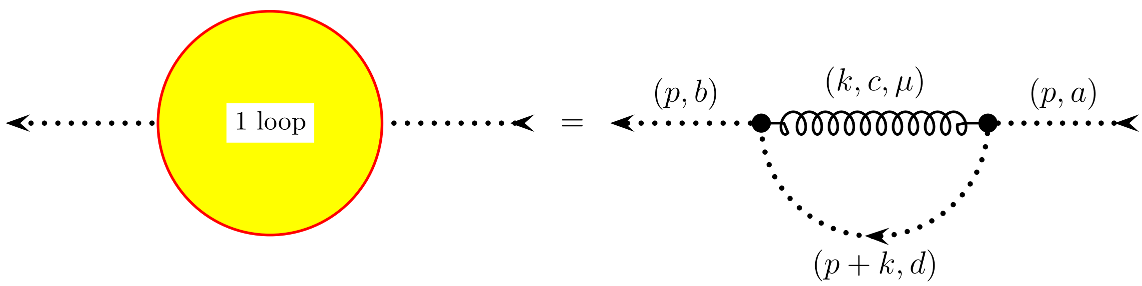
$$\left(\frac{d\sigma(\|)}{d\Omega}\right)_{\text{c.m.}} \approx \left(\frac{d\sigma(\perp)}{d\Omega}\right)_{\text{c.m.}} \approx \frac{g^4}{256\pi^2 m^2\beta}\,\frac{C(r)\big(4C(r)-C(G)\big)}{\dim(r)}\,. \tag{S.73}$$

In the opposite limit of ultra-relativistic scalars, $\beta \approx 1$ leads to $A \approx 1$ (except for $\theta \approx 0$) and therefore

$$\begin{aligned} \left(\frac{d\sigma(\perp)}{d\Omega}\right)_{\text{c.m.}} &\approx \frac{g^4}{16\pi^2 E_{\text{c.m.}}^2}\,\frac{C(r)}{\dim(r)}\left[C(r)\,-\,C(G)\,\frac{\sin^2\theta}{4}\right], \\ \left(\frac{d\sigma(\|)}{d\Omega}\right)_{\text{c.m.}} &\approx \frac{g^4}{16\pi^2 E_{\text{c.m.}}^2}\,\frac{C(r)}{\dim(r)}\left[C(r)\,+\,C(G)\,\frac{9\cos^2\theta-1}{4}\right]. \end{aligned} \tag{S.74}$$

Problem **3**(a–c):

At the one-loop level, the $\delta_2^{(\text{gh})}$ cancels the divergence of a single diagram



which evaluates (in the Feynman gauge) to

$$-i\Sigma^{ba}_{1\,\text{loop}}(p)\;=\;\int\!\frac{d^4k}{(2\pi)^4}\,\frac{i}{(p+k)^2+i0}\times\frac{-i}{k^2+i0}\times -gf^{cad}(p+k)_\mu\times -gf^{cdb}p^\mu. \tag{S.75}$$

In particular, the group factor here is

$$\sum_{c,d}f^{cad}f^{cdb}\;=\;\sum_{c,d}\bigl(-iT^a_{\text{adj}}\bigr)^{dc}\bigl(-iT^b_{\text{adj}}\bigr)^{cd}\;=\;-\operatorname{tr}_{\text{adj}}\bigl(T^aT^b\bigr)\;=\;-R(\text{adj})\times\delta^{ab}. \tag{S.76}$$

Using $R(\text{adj})=C(\text{adj})\equiv C(G)$ and taking care of all the signs and $\pm i$ factors, we arrive at

$$\Sigma^{ba}_{1\,\text{loop}}(p)\;=\;-ig^2C(G)\delta^{ba}\times p^\mu\times\int\!\frac{d^4k}{(2\pi)^4}\,\frac{(p+k)_\mu}{(k^2+i0)\times((p+k)^2+i0)}\,. \tag{S.77}$$

Note: the p^μ factor from the outgoing ghost vertex can be pulled outside the integral, which reduces its UV divergence form quadratic to linear. Moreover, by Lorentz symmetry the linear divergence cancels out, and the remaining integral becomes $p_\mu\times O(\log\Lambda)$. Consequently, the whole amplitude has form

$$\Sigma^{ba}_{1\,\text{loop}}(p)\;=\;\delta^{ba}\times p^2\times\Pi_{1\,\text{loop}}(p) \tag{S.78}$$

and we do not need the ghost-mass counterterm. Also, the logarithmic divergence of $\Pi(p)$ may be canceled by the $\delta^{(\text{gh})}_2$ counterterm as

$$\Pi(p)\;=\;\Pi_{\text{loop}}(p)\;-\;\delta^{(\text{gh})}_2. \tag{S.79}$$

Indeed, introducing the Feynman parameter x into the momentum integral (S.77), we have

$$\begin{aligned}\int\!\frac{d^4k}{(2\pi)^4}\,\frac{(p+k)_\mu}{(k^2+i0)\times((p+k)^2+i0)}\;&=\int_0^1\!dx\int\!\frac{d^4\ell}{(2\pi)^4}\frac{\ell_\mu+(1-x)p_\mu}{[\ell^2+x(1-x)p^2+i0]^2}\\ &=\int_0^1\!dx\,(1-x)p_\mu\times\int\!\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2+x(1-x)p^2+i0]^2}\end{aligned} \tag{S.80}$$

where the second equality follows from the $\ell\to-\ell$ symmetry of the integral. Using dimensional regularization for the UV divergence of the remaining integral, we have

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 + x(1-x)p^2 + i0]^2} \;=\; \frac{i}{16\pi^2}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-x(1-x)p^2} + \text{finite constant}\right), \tag{S.81}$$

hence

$$\int_0^1 dx\,(1-x)p_\mu \times [\cdots] \;=\; \frac{ip_\mu}{32\pi^2}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2} + \text{finite constant}\right) \tag{S.82}$$

and consequently

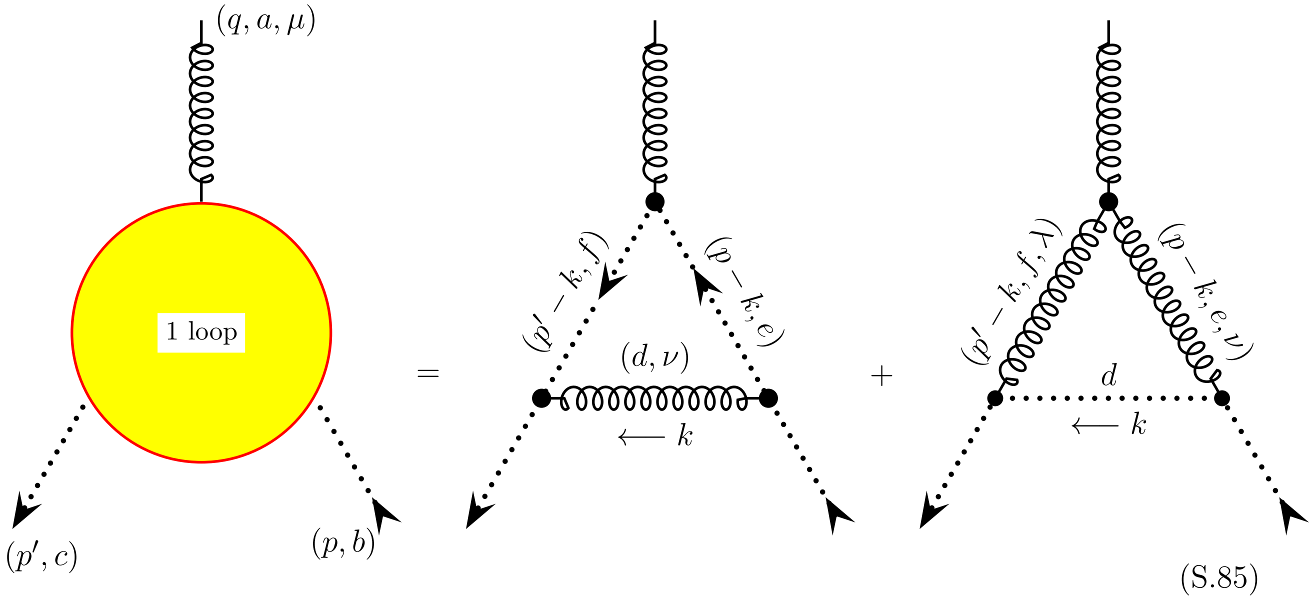
$$\Sigma^{ba}_{1\,\text{loop}}(p) \;=\; +\frac{g^2 C(G)}{32\pi^2} \times \delta^{ba} p^2 \times \left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2} + \text{finite constant}\right). \tag{S.83}$$

To cancel the UV divergence here, we need

$$\delta_2^{(\text{gh})}[1\,\text{loop}] \;=\; +\frac{g^2 C(G)}{32\pi^2} \times \frac{1}{\epsilon}\,. \tag{S.84}$$

$\star \quad \star \quad \star$

Now consider the $\delta_1^{(\text{gh})}$ counterterm. At the one-loop level, it cancels the UV divergence of two diagrams



(S.85)

The first diagram here evaluates to

$$
\begin{aligned}
-igG^{\mu,abc}(1^{\text{st}}) \ = \int\frac{d^4k}{(2\pi)^4}\,&\frac{i}{(p-k)^2+i0}\times\frac{i}{(p'-k)^2+i0}\times\frac{-i}{k^2+i0}\times\\
&\times -gf^{dbe}(p-k)^\nu\times -gf^{aef}(p'-k)^\mu\times -gf^{dfc}p'_\nu\,.
\end{aligned}
\tag{S.86}
$$

In particular, the group factor on the second line here amounts to

$$
\begin{aligned}
X_1^{abc} \ \equiv \sum_{d,e,f} f^{dbe}f^{aef}f^{dfc} \ &= \ -\sum_{d,e,f} f^{dcf}f^{afe}f^{deb}\\
&= \ -\sum_{d,e,f}\bigl(-iT^d_{\text{adj}}\bigr)^{cf}\bigl(-iT^a_{\text{adj}}\bigr)^{fe}\bigl(-iT^d_{\text{adj}}\bigr)^{eb}\\
&= \ -i\sum_d\bigl(T^d_{\text{adj}}T^a_{\text{adj}}T^d_{\text{adj}}\bigr)^{cb}.
\end{aligned}
\tag{S.87}
$$

The simplest way to take the last sum here is to use the abstract generators $\hat{T}^a$ of the Lie algebra instead of the specific matrices representing them in the adjoint multiplet:

$$
\begin{aligned}
\sum_d \hat{T}^d\hat{T}^a\hat{T}^d \ &= \ \sum_d \hat{T}^d\hat{T}^d\hat{T}^a \ + \ \sum_d \hat{T}^d\bigl[\hat{T}^a,\hat{T}^d\bigr]\\
&= \ \left(\sum_d \hat{T}^d\hat{T}^d\right)\times\hat{T}^a \ + \ \sum_{d,h}\hat{T}^d\times if^{adh}\hat{T}^h\\
&= \ \hat{C}_2\times\hat{T}^a \ + \ \frac{i}{2}\sum_{d,h} f^{adh}\times\bigl[\hat{T}^d,\hat{T}^h\bigr]\\
&= \ \hat{C}_2\times\hat{T}^a \ - \ \frac{1}{2}\sum_{d,h,j} f^{adh}f^{dhj}\hat{T}^j\\
&= \ \hat{C}_2\times\hat{T}^a \ - \ \frac{C(G)}{2}\times\hat{T}^a
\end{aligned}
\tag{S.88}
$$

where the last equality follows from eq. (S.76), $\sum_{dh} f^{adh}f^{dhj} = C(G)\times\delta^{aj}$. Consequently, in the adjoint representation of the Lie algebra

$$
\sum_d T^d_{\text{adj}}T^a_{\text{adj}}T^d_{\text{adj}} \ = \ C(\text{adj})\times T^a_{\text{adj}} \ - \ \frac{C(G)}{2}\times T^a_{\text{adj}} \ = \ \frac{C(G)}{2}\times T^a_{\text{adj}}
\tag{S.89}
$$

and hence

$$X_1^{abc} \;=\; -i\frac{C(G)}{2}\times\left(T^a_{\rm adj}\right)^{cb} \;=\; +\frac{C(G)}{2}\times f^{acb} \;=\; -\frac{C(G)}{2}\times f^{abc}. \tag{S.90}$$

Plugging this group factor into the loop amplitude (S.86) and pulling all the constant factors outside the integral, we obtain

$$-igG^{\mu,abc}(1^{\rm st}) \;=\; -\frac{g^3C(G)}{2}\times f^{abc}p'_\nu\times H_1^{\nu\mu} \tag{S.91}$$

where

$$H_1^{\mu\nu} \;=\; -i\int\!\frac{d^4k}{(2\pi)^4}\,\frac{(p-k)^\nu(p'-k)^\mu}{[(p-k)^2+i0]\times[(p'-k)^2+i0]\times[k^2+i0]}\,. \tag{S.92}$$

Note that thanks to the k–independent factor p'_ν of the left vertex which we pulled out from the momentum integral, the remaining integral $H^{\mu\nu}$ is only logarithmically divergent. Consequently, the infinite part of $H^{\mu\nu}$ depends only on the leading terms of the numerator and the denominator (as polynomials in k), thus

$$\begin{aligned}\left[H_1^{\mu\nu}\right]_\infty \;&=\; -i\int\!\frac{d^4k}{(2\pi)^4}\,\frac{k^\nu k^\mu+\cdots}{(k^2+i0)^3+\cdots} \;=\; -\int\!\frac{d^4k_E}{(2\pi)^4}\,\frac{k_E^\mu k_E^\nu+\cdots}{(k_E^2)^3+\cdots}\\ &=\; +\frac{g^{\mu\nu}}{4}\times\int\!\frac{d^4k_E}{(2\pi)^4}\,\frac{k_E^2+\cdots}{(k_E^2)^3+\cdots} \;=\; +\frac{g^{\mu\nu}}{4}\times\frac{1}{16\pi^2}\times\frac{1}{\epsilon}\,.\end{aligned} \tag{S.93}$$

Altogether, the divergent part of the first diagram amounts to

$$\left[-igG^{\mu,abc}(1^{\rm st})\right]_\infty \;=\; -gf^{abc}p'^\mu\times\frac{g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.94}$$

Note that its dependence on the colors of the 3 external particle, on the index μ of the gluon, and on the ghosts' momenta have just the right form to be canceled by the $\delta_1^{(\rm gh)}$ counterterm vertex,

$$-igG^{\mu,abc}(\text{counterterm}) \;=\; -gf^{abc}p'^{\mu}\times\delta_1^{(\text{gh})}\,. \tag{S.95}$$

In particular, to cancel just the first diagram we need

$$\delta_1^{(\text{gh})}(1^{\text{st}}) \;=\; -\frac{g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.96}$$

Now consider the second diagram (S.85), which evaluates to

$$\begin{aligned}
-igG^{\mu,abc}(2^{\text{nd}}) \;=\; \int\!\frac{d^4k}{(2\pi)^4}\,&\frac{-i}{(p-k)^2+i0}\times\frac{-i}{(p'-k)^2+i0}\times\frac{i}{k^2+i0}\times\\
&\times -gf^{ebd}k_\nu\times -gf^{fdc}p'_\lambda\times\\
&\times -gf^{aef}\begin{bmatrix} g^{\mu\lambda}(q-(k-p'))^\nu\\ +g^{\lambda\nu}((k-p')-(p-k))^\mu\\ +g^{\mu\nu}((p-k)-q)^\lambda\end{bmatrix}.
\end{aligned} \tag{S.97}$$

Despite other complications, the group factor here is the same as in the first diagram,

$$X_2^{abc} \;=\; \sum_{d,e,f} f^{ebd}f^{fdc}f^{aef} \;=\; +\sum_{d,e,f} f^{dbe}f^{aef}f^{dfc} \;=\; X_1^{abc} \;=\; -\frac{C(G)}{2}\times f^{abc} \tag{S.98}$$

Pulling this group factor — as well as other k–independent factors outside of the integral (S.97), we obtain

$$-igG^{\mu,abc}(2^{\text{nd}}) \;=\; -\frac{g^3C(G)}{2}\times f^{abc}p'_\lambda\times H_2^{\lambda\mu} \tag{S.99}$$

where

$$H_2^{\lambda\mu} \;=\; +i\!\int\!\frac{d^4k}{(2\pi)^4}\,\frac{k_\nu\times\left[g^{\lambda\mu}(q+p'-k)^\nu \;+\; g^{\lambda\nu}(2k-p-p')^\mu \;+\; g^{\mu\nu}(p-q-k)^\lambda\right]}{[(p-k)^2+i0]\times[(p'-k)^2+i0]\times[k^2+i0]}\,. \tag{S.100}$$

Again, thanks to the k–independent factor p'_λ of the left vertex which we pulled out from the momentum integral, the remaining integral $H_2^{\lambda\mu}$ is only logarithmically divergent. Although the numerator in the integral (S.100) is much messier than in the integral (S.92) for the first diagram, its leading term for $k \to \infty$ is fairly simple

$$\text{(numerator)} \;=\; g^{\lambda\mu} \times (-k^2) \;+\; k^\lambda \times (2k)^\mu \;+\; k^\mu \times (-k)^\lambda \;+\; \cdots \;=\; -g^{\lambda\mu}k^2 \;+\; k^\lambda k^\mu \;+\; \cdots \tag{S.101}$$

and that's all we need to get the infinite part of the integral. Specifically,

$$\begin{aligned}
\left[H_2^{\lambda\mu}\right]_\infty \;&=\; +i\int\frac{d^4k}{(2\pi)^4}\,\frac{-g^{\mu\lambda}k^2 \;+\; k^\lambda k^\mu \;+\; \cdots}{(k^2+i0)^3 \;+\; \cdots} \\
&=\; +\int\frac{d^4k_E}{(2\pi)^4}\,\frac{+g^{\lambda\mu}_{\text{Mink}} \times k_E^2 \;+\; k_E^\lambda k_E^\mu \;+\; \cdots}{(k_E^2)^3 \;+\; \cdots} \\
&=\; +\left(g^{\lambda\mu}_{\text{Mink}} \;-\; \frac{1}{4}\,g^{\lambda\mu}_{\text{Mink}}\right) \times \int\frac{d^4k_E}{(2\pi)^4}\,\frac{k_E^2+\cdots}{k_E^6+\cdots} \\
&=\; \frac{3}{4}\,g^{\lambda\mu} \times \frac{1}{16\pi^2} \times \frac{1}{\epsilon}
\end{aligned} \tag{S.102}$$

and hence the infinite part of the second diagram's amplitude

$$\left[-igG^{\mu,abc}(2^{\text{nd}})\right]_\infty \;=\; -gf^{abc}p'^\mu \times \frac{3g^2C(G)}{128\pi^2} \times \frac{1}{\epsilon}\,. \tag{S.103}$$

Again, this divergence has exactly the right form to be canceled by the $\delta_1^{(\text{gh})}$ counterterm vertex. This time, to cancel just the divergence of the second diagram we need

$$\delta_1^{(\text{gh})}(2^{\text{nd}}) \;=\; -\frac{3g^2C(G)}{128\pi^2} \times \frac{1}{\epsilon}\,. \tag{S.104}$$

Finally, combining the two diagrams' contributions, we get the net one-loop counterterm coefficient

$$\delta_1^{(\text{gh})}[1\,\text{loop}] \;=\; -\frac{g^2C(G)}{128\pi^2} \times \frac{1}{\epsilon} \;-\; \frac{3g^2C(G)}{128\pi^2} \times \frac{1}{\epsilon} \;=\; -\frac{g^2C(G)}{32\pi^2} \times \frac{1}{\epsilon}\,. \tag{S.105}$$

Comparing to the $\delta_2^{(\text{gh})}$ for the ghosts' wave function renormalization

$$\delta_2^{(\text{gh})}[1\,\text{loop}] \;=\; +\frac{g^2C(G)}{32\pi^2} \times \frac{1}{\epsilon}\,, \tag{S.84}$$

we immediately obtain the difference

$$\delta_1^{(\mathrm{gh})}[1\,\mathrm{loop}] \ - \ \delta_2^{(\mathrm{gh})}[1\,\mathrm{loop}] \ = \ -\frac{g^2 C(G)}{16\pi^2} \times \frac{1}{\epsilon}\,. \tag{S.106}$$

As promised, this difference agrees with the $\delta_1 - \delta_2$ difference for the quarks we had calculated in class

$$\delta_1^{(\mathrm{q})}[1\,\mathrm{loop}] \ - \ \delta_2^{(\mathrm{q})}[1\,\mathrm{loop}] \ = \ -\frac{g^2 C(G)}{16\pi^2} \times \frac{1}{\epsilon}\,. \tag{S.107}$$