RADIATION BY COMPACT ANTENNAS

Introduction

In these notes we consider radiation of EM waves by compact antennas fed by the harmonic current $I(t) = I_0 e^{-i\omega t}$. The current density $\mathbf{J}$ and the charge density ρ in such antennas may have complicated space dependence, but their time dependence is purely harmonic,

$$\mathbf{J}(\mathbf{x},t) \;=\; \mathbf{J}(\mathbf{x})e^{-i\omega t}, \qquad \rho(\mathbf{x},t) \;=\; \rho(\mathbf{x})e^{-i\omega t}, \tag{1}$$

so the EM field emitted by the antenna also have harmonic time dependence,

$$\mathbf{E}(\mathbf{x},t) \;=\; \mathbf{E}(\mathbf{x})e^{-i\omega t}, \qquad \mathbf{H}(\mathbf{x},t) \;=\; \mathbf{H}(\mathbf{x})e^{-i\omega t}. \tag{2}$$

By *compact* antenna I mean an antenna whose size L is much smaller that the wavelength $\lambda = 2\pi c/\omega$. For simplicity, let's also assume there is nothing of interest outside the antenna itself — no electric conductors, no dielectrics, no magnetic materials, just vacuum and the EM radiation emitted by the antenna. In practice, an antenna is often surrounded by air, but since the air has $\epsilon \approx 1$, $\mu \approx 1$, and $\sigma \approx 0$, it may be approximated by the vacuum.

With these assumptions, outside the antenna itself

$$\nabla\times\mathbf{H} \;=\; \epsilon_0\frac{\partial\mathbf{E}}{\partial t} \;=\; -i\omega\epsilon_0\mathbf{E} \;=\; -i\frac{k}{Z_0}\mathbf{E}$$

where $k = \omega/c$ and $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\ \Omega$. Consequently, the magnetic field outside the antenna completely determines the electric field as

$$\mathbf{E}(\mathbf{x}) \;=\; \frac{iZ_0}{k}\nabla\times\mathbf{H}(\mathbf{x}). \tag{3}$$

Or in terms of the vector potential

$$\mathbf{A}(\mathbf{x},t) \;=\; \mathbf{A}(\mathbf{x})e^{-i\omega t}, \tag{4}$$

$$\mathbf{H}(x) \;=\; \frac{1}{\mu_0}\nabla\times\mathbf{A}(\mathbf{x}), \tag{5}$$

$$\mathbf{E}(\mathbf{x}) \;=\; \frac{iZ_0}{\mu_0 k}\,\nabla\times(\nabla\times\mathbf{A}(\mathbf{x})). \tag{6}$$

In the Landau gauge, the vector potential obeys

$$\Box\,\mathbf{A}(\mathbf{x},t) \;=\; \mu_0\mathbf{J}(\mathbf{x},t), \tag{7}$$

so it can be found using the retarded Green's function of the wave equation as

$$\mathbf{A}(\mathbf{x},t) \;=\; \frac{\mu_0}{4\pi}\iiint d^3\mathbf{y}\,\frac{\mathbf{J}(\mathbf{y},t_{\rm ret})}{|\mathbf{x}-\mathbf{y}|} \tag{8}$$

where the current inside the integral is at the retarded time

$$t_{\rm ret} \;=\; t\;-\;\frac{|\mathbf{x}-\mathbf{y}|}{c}. \tag{9}$$

For a harmonic current,

$$\exp(-i\omega t_{\rm ret}) \;=\; \exp(-i\omega t)\times\exp(+ik|\mathbf{x}-\mathbf{y}|), \tag{10}$$

hence

$$\mathbf{A}(\mathbf{x},t) \;=\; \mathbf{A}(\mathbf{x})e^{-i\omega t}, \tag{11}$$

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\iiint d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\,\frac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}\,. \tag{12}$$

There are no exact formulae for this integral for general currents and general locations $\mathbf{x}$. Instead, we are useful approximations valid for different distances $r=|\mathbf{x}|$ from the antenna. Specifically, consider 3 different zones of distance:

1. The near zone (AKA the static zone) of

$$r \;\sim\; \text{antenna size } L \;\ll\; \frac{1}{k}\,. \tag{13}$$

2. The intermediate zone (AKA the induction zone) of

$$r \;\sim\; \frac{1}{k}\,. \tag{14}$$

3. The far zone (AKA the radiation zone) of

$$r \;\gg\; \frac{1}{k}\,. \tag{15}$$

In the near zone both $\mathbf{x}$ and $\mathbf{y}$ are much smaller that $1/k$, hence

$$k|\mathbf{x}-\mathbf{y}| \;\ll\; 1 \quad\Longrightarrow\quad \exp(ik|\mathbf{x}-\mathbf{y}|) \;\approx\; 1 \tag{16}$$

and therefore

$$\mathbf{A}(\mathbf{x}) \;\approx\; \frac{\mu_0}{4\pi}\iiint d^3\mathbf{y}\,\frac{\mathbf{J}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,, \tag{17}$$

similar to vector potential of a stationary current. Consequently, the magnetic field $\mathbf{H}(\mathbf{x})$ obtains from the Biot–Savart–Laplace formula just as if $\mathbf{J}(\mathbf{x})$ was a stationary current. Likewise, the electric field in the near zone obtains from the quasi-static Coulomb formula

$$\mathbf{E}(\mathbf{x}) \;\approx\; -\nabla\Phi(\mathbf{x}) \quad \text{where} \quad \Phi(\mathbf{x}) \;=\; \frac{1}{4\pi\epsilon_0}\iiint d^3\mathbf{y}\,\frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,. \tag{18}$$

On the other hand, in the intermediate and far zones the distance $r=|\mathbf{x}|$ to the observer is much larger than the antenna's size, hence

$$|\mathbf{x}| \;\gtrsim\; \frac{1}{k} \;\gg\; L \;\geq\; |\mathbf{y}| \tag{19}$$

and therefore

$$|\mathbf{x}-\mathbf{y}| \;\approx\; r \;-\; \mathbf{n}\cdot\mathbf{y} \quad \text{for } r \;=\; |\mathbf{x}| \text{ and } \mathbf{n} \;=\; \frac{\mathbf{x}}{r}\,. \tag{20}$$

In the context of the integral (12), this approximation means

$$\begin{aligned} \exp(ik|\mathbf{x}-\mathbf{y}|) \;&\approx\; \exp(ikr)\times\exp(-ik\mathbf{n}\cdot\mathbf{y}) \\ \text{while } \frac{1}{|\mathbf{x}-\mathbf{y}|} \;&\approx\; \frac{1}{r}\,, \end{aligned} \tag{21}$$

hence

$$\mathbf{A}(\mathbf{x}) \;\approx\; \frac{e^{ikr}}{r}\,\mu_0\mathbf{f}(\mathbf{n}) \tag{22}$$

for

$$\mathbf{f}(\mathbf{n}) \;=\; \frac{1}{4\pi}\iiint d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\exp(-ik\mathbf{n}\cdot\mathbf{y}). \tag{23}$$

This vector potential has a form of a *spherical divergent wave* I shall explain in a moment.

Spherical Waves

Before addressing the spherical electromagnetic waves, let me explain the spherical waves of a scalar field $\psi(\mathbf{x})$. A spherically symmetric divergent wave has a form

$$\psi(\mathbf{x}) \;=\; \frac{A\,e^{+ikr}}{r} \quad \text{where } r=|\mathbf{x}| \text{ and } A=\text{const} \tag{24}$$

This wave is an exact eigenstate of the $-\nabla^2$ operator in infinite space with eigenvalue k^2, hence

$$\Psi(\mathbf{x},t) \;=\; \psi(\mathbf{x})e^{-i\omega t} \tag{25}$$

is the exact solution of the wave equation for $\omega = k\times v_{\text{wave}}$. The amplitude of the spherical divergent wave A/r diminishes with distance as $1/r$, so the power carried by the wave diminishes as $1/r^2$. Indeed, the equivalent of the Poynting vector for the scalar wave is

$$\mathbf{S} \;=\; \tfrac{1}{2}\,\mathrm{Im}\big(\psi^*\nabla\psi\big) \;=\; \tfrac{1}{2}|\psi|^2\,\nabla(\mathrm{phase}(\psi))\,, \tag{26}$$

hence for the spherical divergent wave (24),

$$\mathbf{S} \;=\; \frac{|A|^2}{2r^2}\,k\mathbf{n}\;: \tag{27}$$

the power flows uniformly in all direction, but the flux diminishes with distance as $1/r^2$. Physically, this means a fixed power emitted into a unit of solid angle $d\Omega$:

$$dP \;=\; \mathbf{S}\cdot d\mathbf{area} \;=\; \frac{k|A|^2}{r^2}\mathbf{n}\cdot(r^2\,d\Omega\mathbf{n}) \;=\; \frac{k|A|^2}{2}\,d\Omega\,, \tag{28}$$

thus

$$\frac{dP}{d\Omega} \;=\; \frac{k|A|^2}{2}\,. \tag{29}$$

More general spherical divergent waves are not uniform in all directions but have non-trivial directional dependence $f(\theta,\phi)$, thus

$$\psi(r,\theta,\phi) \;\approx\; \frac{Ae^{ikr}}{r}\times f(\theta,\phi). \tag{30}$$

Alas, such waves are not exact eigenstates of the $-\nabla^2$ operator, although they asymptote to the eigenstates for $kr\to\infty$. Instead, the exact eigenstates are power series in $1/kr$ where the approximate waves (30) are merely the leading terms for $kr\gg 1$, thus

$$\psi(r,\theta,\phi) \;=\; \frac{Ae^{ikr}}{r}\times\left(f(\theta,\phi)\;+\;\sum_{n=1}^{\infty}\frac{f^{(n)}(\theta,\phi)}{(ikr)^n}\right) \tag{31}$$

where the angular dependencies $f^{(n}(\theta,\phi)$ of the subleading terms obtain by acting with the $\mathbf{L}^2$ operator on the leading $f(\theta,\phi)$. Specifically, for the leading angular dependence $Af(\theta,\phi)$

expanding into spherical harmonics as

$$Af(\theta,\phi) \;=\; \sum_{\ell,m} A_{\ell,m} Y_{\ell,m}(\theta,\phi), \tag{32}$$

the exact spherical divergent wave becomes

$$\psi(r,\theta,\phi) \;=\; \frac{e^{ikr}}{r} \sum_{\ell,m} A_{\ell,m} Y_{\ell,m}(\theta,\phi) \times \left(1 \,+\, \sum_{n=1}^{\ell} \frac{(\ell+n)!}{(\ell-n)!\,n!}\,\frac{1}{(2ikr)^n}\right). \tag{33}$$

But regardless of the details of the subleading terms, for $kr \gg 1$ we have

$$|\psi|^2 \;=\; \frac{|A|^2}{r^2} \times |f(\theta,\phi)|^2 \times \left(1 \,+\, O\left(\frac{1}{k^2r^2}\right)\right) \tag{34}$$

while

$$\nabla\,\mathrm{phase}(\psi) \;=\; k\mathbf{n} \,+\, \frac{1}{r}\nabla\,\mathrm{phase}(f) \,+\, O\left(\frac{1}{kr^2}\right) \;=\; k\left(\mathbf{n} + O\left(\frac{1}{kr}\right)\right), \tag{35}$$

hence the power density at long distances $kr \gg 1$ is

$$\mathbf{S} \;=\; \frac{k|A|^2\,|f(\theta,\phi)|^2}{2r^2}\left(\mathbf{n} \,+\, O\left(\frac{1}{kr}\right)\right), \tag{36}$$

and the power flowing into a small solid angle $d\Omega$

$$\frac{dP}{d\Omega} \;=\; \frac{k|A|^2\,|f(\theta,\phi)|^2}{2}\left(1 \,+\, O\left(\frac{1}{kr}\right)\right). \tag{37}$$

Thus, at long distances $r \gg (1/k)$ from the source of the spherical divergent wave, the angular distribution of the wave power is proportional to the $|f(\theta,\phi)|^2$,

$$\frac{dP}{d\Omega} \;=\; \frac{k|A|^2}{2} \times |f(\theta,\phi)^2|. \tag{38}$$

The divergent spherical waves of a scalar field have direct application to the scattering theory in quantum mechanics where $\psi(\mathbf{x})$ is the wave function of the particle being scattered.

In that setting, the incident wave $\psi = e^{ikz}$ is the plane wave representing the uniform beam of the incoming particles, while the outgoing wave is the divergent spherical wave representing the particles scattered in all possible directions. Altogether, we look for a solution to the Schrödinger equation of the form

$$\psi(\mathbf{x}) \;=\; e^{ikz} \;+\; \frac{e^{ikr}}{r}\, f(\theta,\phi) \;+\; O\left(\frac{1}{r^2}\right), \tag{39}$$

where the angular function $f(\theta,\phi)$ — or rather its magnitude^2 — gives us the partial cross-section for scattering in a particular direction,

$$\frac{d\sigma}{d\Omega} \;=\; |f(\theta,\phi)|^2. \tag{40}$$

Spherical Electromagnetic Waves

Now let's turn out attention to the spherical electromagnetic waves. Earlier in these notes we saw that in the intermediate and far distances from the antenna the vector potential has approximate form

$$\mathbf{A}(\mathbf{x},t) \;\approx\; \frac{e^{ikr-i\omega t}}{r}\,\mu_0\mathbf{f}(\mathbf{n}) \tag{22}$$

for

$$\mathbf{f}(\mathbf{n}) \;=\; \frac{1}{4\pi}\iiint d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\exp(-ik\mathbf{n}\cdot\mathbf{y}). \tag{23}$$

We shall learn how to evaluate the integral (23) later in these notes; for the moment, let $\mathbf{f}(,\theta,\phi)$ be a generic vector-valued function of the direction $\mathbf{n}$ towards the observer at $\mathbf{x}$.

Similarly to what we saw for the scalar waves, the *spherical EM wave* (22) is not an exact solution of the wave equation $\Box\mathbf{A}(x,t)=0$ but rather an asymptotic limit of a solution at large radii $r \gg (1/k)$. An exact solution would be given by a power series in $1/kr$ in which (22) is but the leading term,

$$\mathbf{A}(\mathbf{x},t) \;=\; \mu_0\,\frac{e^{+ikr-i\omega r}}{r}\left(\mathbf{f}(\theta,\phi) \;+\; \sum_{n=1}^{\infty}\frac{\mathbf{f}^{(n)}(\theta,\phi)}{(ikr)^n}\right), \tag{41}$$

although the series becomes a polynomial of finite degree ℓ when $\mathbf{f}(\theta,\phi)$ happens to be a spherical harmonic. Later in class, I'll let you work out in a homework the subleading terms in this expansion — and hence in the similar expansion for the electric and the magnetic fields — for $\ell = 1$ and maybe for $\ell = 2$. But for the present purposes, let's focus on the leading term (22) dominating the vector potential in the far zone $r \gg (1/k)$.

At long distances, the fastest changing factor in eq. (22) is the e^{ikr} factor; indeed,

$$\nabla e^{ikr} \;=\; (ik\mathbf{n})\,e^{ikr} \quad\text{while}\quad \nabla\frac{1}{r} \;=\; \frac{-\mathbf{n}}{r^2} \quad\text{and}\quad \nabla\mathbf{f} \;=\; O\left(\frac{\mathbf{f}}{r}\right), \tag{42}$$

hence the magnetic field of the spherical EM wave is

$$\mathbf{H} \;=\; \frac{1}{\mu_0}\nabla\times\mathbf{A} \;=\; \frac{e^{ikr}}{r}\,ik\mathbf{n}\times\mathbf{f} \;+\; e^{ikr}O(f/r^2) \;=\; ik\,\frac{e^{ikr}}{r}\Big(\mathbf{n}\times\mathbf{f} \;+\; O(f/kr)\Big). \tag{43}$$

Similarly, the electric field of the spherical EM wave obtains as

$$\mathbf{E} \;=\; \frac{iZ_0}{k}\nabla\times\mathbf{H} \;=\; -ikZ_0\frac{e^{ikr}}{r}\Big(\mathbf{n}\times(\mathbf{n}\times\mathbf{f}) \;+\; O(f/kr)\Big). \tag{44}$$

Thus, far away from the radiating antenna, the electric and the magnetic fields of the spherical wave *locally* look like the fields of a plane wave moving in the radial direction $\mathbf{n}$ with an amplitude $\propto 1/r$. Indeed, locally

$$\mathbf{E}\perp\mathbf{n},\quad \mathbf{H}\perp\mathbf{n},\quad \mathbf{E}\perp\mathbf{H} \tag{45}$$

and more specifically

$$\mathbf{E} \;=\; -Z_0\,\mathbf{n}\times\mathbf{H} \tag{46}$$

The Poynting vector of such a wave

$$\mathbf{S} \;=\; \tfrac{1}{2}\operatorname{Im}\big(\mathbf{E}\times\mathbf{H}^*\big) \;=\; \frac{Z_0}{2}\operatorname{Im}\big(\mathbf{H}^*\times(\mathbf{n}\times\mathbf{H})\big) \;=\; \frac{Z_0}{2}\,|\mathbf{H}|^2\,\mathbf{n} \tag{47}$$

has a radial direction (to the leading order in $1/kr$) while its magnitude diminishes with the distance as $|\text{amplitude}|^2 \propto 1/r^2$. Specifically,

$$\mathbf{S}_{\text{leading}} \;=\; \frac{Z_0 k^2}{2r^2}\,\left|\mathbf{n}\times\mathbf{f}(\mathbf{n})\right|^2\,\mathbf{n}, \tag{48}$$

so the EM power radiated into a solid angle $d\Omega$ is

$$\frac{dP}{d\Omega} \;=\; \frac{k^2 Z_0}{2}\,\left|\mathbf{n}\times\mathbf{f}(\mathbf{n})\right|^2. \tag{49}$$

In the following sections of these notes, we shall work out the angular dependence of this radiated power — as well as the net power — for the waves emitted by the oscillating electric dipoles, magnetic dipoles, and electric quadrupoles.

MULTIPOLE EXPANSION

Given the general formulae (43) and (44)— for the EM fields of a spherical wave and eq. (49) for the radiated power, our next task is to learn how to calculate the angular amplitude

$$\mathbf{f}(\mathbf{n}) \;=\; \frac{1}{4\pi}\iiint d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\,\exp(-ik\mathbf{n}\cdot\mathbf{y}) \tag{50}$$

for the wave emitted by a particular source current $\mathbf{J}(\mathbf{y})$. Suppose our source is compact and has size $\ll (1/k)$. In this case, $k\mathbf{n}\cdot\mathbf{y}$ is a small number everywhere within the source, so we may expand

$$\exp(-ik\mathbf{n}\cdot\mathbf{y}) \;=\; 1 \;-\; ik(\mathbf{n}\cdot\mathbf{y}) \;-\; \tfrac{1}{2}k^2(\mathbf{n}\cdot\mathbf{y})^2 \;+\; \cdots \;=\; \sum_{m=0}^{\infty}\frac{(-ik)^m}{m!}\,(\mathbf{n}\cdot\mathbf{y})^m \tag{51}$$

and hence

$$\mathbf{f}(\mathbf{n}) \;=\; \sum_{m=0}^{\infty}\mathbf{f}_m(\mathbf{n}) \tag{52}$$

where

$$\mathbf{f}_m(\mathbf{n}) \;=\; \frac{(-ik)^m}{4\pi\,m!}\iiint d^3\mathbf{y}\,(\mathbf{n}\cdot\mathbf{y})^m\,\mathbf{J}(\mathbf{y}). \tag{53}$$

We shall see later in this section that

- the $f_0(\mathbf{n})$ is related to the oscillating electric dipole moment of the antenna;
- the $f_1(\mathbf{n})$ is related to the oscillating magnetic dipole moment and/or the electric quadrupole moment;
- *etc.*, *etc.*

⋆ In general, the $\mathbf{f}_m(\mathbf{n})$ is related to the magnetic 2^m–pole moment and/or the electric 2^{m+1}–pole moment.

Note that the expansion starts with the electric dipole moment rather than the monopole momenta AKA the net charge $Q_{\rm net}$. The reason is very simple: the net charge is conserved, so it cannot possibly oscillate with a non-zero frequency ω.

Electric Dipole Radiation

For a compact antenna of size $L \ll (1/k)$, we start by approximating $\exp(-ik\mathbf{n}\cdot\mathbf{y}) \approx 1$, hence

$$\mathbf{f}(\mathbf{n}) \;\approx\; \mathbf{f}_0 \;=\; \frac{1}{4\pi}\iiint d^3\mathbf{y}\,\mathbf{J}(\mathbf{y}), \tag{54}$$

same for all space directions $\mathbf{n}$. The integral here is related to the antenna's oscillating dipole moment

$$\mathbf{p} \;=\; \iiint d^3\mathbf{y}\,\mathbf{y}\,\rho(\mathbf{y}) \tag{55}$$

by the continuity equation for the oscillating charge density $\rho(\mathbf{x},t) = \rho(\mathbf{x})e^{-i\omega t}$:

$$\nabla\cdot\mathbf{J} \;=\; -\frac{\partial\rho}{\partial t} \;=\; +i\omega\rho\,, \tag{56}$$

hence

$$\begin{aligned}
p_i \;&=\; \frac{-i}{\omega}\iiint d^3\mathbf{y}\,y_i\,(\nabla\cdot\mathbf{J} = \nabla_j J_j) \\
&\qquad \langle\!\langle\, \text{integrating by parts}\,\rangle\!\rangle \\
&=\; \frac{+i}{\omega}\iiint d^3\mathbf{y}\,J_j(\mathbf{y})\,(\nabla_j y_i = \delta_{ij}) \\
&=\; \frac{+i}{\omega}\iiint d^3\mathbf{y}\,J_i(\mathbf{y}) \\
&=\; \frac{4\pi i}{\omega}\,(f_0)_i\,,
\end{aligned} \tag{57}$$

thus

$$\mathbf{f}_0 = \frac{-i\omega}{4\pi}\,\mathbf{p}. \tag{58}$$

In the electric dipole approximation, the spherical wave radiated by the antenna becomes

$$\mathbf{H}(\mathbf{x}) = \frac{k\omega}{4\pi}\,\frac{e^{ikr}}{r}\,(-\mathbf{n}\times\mathbf{p})\,\Big(1+O(1/kr)\Big), \tag{59}$$

$$\mathbf{E}(\mathbf{x}) = \frac{Z_0 k\omega}{4\pi}\,\frac{e^{ikr}}{r}(\mathbf{n}\times(\mathbf{n}\times\mathbf{p}))\,\Big(1+O(1/kr)\Big), \tag{60}$$

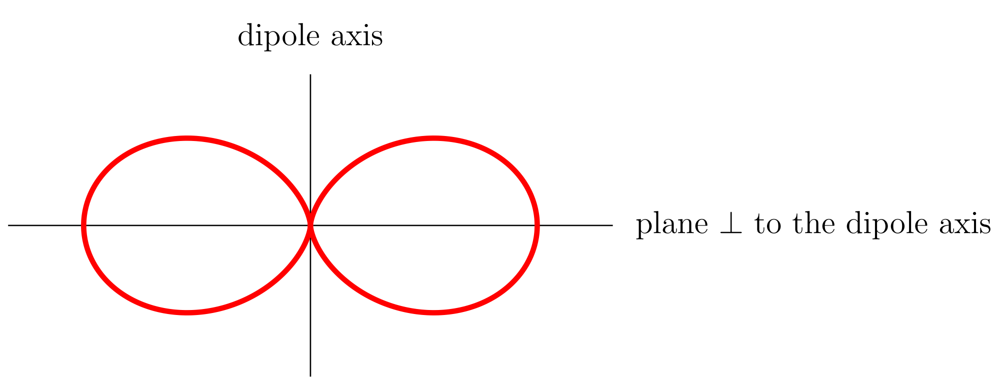
and the EM power radiated into a solid angle $d\Omega$ is

$$\frac{dP}{d\Omega} = \frac{Z_0 k^2\omega^2}{32\pi^2}\,\|\mathbf{n}\times\mathbf{p}\|^2\,. \tag{61}$$

To understand the angular distribution of this radiation, please note that $\mathbf{p}$ is the complex amplitude of the oscillating dipole moment $\mathbf{p}e^{-i\omega t}$. In general, the 3 components (p_x, p_y, p_z) of the dipole moment may oscillate with different phases; for example, a charged particle moving in a circle in the (x, y) plane has $p_x(t)$ and $p_y(t)$ oscillating with phases different by 90°, thus $\mathbf{p} = p(1, \pm i, 0)$. On the other hand, for a linear dipole — that is, a dipole moment oscillating back and forth along a fixed axis, — all 3 components (p_x, p_y, p_z) of the complex amplitude have the same phase. For such a linear dipole,

$$\|\mathbf{n}\times\mathbf{p}\|^2 = \|\mathbf{p}\|^2\times\sin^2\theta \tag{62}$$

where θ is the angle between the dipole's axis and the direction $\mathbf{n}$ of a distant detector of the EM wave. Consequently, the power emitted in any particular direction is proportional to $\sin^2\theta$, as illustrated by this *radiation power diagram*



Note: the radiation is strongest in the directions $\perp$ to the dipole axis, weaker in other directions, and there is no radiation at all in the direction of the dipole axis itself. For example, a vertical antenna has vertical dipole moment $\mathbf{p}$, hence the radiation is strongest in the horizontal direction, weaker in directions at some angles above or below the horizon, and no radiation at all goes vertically up or vertically down.

As to the net power of the dipole radiation, it obtains by integration the directional power (61) over the 4π directions of $\mathbf{n}$,

$$P_{\text{net}} = \oiint d^2\Omega(\mathbf{n}) \, \frac{Z_0 k^2 \omega^2}{32\pi^2} \, \|\mathbf{n} \times \mathbf{p}\|^2 = \frac{Z_0 k^2 \omega^2}{32\pi^2} \, \|\mathbf{p}\|^2 \times \oiint d^2\Omega(\theta, \phi) \, \sin^2\theta \tag{63}$$

where

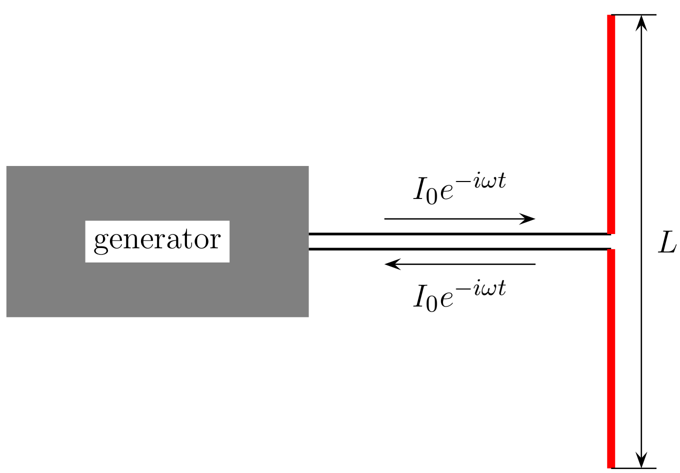
$$\oiint d^2\Omega(\theta, \phi) \, \sin^2\theta = \frac{2}{3} \times 4\pi = \frac{8\pi}{3}, \tag{64}$$

hence

$$P_{\text{net}} = \frac{Z_0 k^2 \omega^2}{12\pi} \, \|\mathbf{p}\|^2 = \frac{Z_0}{12\pi c^2} \, \omega^4 \|\mathbf{p}\|^2. \tag{65}$$

Linear Antenna Example

As an example of dipole radiation, consider a center-fed linear antenna of length $L \ll \lambda$,



Neglecting the antenna's thickness, we can write the current in the antenna as

$$\mathbf{J}(x, y, z, t) = I(z)\delta(x)\delta(y)\mathbf{n}_z \, e^{-i\omega t} \tag{66}$$

hence oscillating charge density

$$\rho(x,y,z,t) \;=\; \frac{1}{i\omega}\,\nabla\cdot\mathbf{J} \;=\; \frac{1}{i\omega}\,\frac{dI}{dz}\,\delta(x)\delta(y)\,e^{-i\omega t}. \tag{67}$$

For an antenna of uniform thickness — and hence uniform capacitance, — this charge density is uniform along each half of the antenna, which means uniform dI/dz over each half. Further more, no current flows through the ends of the antenna at $z=\pm\frac{L}{2}$ while the current at $z=0$ is the feed current $I_0e^{-i\omega t}$, thus

$$I(z) \;=\; I_0\left(1-\frac{2|z|}{L}\right) \tag{68}$$

and hence

$$\rho(x,y,z,t) \;=\; \frac{2iI_0}{\omega L}\,\mathrm{sign}(z)\delta(x)\delta(y)e^{-i\omega t}. \tag{69}$$

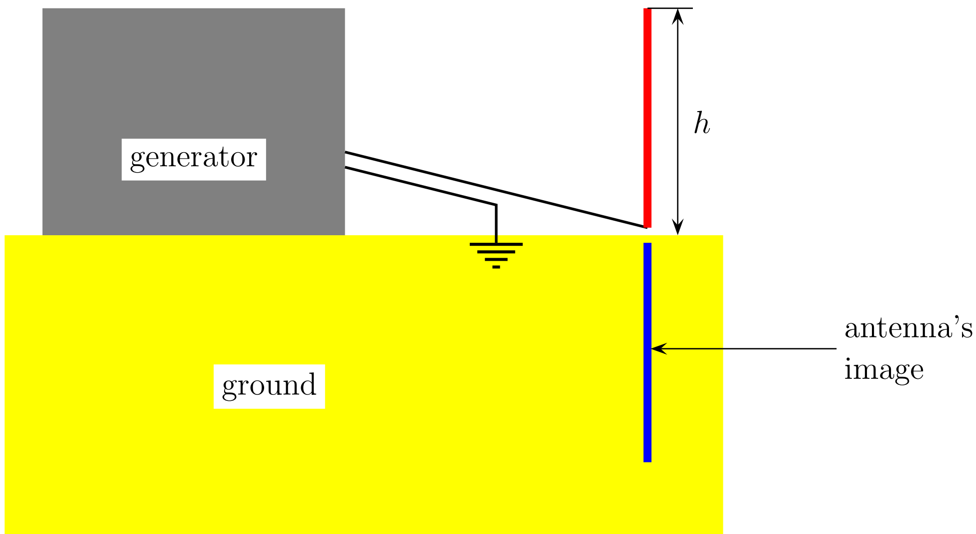
For this antenna, the electric dipole moment is obviously vertical, $\mathbf{p}=(0,0,p_z)$, and the amplitude of its vertical component is

$$p_z \;=\; \frac{2iI_0}{\omega L}\int\limits_{-L/2}^{+L/2}dz\,z\,\mathrm{sign}(z) \;=\; \frac{2iI_0}{\omega L}\times 2\int\limits_{0}^{L/2}dz\,z \;=\; \frac{2iI_0}{\omega L}\times 2\times\frac{L^2}{8} \;=\; \frac{iLI_0}{2\omega}\,, \tag{70}$$

so the net EM power radiated by this antenna is

$$P_{\mathrm{net}} \;=\; \frac{Z_0}{12\pi c^2}\,\omega^4\times\frac{L^2|I_0|^2}{4\omega^2} \;=\; \frac{|I_0|^2}{2}\times\frac{Z_0\omega^2L^2}{24\pi c^2} \;=\; \frac{|I_0|^2}{2}\times\frac{\pi Z_0}{6}\times\frac{L^2}{\lambda^2}\,. \tag{71}$$

A close cousin to the center-fed monopole antenna is the “monopole antenna” rising vertically from the ground:



The ground acts as a good conductor, so the charged antenna above the ground is accompanied by its mirror image (of the opposite charge) below the ground. Together, the antenna and its image act as a combined center-fed dipole antenna of net length $L = 2h$, hence the dipole moment amplitude

$$p_z = \frac{i(2h)I_0}{2\omega}. \tag{72}$$

Above the ground, this dipole moment radiates as any other electric dipole, thus

$$\frac{dP}{d\Omega} = \frac{Z_0\omega^4|p_z|^2}{32\pi^2c^2} \times \sin^2\theta, \tag{73}$$

but only for the above-the-ground $\theta < \frac{\pi}{2}$; in the directions below the horizontal, there is no radiation at all. Consequently, the net radiated power is

$$P_{\text{net}} = \frac{Z_0\omega^4|p_z|^2}{32\pi^2c^2} \times \left(\frac{4\pi}{3} \text{ instead of } \frac{8\pi}{3}\right) = \frac{Z_0\omega^4|p_z|^2}{24\pi^2c^2}, \tag{74}$$

or in terms of the "monopole" antenna's height h and the radiation's wavelength $\lambda = 2\pi c/\omega$,

$$P_{\text{net}} = \frac{|I_0|^2}{2} \times \frac{\pi Z_0}{3} \times \frac{h^2}{\lambda^2}. \tag{75}$$

From the RF generator's point of view, the antenna is a load of some impedance Z, which consumes RF power

$$P = \frac{|I_0|^2}{2} \times \mathrm{Re}(Z). \tag{76}$$

Neglecting the antenna's ohmic resistance, the power it consumes from the RF generator is the net power it radiates, so the real part of the antenna's impedance as a load is called its *radiation resistance*

$$R_{\mathrm{rad}} = \mathrm{Re}(Z_{\mathrm{rad}}) = \frac{2P_{\mathrm{radiation}}^{\mathrm{net}}}{|I_0|^2}\,. \tag{77}$$

Interpreting eqs. (71) and (75) in terms of the radiation resistance, we have:

For a center-fed dipole antenna of length $L \ll \lambda$,

$$R_{\mathrm{rad}} = \frac{\pi Z_0}{6} \times \frac{L^2}{\lambda^2} \approx (197\ \Omega) \times \frac{L^2}{\lambda^2}. \tag{78}$$

For a vertical "monopole" antenna of height $h \ll \lambda$,

$$R_{\mathrm{rad}} = \frac{\pi Z_0}{3} \times \frac{h^2}{\lambda^2} \approx (394\ \Omega) \times \frac{h^2}{\lambda^2}. \tag{79}$$

For example, consider a mast antenna of height $h = 30$ m ≈ 100 ft broadcasting AT radio at frequency 750 kHz and wavelength $\lambda = 400$ m. For this antenna, the radiation resistance is

$$R_{\mathrm{rad}} \approx 2.22\ \Omega, \tag{80}$$

so in order to radiate its quota of 50 kW of RF power, the generator must feed it with the current of amplitude $|I_0| = 212$ A (*i.e.*, RMS current of 150 A).

PS: The impedance of an antenna as a load to the RF generator generally has both real and imaginary parts. The real part — called the radiation resistance — is related to the power broadcast by the antenna as we saw above, but the imaginary part — called the radiation reactance — is much harder to calculate. In principle, the net impedance obtains as

$$Z = \frac{\text{voltage } V_0 \text{ in the gap at } z = 0}{\text{feed current } I_0}\,, \tag{81}$$

but calculating the voltage in the gap involves understanding the near-range electric field right next to the antenna. Moreover, this near-range field depends on the exact geometry of the antenna's ends right next to the gap — the gap length, the antenna's diameter, whether the antenna's ends at the gap are flat or hemispherical (or whatever), *etc.*, *etc.*, — so the calculation of the gap voltage is generally quite hard. Thus, the only thing I can say about the radiation reactance is that for short dipole or 'monopole" antennas the capacitance is more important than the inductance, so the sign of $\mathrm{Im}(Z)$ should be negative. But calculating the magnitude of this reactance is beyond the scope of this class.

Back to the Electric Dipole Radiation

Now consider the non-linear electric dipole moments $\mathbf{p}$ whose components (p_x, p_y, p_z) have different phases. The net EM power radiated by such a dipole is given by exactly the same formula

$$P_{\text{net}} = \frac{Z_0 k^2 \omega^2}{12\pi} \|\mathbf{p}\|^2 = \frac{Z_0}{12\pi c^2} \omega^4 \|\mathbf{p}\|^2 \tag{82}$$

as for the linear dipoles. Indeed, the directional power radiated by the dipole is

$$\frac{dP}{d\Omega} = \frac{Z_0 \omega^4}{32\pi^2 c^2} \|\mathbf{n} \times \mathbf{p}\|^2 \tag{61}$$

where

$$\begin{aligned} \|\mathbf{n} \times \mathbf{p}\|^2 &= \epsilon_{ijk} n_j p_k \times \epsilon_{i\ell m} n_\ell p_m^* \\ &= (\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}) n_j n_\ell p_k p_m^* \\ &= (\delta_{km} - n_k n_m) p_k p_m^* , \end{aligned} \tag{83}$$

hence

$$\oiint d^2\Omega(\mathbf{n}) \; \|\mathbf{n} \times \mathbf{p}\|^2 = p_k p_m^* \times \oiint d^2\Omega(\mathbf{n}) \, (\delta_{km} - n_k n_m). \tag{84}$$

By the rotational symmetry, the integral here has form

$$\oiint d^2\Omega(\mathbf{n}) \, (\delta_{km} - n_k n_m) = A \times \delta_{km} \tag{85}$$

where the coefficient A obtains by multiplying both sides of this equation by δ_{km}: on the

RHS

$$A\delta_{km} \times \delta_{km} \;=\; 3A \tag{86}$$

while on the LHS

$$(\delta_{km} \,-\, n_k n_m) \times \delta_{km} \;=\; 3 \,-\, \mathbf{n}^2 \;=\; 2, \tag{87}$$

hence

$$\delta_{km} \times \oiint d^2\Omega(\mathbf{n})\,(\delta_{km} \,-\, n_k n_m) \;=\; 2\oiint d^2\Omega(\mathbf{n}) \;=\; 2 \times 4\pi \tag{88}$$

and therefore

$$A \;=\frac{8\pi}{3}\,. \tag{89}$$

In the context of eq. (84), this means that

$$\oiint d^2\Omega(\mathbf{n})\; \|\mathbf{n}\times\mathbf{p}\|^2 \;=\; p_k p_m^* \times\; \frac{8\pi}{3}\delta_{km} \;=\; \frac{8\pi}{3}\,\|\mathbf{p}\|^2 \tag{90}$$

and therefore

$$P_{\rm net} \;=\oiint d^2\Omega\,\frac{dP}{d\Omega} \;=\; \frac{Z_0\omega^4}{32\pi^2c^2}\times\frac{8\pi}{3}\,\|\mathbf{p}\|^2 \;=\; \frac{Z_0\omega^4}{12\pi c^2}\,\|\mathbf{p}\|^2. \tag{91}$$

For an example, consider the Rutherford model of a hydrogen atom: a classical electron in a circular orbit around the nucleus. This atom has a rotating dipole moment

$$\mathbf{p}(t) \;=\; -er(\cos(\omega t),\sin(\omega t),0) \;=\; -er\,\mathrm{Re}\Big(e^{-i\omega t}(1,i,0)\Big) \tag{92}$$

which we may interpret as oscillating dipole moment with a non-linear complex amplitude

$$\mathbf{p} \;=\; -er(1,i,0). \tag{93}$$

This dipole moment radiates EM waves at net power

$$P \;=\; \frac{Z_0\,\omega^4}{12\pi c^2}\times\Big(\|\mathbf{p}\|^2 \;=\; 2e^2r^2\Big) \;=\; \frac{Z_0e^2}{6\pi c^2}\times\omega^4r^2, \tag{94}$$

where the frequency ω and the orbital radius r of the atom are related by the Kepler-like formula

$$m_e\omega^2 r \;=\; \frac{e^2}{4\pi\epsilon_0 r^2} \quad\Longrightarrow\quad \omega^2\times r^3 \;=\; \frac{e^2}{4\pi\epsilon_0 m_e} \;=\; \text{const}, \tag{95}$$

hence

$$P \;=\; \frac{Z_0 e^6}{96\pi^3\epsilon_0^2\, c^2 m_e^2}\times\frac{1}{r^4}\,. \tag{96}$$

Thus power is radiated at the expense of the atom's energy

$$U \;=\; \frac{m_e\omega^2 r^2}{2} \;-\; \frac{e^2}{4\pi\epsilon_0 r} \;=\; -\frac{e^2}{8\pi\epsilon_0 r}\,, \tag{97}$$

which therefore changes with time as

$$\frac{dU}{dt} \;=\; -P. \tag{98}$$

In terms of the orbital radius, this formula means the radius shrinks with time according to

$$\frac{e^2}{8\pi\epsilon_0}\,\frac{d}{dt}\left(\frac{-1}{r}\right) \;=\; -\frac{Z_0 e^6}{96\pi^3\epsilon_0^2\, c^2 m_e^2}\times\frac{1}{r^4}\,, \tag{99}$$

hence

$$\frac{1}{r^2}\,\frac{dr}{dt} \;=\; -\frac{Z_0 e^4}{12\pi^2\epsilon_0\, c^2 m_e^2}\times\frac{1}{r^4}\,, \tag{100}$$

$$\frac{d(r^3/3)}{dt} \;=\; r^2\times\frac{dr}{dt} \;=\; -\frac{Z_0 e^4}{12\pi^2\epsilon_0\, c^2 m_e^2} \;=\; \text{const}, \tag{101}$$

and therefore

$$r^3(t) \;=\; r_0^3 \;-\; \frac{Z_0 e^4}{4\pi^2\epsilon_0\, c^2 m_e^2}\times t. \tag{102}$$

Note: this orbital radius shrinks all the way to zero in a finite time, so the classical Rutherford atom has a finite lifetime

$$T \;=\; r_0^3\times\frac{4\pi^2\epsilon_0\, c^2 m_e^2}{Z_0 e^4} \tag{103}$$

before the electron spirals down all the way to the nucleus! Numerically, this lifetime is

$$T \;=\; (1.05\cdot 10^{-10}\ \text{s})\times\bigl(\text{r}_0[\text{in Å}]\bigr)^3; \tag{104}$$

for example, for $r_0 = 0.53$ Å (Bohr radius of the quantum atom in the ground state), the classical lifetime is only $T = 1.6\cdot 10^{-11}$ s.

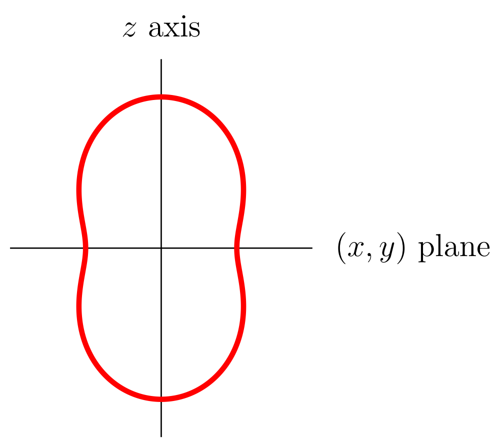
Going back to the general non-linear electric dipoles, while the net EM power radiated by such dipoles is the same as for the linear dipoles with similar $\|\mathbf{p}\|^2$, the angular distribution of the radiated power is quite different from the $\sin^2\theta$ rule for the linear dipoles. In general,

$$\frac{dP}{d\Omega} \propto \|\mathbf{p}\|^2 - (\mathbf{n}\cdot\operatorname{Re}\mathbf{p})^2 - (\mathbf{n}\cdot\operatorname{Im}\mathbf{p})^2. \tag{105}$$

For example, for the dipole moment rotating in the (x, y) plane — like an electron circling the nucleus, — we have

$$\mathbf{p} = p(1, i, 0) \quad\Longrightarrow\quad \frac{dP}{d\Omega} \propto 2 - n_x^2 - n_y^2 = 1 + n_z^2 = 1 + \cos^2\theta. \tag{106}$$

Here is the radiation power diagram for this rotating dipole:



Next Order of the Multipole Expansion

Thus far, we have focused on the leading term in the multipole expansion for the radiation by a compact antenna — the electric dipole radiation. But for antennas with zero electric dipole moments, we need to consider the subleading orders of the multipole expansion

$$\mathbf{f}(\mathbf{n}) = \sum_{m=0}^{\infty} \mathbf{f}_m(\mathbf{n}) \tag{52}$$

where

$$\mathbf{f}_m(\mathbf{n}) \;=\; \frac{(-ik)^m}{4\pi\, m!} \iiint d^3\mathbf{y}\,(\mathbf{n}\cdot\mathbf{y})^m\,\mathbf{J}(\mathbf{y}). \tag{107}$$

So let's focus on the first subleading order $\mathbf{f}_1(\mathbf{n})$ which is related to the magnetic dipole moment and the electric quadrupole moment of the antenna. Specifically,

$$\mathbf{f}_1(\mathbf{n}) \;=\; \frac{-ik}{4\pi} \int d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})(\mathbf{n}\cdot\mathbf{y}) \;=\; \mathbf{f}_{Md} \;+\; \mathbf{f}_{Eq} \tag{108}$$

where

$$\mathbf{f}_{Md}(\mathbf{n}) \;=\; \frac{-ik}{8\pi} \int d^3\mathbf{y}\Big(\mathbf{J}(\mathbf{y}\cdot\mathbf{n}) \;-\; \mathbf{y}(\mathbf{J}\cdot\mathbf{n})\Big) \tag{109}$$

and

$$\mathbf{f}_{Eq}(\mathbf{n}) \;=\; \frac{-ik}{8\pi} \int d^3\mathbf{y}\Big(\mathbf{J}(\mathbf{y}\cdot\mathbf{n}) \;+\; \mathbf{y}(\mathbf{J}\cdot\mathbf{n})\Big). \tag{110}$$

The relation of these integrals to the magnetic dipole moment and the electric quadrupole moment will become clear in a moment.

Magnetic Dipole Radiation

Let's start with the integral (109) and its relation to the magnetic dipole moment. By the double-cross-product formula

$$\mathbf{J}(\mathbf{y}\cdot\mathbf{n}) \;-\; \mathbf{y}(\mathbf{J}\cdot\mathbf{n}) \;=\; \mathbf{n}\times(\mathbf{J}\times\mathbf{y}), \tag{111}$$

hence

$$\mathbf{f}_{Md}(\mathbf{n}) \;=\; \frac{-ik}{8\pi}\,\mathbf{n}\times \int d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\times\mathbf{y}, \tag{112}$$

where the integral is precisely $2\mathbf{m}$ — (twice) the magnetic dipole moment of the antenna, thus

$$\mathbf{f}_{Md}(\mathbf{n}) \;=\; \frac{-ik}{4\pi}\,(\mathbf{n}\times\mathbf{m}). \tag{113}$$

Consequently, in the radiation zone far away from the antenna,

$$\mathbf{H} \;\approx\; -\left(\frac{k^2}{4\pi} \;=\; \frac{\omega^2}{4\pi c^2}\right)\,\big(\mathbf{n}\times(\mathbf{n}\times\mathbf{m})\big)\,\frac{e^{ikr-i\omega t}}{r}\,, \tag{114}$$

$$\begin{aligned}\mathbf{E} &\approx +\frac{Z_0\,\omega^2}{4\pi c^2}\big(\mathbf{n}\times(\mathbf{n}\times(\mathbf{n}\times\mathbf{m}))\big)\,\frac{e^{ikr-i\omega t}}{r}\\ &= -\frac{Z_0\,\omega^2}{4\pi c^2}\,(\mathbf{n}\times\mathbf{m})\,\frac{e^{ikr-i\omega t}}{r}\,. \end{aligned}\tag{115}$$

Note that the magnetic field (114) of the magnetic dipole behaves exactly like the electric field (59) of the electric dipole, while the electric field (115) of the magnetic dipole behaves exactly like the magnetic field (60) of the electric dipole.

Likewise, the power of the magnetic dipole radiation

$$\frac{dP}{d\Omega} = \frac{Z_0\,\omega^4}{32\pi^2c^4}\,\big\|\mathbf{n}\times(\mathbf{n}\times\mathbf{m})\big\|^2 = \frac{Z_0\,\omega^4}{32\pi^2c^4}\,\big\|\mathbf{n}\times\mathbf{m}\big\|^2 \tag{116}$$

has similar angular distribution and similar net power

$$P_{\rm net} = \frac{Z_0\,\omega^4}{12\pi c^4}\,\|\mathbf{m}\|^2 \tag{117}$$

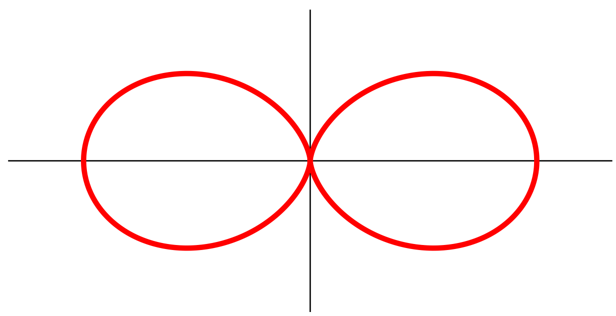
to the radiation of electric dipole

$$\mathbf{p} = \frac{\mathbf{m}}{c}\,. \tag{118}$$

As an example, consider a loop antenna — a flat loop of wire of area A. Or more often, a flat coil having N turns of area A. When this coil is fed a harmonic current of frequency ω and amplitude I_0, it has oscillating magnetic moment of amplitude $m = NAI_0$ in the direction of the coil's axis, *i.e.* $\perp$ to the plane of the flat coil. As a complex vector, $\mathbf{m}$ is real up to an overall phase, hence the angular distribution of the loop antenna's radiation is

$$\frac{dP}{d\Omega} \propto \sin^2\theta \tag{119}$$

where θ is the angle between the antenna's axis and the direction of the radiation. Thus, the radiation is strongest within the plane of the loop, while no power is emitted along the loop's axis. Graphically, the angular distribution (119) is illustrated by the *radiation power* *diagram*



The net power emitted by the loop antenna is

$$P_{\rm net} \;=\; \frac{Z_0\,\omega^4}{12\pi c^4}\,(NA\,|I_0|)^2. \tag{120}$$

From the point of view of the RF generator feeding the current of amplitude I_0 to the antenna, the antenna is a load of some impedance Z, and the power radiated by the antenna is the power consumed by this load,

$$P_{\rm net} \;=\; \frac{|I_0|^2}{2}\times{\rm Re}(Z). \tag{121}$$

In light of eq. (120), the real (active) part of this impedance — also called the radiation resistance — is

$$R_{\rm rad} \;=\; {\rm Re}(Z) \;=\; \frac{Z_0}{6\pi}\times(\omega/c)^4\times(NA)^2 \;=\; \frac{Z_0}{6\pi}\times\left(NA\times\left(\frac{2\pi}{\lambda}\right)^2\right)^2. \tag{122}$$

For example, take antenna made of 10 turns of area $A = 1\ {\rm m}^2$, and let it radiate short-wave radio signal at wavelength $\lambda = 20$ m (frequency $\omega = 2\pi\times 15$ MHz). For this antenna, $R_{\rm rad}\approx 20\ \Omega$, so if we feed it with current of amplitude $I_0 = 10$ A, it would radiate 1 kW of net radio power.

Electric Quadrupole Radiation

Warning: In different textbooks, the definitions of the electric quadrupole tensor differ by a factor of 2. In my convention — which I borrowed from Griffith's textbook, —

$$\mathcal{Q}_{ij} \;=\; \int\! d^3\mathbf{y}\, \rho(\mathbf{y}) \Big(\tfrac{3}{2} y_i y_j \,-\, \tfrac{1}{2}\delta_{ij}\mathbf{y}^2\Big), \tag{123}$$

while Jackson's textbook uses

$$\mathcal{Q}_{ij} \;=\; \int\! d^3\mathbf{y}\, \rho(\mathbf{y}) \Big(3 y_i y_j \,-\, \delta_{ij}\mathbf{y}^2\Big). \tag{124}$$

Consequenly, the formulae I'll derive in this section for the EM fields radiated by an oscillating electric quadrupole are going to differ from Jackson's by the factor of 2, and the formular for the radiated power — by the factor of 4.

Anyhow, the electric quadrupole contribution to the EM radiation stems from the

$$\mathbf{f}_{Eq}(\mathbf{n}) \;=\; \frac{-ik}{8\pi} \int\! d^3\mathbf{y} \Big(\mathbf{J}(\mathbf{y}\cdot\mathbf{n}) \,+\, \mathbf{y}(\mathbf{J}\cdot\mathbf{n})\Big) \tag{110}$$

contribution to the first subleading term in the expansion of $\mathbf{f}$ into powers of $k \times$ (size). To see the relation of this $\mathbf{f}_{Ed}$ to the electric quadrupole moment tensor (123), we use the continuity equation for the harmonic charge and current densities,

$$\nabla\cdot\mathbf{J} \;=\; -\frac{\partial\rho}{\partial t} \;=\; +i\omega\rho. \tag{125}$$

In light of this formula,

$$\begin{aligned}
i\omega\,\mathcal{Q}_{ij} \;&=\; \int\! d^3\mathbf{y}\, (\nabla\cdot\mathbf{J}) \Big(\tfrac{3}{2} y_i y_j \,-\, \tfrac{1}{2}\delta_{ij}\mathbf{y}^2\Big) \\
&\qquad \langle\!\langle \text{integrating by parts} \rangle\!\rangle \\
&=\; -\int\! d^3\mathbf{y}\, J_k(\mathbf{y})\, \nabla_k \Big(\tfrac{3}{2} y_i y_j \,-\, \tfrac{1}{2}\delta_{ij}\mathbf{y}^2\Big) \\
&=\; -\int\! d^3\mathbf{y}\, J_k \Big(\tfrac{3}{2}\delta_{ki} y_j \,+\, \tfrac{3}{2} y_i \delta_{kj} \,-\, \delta_{ij} y_k\Big) \\
&=\; -\int\! d^3\mathbf{y}\, \Big(\tfrac{3}{2} J_i y_j \,+\, \tfrac{3}{2} y_i J_j \,-\, \delta_{ij}(\mathbf{y}\cdot\mathbf{J})\Big),
\end{aligned} \tag{126}$$

and hence

$$\begin{aligned} i\omega\mathcal{Q}_{ij}n_j \ &= \ -\int d^3\mathbf{y}\left(\tfrac{3}{2}J_i(\mathbf{y}\cdot\mathbf{n})\ +\ \tfrac{3}{2}y_i(\mathbf{J}\cdot\mathbf{n})\ -\ n_i(\mathbf{y}\cdot\mathbf{J})\right) \\ &= \ -\frac{3}{2}\int d^3\mathbf{y}\left(\mathbf{J}(\mathbf{y}\cdot\mathbf{n})\ +\ \mathbf{y}(\mathbf{J}\cdot\mathbf{n})\right)_i +\ \mathbf{n}_i\int d^3\mathbf{y}\,(\mathbf{y}\cdot\mathbf{J}). \end{aligned} \tag{127}$$

The first term on the second line here is similar to eq. (110) for the $\mathbf{f}_{Eq}$ — except for the overall coefficient — while the second term has a form $\mathbf{n}$ times a scalar. Therefore,

$$\mathbf{f}_{Eq}(\mathbf{n}) \ = \ -\frac{\omega k}{12\pi}\,(\mathcal{Q}\circ\mathbf{n})\ +\ (\text{scalar})\,\mathbf{n} \tag{128}$$

where $(\mathcal{Q}\circ\mathbf{n})$ is a vector with components $(\mathcal{Q}\circ\mathbf{n})_i = \mathcal{Q}_{ij}n_j$.

Moreover, the second term (scalar) $\mathbf{n}$ in eq. (128) does not affect the EM fields $\mathbf{E}$ and $\mathbf{H}$ or the power of the EM waves, at least not in the radiation zone of $r \gg \lambda$. Indeed, in that zone

$$\mathbf{H} \ = \ ik\,\frac{e^{ikr}}{r}\,\mathbf{n}\times\mathbf{f}(\mathbf{n}), \tag{43}$$

$$\mathbf{E} \ = \ -ikZ_0\frac{e^{ikr}}{r}\,\mathbf{n}\times(\mathbf{n}\times\mathbf{f}(\mathbf{n})), \tag{44}$$

$$\frac{dP}{d\Omega} \ = \ \frac{k^2Z_0}{2}\,\|\mathbf{n}\times\mathbf{f}(\mathbf{n})\|^2, \tag{49}$$

and in all these formulae the$\mathbf{f}(\mathbf{n})$ appears only in the combination $\mathbf{n}\times\mathbf{f}(\mathbf{n})$. Consequently, from the radiation point of view,

$$\mathbf{f}(\mathbf{n})\ +\ (\text{any scalar})\,\mathbf{n}\ \cong\ \mathbf{f}(\mathbf{n}). \tag{129}$$

In particular, for the electric quadrupole radiation,

$$\mathbf{f}_{Eq}(\mathbf{n})\ \cong\ -\frac{\omega k}{12\pi}\,(\mathcal{Q}\circ\mathbf{n}). \tag{130}$$

In terms of the EM fields in the radiation zone, this means

$$\mathbf{H}\ \approx\ +i\frac{k^2\omega}{12\pi}\left(\mathbf{n}\times(\mathcal{Q}\circ\mathbf{n})\right)\frac{e^{ikr-i\omega t}}{r}\,, \tag{131}$$

$$\mathbf{E} \;\approx\; -i\frac{Z_0 k^2 \omega}{12\pi}\,\big(\mathbf{n}\times(\mathbf{n}\times(\mathcal{Q}\circ\mathbf{n}))\big)\,\frac{e^{ikr-i\omega t}}{r}\,, \tag{132}$$

while the EM power radiated per unit of solid angle is

$$\frac{dP}{d\Omega} \;=\; \frac{Z_0 k^4 \omega^2}{288\pi^2}\,\big\|\mathbf{n}\times(\mathcal{Q}\circ\mathbf{n})\big\|^2 \;=\; \frac{Z_0\,\omega^6}{288\pi^2 c^4}\Big((\mathcal{Q}^*_{ij}n_j)(\mathcal{Q}_{ik}n_k) \,-\, |n_i\mathcal{Q}_{ij}n_j|^2\Big). \tag{133}$$

To calculate the net radiated power, we need to integrate eq. (133) over the 4π solid angle. In components,

$$\oiint d^2\Omega\Big((\mathcal{Q}^*\cdot\mathbf{n})\cdot(\mathcal{Q}\cdot\mathbf{n}) - \big|\mathbf{n}\cdot\mathcal{Q}\cdot\mathbf{n}\big|^2\Big) \;=\; Q^*_{ij}Q_{ik}\oiint d^2\Omega\, n_j n_k \;-\; Q^*_{ij}Q_{k\ell}\oiint d^2\Omega\, n_i n_j n_k n_\ell \tag{134}$$

where the remaining integrals on the RHS must be rotationally invariant and also totally symmetric in the indices of all the **n** vectors. Thus

$$\oiint d^2\Omega\, n_j n_k \;=\; A_2\delta_{jk}\,, \quad \oiint d^2\Omega\, n_i n_j n_k n_\ell \;=\; A_4\big(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\big) \tag{135}$$

for some overall coefficients A_2 and A_4 which obtain by setting all the indices to 3 (*i.e.*, z):

$$A_2 \;=\oiint d^2\Omega\,\cos^2\theta \;=\; \frac{4\pi}{3}\,, \quad 3A_4 \;=\oiint d^2\Omega\,\cos^4\theta \;=\; \frac{4\pi}{5}\,. \tag{136}$$

Consequently,

$$\begin{aligned}
&\oiint d^2\Omega\,\Big((\mathcal{Q}^*\circ\mathbf{n})\cdot(\mathcal{Q}\circ\mathbf{n}) \,-\, \big|\mathbf{n}\circ\mathcal{Q}\circ\mathbf{n}\big|^2\Big) \;= \\
&\quad= \; Q^*_{ij}Q_{ik}\times\frac{4\pi}{3}\,\delta_{jk} \;-\; Q^*_{ij}Q_{k\ell}\times\frac{4\pi}{15}\big(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\big) \\
&\quad= \; \frac{4\pi}{3}\,Q^*_{ij}Q_{ij} \;-\; \frac{4\pi}{15}\Big(Q^*_{ii}Q_{kk} + Q^*_{ij}Q_{ji} + Q^*_{ij}Q_{ij}\Big) \\
&\qquad\quad \langle\!\langle\text{ using symmetry and tracelessness of the quadrupole moment tensor }\rangle\!\rangle \\
&\quad= \; \frac{4\pi}{3}\,Q^*_{ij}Q_{ij} \;-\; \frac{4\pi}{15}\Big(0 + 2Q^*_{ij}Q_{ij}\Big) \\
&\quad= \; \frac{4\pi}{5}\,Q^*_{ij}Q_{ij} \;=\; \frac{4\pi}{5}\,\mathrm{tr}\big(\mathcal{Q}^\dagger\mathcal{Q}\big),
\end{aligned} \tag{137}$$

and therefore the net radiated power is

$$P_{\rm net} \;=\; \frac{Z_0}{360\pi c^4}\,\omega^6\,{\rm tr}\big(\mathcal{Q}^\dagger\mathcal{Q}\big). \tag{138}$$

The angular distribution of the quadrupole radiation depends on the structure of the quadrupole moment tensor, which can range from a linear quadrupole (all charges arranged along a line) to planar quadrupole (all charges in the same plane) to complicated 3D setups where the charges move in different directions with different phases. For specific examples, let's consider the quadrupole moment tensors proportional to the spherical harmonics $Y_{\ell,m}$ with $\ell=2$, namely

$$\begin{aligned}
\mathcal{Q}^{(m=0)} &= \frac{Q}{\sqrt{3/2}}\begin{pmatrix} -\frac12 & 0 & 0\\ 0 & -\frac12 & 0\\ 0 & 0 & +1\end{pmatrix},\\
\mathcal{Q}^{(m=\pm1)} &= \frac{Q}{2}\begin{pmatrix} 0 & 0 & +1\\ 0 & 0 & \pm i\\ +1 & \pm i & 0\end{pmatrix},\\
\mathcal{Q}^{(m=\pm2)} &= \frac{Q}{2}\begin{pmatrix} +1 & \pm i & 0\\ \pm i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}.
\end{aligned} \tag{139}$$

- The $m=0$ quadrupole mode included all linear quadrupoles as well as other configurations with similar symmetries. (An axial symmetry, or at least a symmetry of 90° rotations around the z axis.) For this mode

$$\mathcal{Q}\circ\mathbf{n} \;=\; \frac{Q}{\sqrt{3/2}}\begin{pmatrix} -\frac12 n_x\\ -\frac12 n_y\\ +n_z\end{pmatrix} \quad\Longrightarrow\quad \begin{aligned}\|\mathcal{Q}\circ\mathbf{n}\|^2 &= \frac{Q^2}{3/2}\left(\tfrac14 n_x^2+\tfrac14 n_y^2+n_z^2\right)\\ &= \frac{Q^2}{6}\Big(1+3n_z^2 \;=\; 1+3\cos^2\theta\Big)\end{aligned} \tag{140}$$

while

$$\mathbf{n}\circ\mathcal{Q}\circ\mathbf{n} \;=\; \frac{Q}{\sqrt{3/2}}\Big(-\tfrac12 n_x^2-\tfrac12 n_y^2+n_z^2 \;=\; \tfrac32 n_z^2-\tfrac12 \;=\; \tfrac32\cos^2\theta-\tfrac12\Big). \tag{141}$$

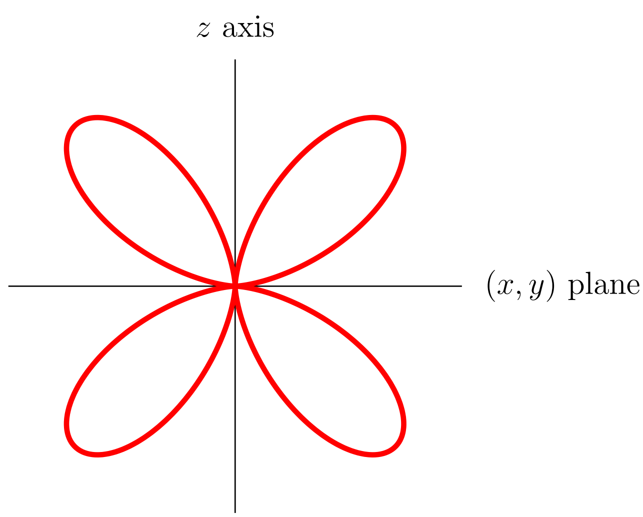
Consequently,

$$\begin{aligned}\|\mathcal{Q}\circ\mathbf{n}\|^2 \;-\; |\mathbf{n}\circ\mathcal{Q}\circ\mathbf{n}|^2 \;&=\; \frac{Q^2}{6}\left(1+3\cos^2\theta\right) \;-\; \frac{Q^2}{3/2}\left(\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2}\right)^2 \\ &=\; \frac{Q^2}{6}\Big(1+3\cos^2\theta-1+6\cos^2\theta-9\cos^4\theta\Big) \\ &=\; \frac{Q^2}{6}\left(9\cos^2\theta-9\cos^4\theta\right) \\ &=\; \frac{3Q^2}{2}\times\cos^2\theta\sin^2\theta\end{aligned} \tag{142}$$

and therefore the angular distribution of the radiated power is

$$\frac{dP}{d\Omega}\;\propto\;\cos^2\theta\sin^2\theta. \tag{143}$$

Here is the radiation power diagram for this distribution:



- Next, consider the $m=\pm1$ quadrupole modes, for which

$$\mathcal{Q}\circ\mathbf{n}\;=\;\frac{Q}{2}\begin{pmatrix}+n_z\\ \pm in_z\\ n_x\pm in_y\end{pmatrix}\quad\Longrightarrow\quad\begin{aligned}\|\mathcal{Q}\circ\mathbf{n}\|^2\;&=\;\frac{Q^2}{4}\left(2n_z^2+n_x^2+n_y^2\right)\\ &=\;\frac{Q^2}{4}\left(1+\cos^2\theta\right)\end{aligned} \tag{144}$$

while

$$\mathbf{n}\circ\mathcal{Q}\circ\mathbf{n} \;=\; \frac{Q}{2}\times 2n_z(n_x\pm in_y) \;=\; Q\times\cos\theta\sin\theta\,e^{\pm i\phi}. \tag{145}$$

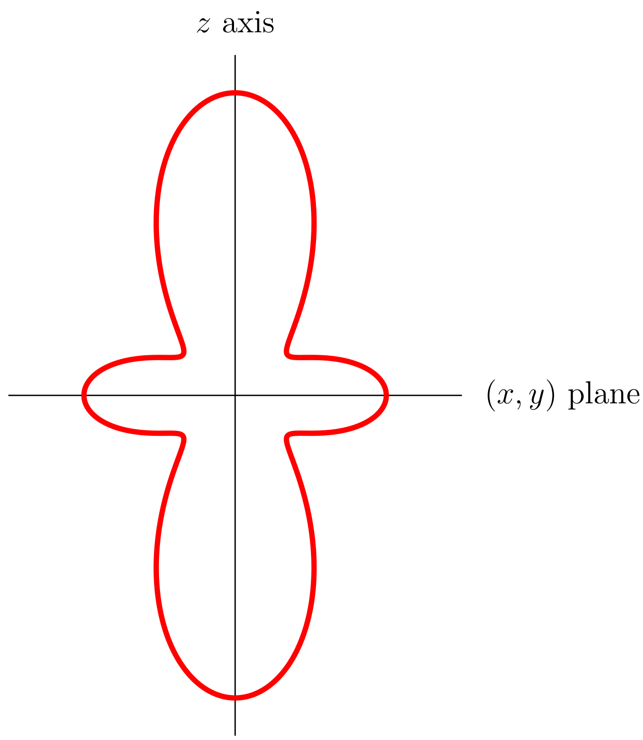
Consequently,

$$\begin{aligned} \|\mathcal{Q}\circ\mathbf{n}\|^2 \;-\; |\mathbf{n}\circ\mathcal{Q}\circ\mathbf{n}|^2 \;&=\; \frac{Q^2}{4}\,(1+\cos^2\theta) \;-\; Q^2\cos^2\theta\sin^2\theta \\ &=\; \frac{Q^2}{4}\times\left(1-3\cos^2\theta+4\cos^4\theta\right) \end{aligned} \tag{146}$$

and therefore

$$\frac{dP}{d\Omega} \;\propto\; 1 \;-\; 3\cos^2\theta \;+\; 4\cos^4\theta. \tag{147}$$

The radiation power diagram for this distribution looks like



- Finally, the $m=\pm2$ quadrupole modes, which include the planar quadrupoles. For these modes

$$\mathcal{Q}\circ\mathbf{n} \;=\; \frac{Q}{2}\begin{pmatrix} n_x \pm in_y \\ \pm in_x - n_y \\ 0 \end{pmatrix} \quad\Longrightarrow\quad \begin{aligned} \|\mathcal{Q}\circ\mathbf{n}\|^2 \;&=\; \frac{Q^2}{4}\times 2|n_x \pm in_y|^2 \\ &=\; \frac{Q^2}{2}\,\sin^2\theta \end{aligned} \tag{148}$$

while

$$\mathbf{n}\circ\mathcal{Q}\circ\mathbf{n} \;=\; \frac{Q}{2}\times(n_x \pm in_y)^2 = \;\frac{Q}{2}\times\sin^2\theta\, e^{\pm 2i\phi}. \tag{149}$$

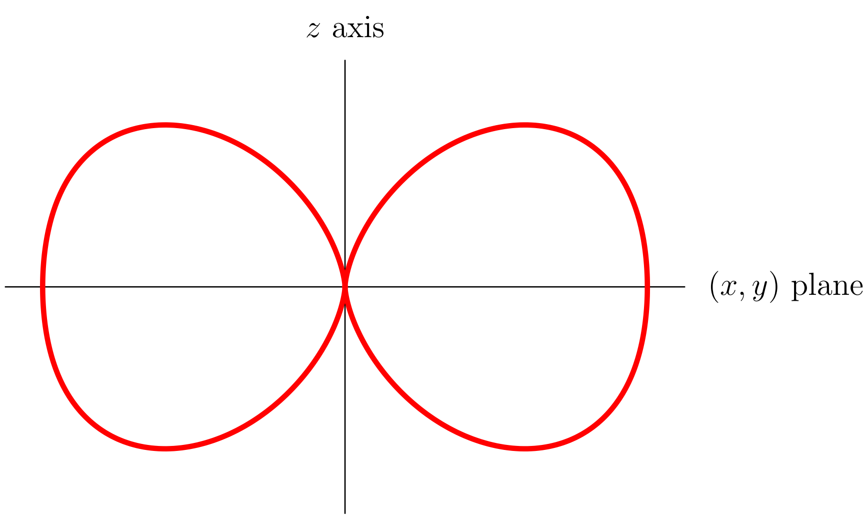
Consequently,

$$\begin{aligned} \|\mathcal{Q}\circ\mathbf{n}\|^2 \;-\; |\mathbf{n}\circ\mathcal{Q}\circ\mathbf{n}|^2 \;&=\; \frac{Q^2}{2}\times\sin^2\theta \;-\; \frac{Q^2}{4}\times\sin^4\theta \\ &=\; \frac{Q^2}{4}\times(\sin^2 \;=\; 1-\cos^2\theta)\times(2-\sin^2\theta \;=\; 1+\cos^2\theta) \\ &=\; \frac{Q^2}{4}\times(1-\cos^4\theta) \end{aligned} \tag{150}$$

and therefore

$$\frac{dP}{d\Omega} \;\propto\; 1-\cos^4\theta. \tag{151}$$

Here is the radiation power diagram for this angular distribution.



Higher orders of the Multipole Expansion

Let me finish these notes with a few words about the higher orders

$$\mathbf{f}_m(\mathbf{n}) \;=\; \frac{(-ik)^m}{4\pi m!} \iiint d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\,(\mathbf{n}\cdot\mathbf{y})^m \tag{152}$$

of the multipole expansion. These higher orders become relevant when all the lower order terms happen to vanish for some compact antenna. Or if the antenna is not so compact, and we need to sum up the whole expansion series for the

$$\mathbf{f}(\mathbf{n}) \;=\; \sum_{m=0}^{\infty} \mathbf{f}_m(\mathbf{n}). \tag{153}$$

Similar to the $\mathbf{f}_1(\mathbf{n})$ — which is related to the magnetic dipole moment and the electric quadrupole moment of the antenna, — each higher-order $\mathbf{f}_m$ is related to the magnetic 2^m–pole moment and the electric 2^{m+1}–pole moment, thus

$$\mathbf{f}_m(\mathbf{n}) \;=\; f_{M:\ell=m}(\mathbf{n}) \;+\; f_{E:\ell=m+1}(\mathbf{n}). \tag{154}$$

To save time, I am not going to work out the details of all these higher-order moments or their precise relations to the $\mathbf{f}_m(\mathbf{n})$. If you ever need them, you can sweat them out by yourself.

THE END