Dispersion of Waves

In these notes we study waves (of any kind) propagating in linear but dispersive media, thus wave equation for the harmonic waves is

$$\Psi(\mathbf{x},t) \;=\; \Psi(\mathbf{x})\times e^{-i\omega t}, \qquad \left(\nabla^2 \,+\, \frac{\omega^2 n^2(\omega)}{c^2}\right)\Psi(\mathbf{x}) \;=\; 0. \tag{1}$$

For simplicity, let's assume that the refraction index is real, or rather a real function of ω, so the wave equation (1) has non-attenuating plane-wave solutions

$$\Psi(\mathbf{x},t) \;=\; \Psi_0\,\exp(i\mathbf{k}\cdot\mathbf{x} \,-\, i\omega t) \quad \text{with } |\mathbf{k}| \;=\; \frac{\omega n(\omega)}{c}\,. \tag{2}$$

Also for simplicity, let's focus on the one-dimensional waves.

For a frequency-independent n, the wave equation (1) Fourier-transforms to a PDE for waves with arbitrary time-dependence,

$$\left(\nabla^2 \,+\, \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\Psi(\mathbf{x},t) \;=\; 0, \qquad v \;\stackrel{\text{def}}{=}\; \frac{c}{n}\,, \tag{3}$$

and in 1 space dimension, the most general solution of this equation is the superposition of two pulses of arbitrary shape, one traveling right at velocity $+v$ and the other traveling left at velocity $-v$,

$$\Psi(x,t) \;=\; \psi_1(x-vt) \;+\; \psi_2(x+vt). \tag{4}$$

Alas, for the frequency-dependent $n(\omega)$ the situation is more complicated: not only the wave velocity depends on the frequency, but also one must distinguish between the *phase velocity* and the *group velocity*. The phase velocity is the velocity of a plane wave's phase:

$$\Psi(x,t) \;=\; \Psi_0\exp(ikx - i\omega t), \qquad \text{phase } \varphi \;=\; kx \,-\, \omega t \;=\; k\left(x \,-\, \frac{\omega}{k}\,t\right), \tag{5}$$

thus the phase ϕ moves at the velocity

$$v_{\text{phase}} \;=\; \frac{\omega}{k} \;=\; \frac{c}{n(\omega)}\,. \tag{6}$$

The trouble with a perfectly uniform plane wave is that it transmits no information. To send a signal, the wave must be modulated in some fashion. For example, at time $t = 0$ we start with a modulated wave packet

$$\Psi(x;t=0) \;=\; F(x)\times e^{ik_0x} \tag{7}$$

where the amplitude profile $F(x)$ changes with x much slower than the phase e^{ik_0x}. Then — as we shall see in a moment — at later times t, the wave pulse looks like

$$\Psi(x;t) \;\approx\; F(x-v_{\rm group}t)\times\exp(ik_0x-i\omega_0t), \tag{8}$$

with the similar profile F but shifted to the right through the distance $v_{\rm group}\times t$. Thus, the amplitude of the wave pulse moves with the group velocity

$$v_{\rm group} \;=\; \frac{d\omega}{dk} \;\neq\; \frac{\omega}{k} \;=\; v_{\rm phase}\,. \tag{9}$$

To verify eq. (8) for a traveling wave pulse — as well as eq. (9) for the group velocity, — consider a Gaussian wave packet

$$\Psi(x;t_0=0) \;=\; \Psi_0\exp\left(-\frac{(x-x_0)^2}{4a^2}\right)\times e^{ik_0x} \tag{10}$$

of real width a much larger than the wavelength $\lambda=2\pi/k_0$. Fourier transforming this wave packet from the x space to the k space, we have

$$\widetilde{\Psi}(k;t_0=0) \;=\; \int\! dx\,e^{-ikx}\times\Psi(x;t_0=0) \;=\; 2\sqrt{\pi}a\Psi_0\times\exp\bigl(-a^2(k-k_0)^2\bigr)\times e^{-ix_0(k-k_0)}, \tag{11}$$

cf. my notes on Gaussian integrals and Gaussian wave packets. The RMS width of this k-space wave packet is $\Delta k=1/(2a)$, so the wider the original pulse is in the x-space, the smaller is the effective range of the wave-numbers k it contains.

At a future time $t>0$, the k-space wave-packet becomes

$$\widetilde{\Psi}(k;t) \;=\; \widetilde{\Psi}(k;t_0=0)\times\exp\bigl(-i\omega(k)t\bigr) \tag{12}$$

where $\omega(k)$ is a non-linear function of the wave number k. In general, this function — called the *dispersion relation* — may be very complicated, but for a wave-packet spanning a narrow range of $k = k_0 \pm \Delta k$, $\Delta k \ll k_0$, we may approximate

$$\omega(k) \;\approx\; \omega(k_0) \;+\; \frac{d\omega}{dk} \times (k - k_0) \;=\; \omega_0 \;+\; v_g \times (k - k_0) \tag{13}$$

where $\omega_0 = \omega(k_0)$ and $v_g = d\omega/dk$ is the group velocity, exactly as in eq. (9). Consequently, the future-time wave-packet (12) becomes

$$\begin{aligned} \widetilde{\Psi}(k;t) \;&\approx\; \widetilde{\Psi}(k;t_0=0) \times \exp\bigl(-i\omega_0 t \;-\; iv_g(k-k_0)t\bigr) \\ &=\; 2\sqrt{\pi}a\Psi_0 \times \exp\bigl(-a^2(k-k_0)^2\bigr) \times e^{-ix_0(k-k_0)} \times \exp\bigl(-i\omega_0 t \;-\; iv_g(k-k_0)t\bigr) \\ &=\; 2\sqrt{\pi}a\Psi_0 e^{-i\omega_0 t} \times \exp\left(-a^2 \times (k-k_0)^2 \;-\; i(x_0+v_g t) \times (k-k_0)\right). \end{aligned} \tag{14}$$

Now let's Fourier transform this future wave-packet back to the coordinate space. Since the only time-dependence of this packet is the overall phase $e^{-i\omega_0 t}$ and the parameter shift $x_0 \to x_0 + v_g t$, we get back to the original wave packet modulo these overall phase factor and the x_0 shift,

$$\Psi(x;t) \;=\; \Psi_0 \exp\bigl(ik_0 x - i\omega_0 t\bigr) \times \exp\left(-\frac{(x-x_0-v_g t)^2}{4a^2}\right). \tag{15}$$

Thus, the Gaussian wave pulse remains a Gaussian wave pulse of the same width $\Delta x = a$, but its center x_0 moves to the right with the group velocity

$$v_g \;=\; \frac{d\omega}{dk}\,, \tag{9}$$

Quod erat demonstrandum.

As an example of different phase and group velocities, consider the EM waves in plasma. As we saw in class, for such waves

$$c^2k^2 \;=\; \omega^2 \;-\; \omega_p^2 \tag{16}$$

where ω_p is the plasma frequency which depends only on the plasma's electron density n_e.

For the dispersion relation (16), the phase velocity is

$$v_{\text{phase}} \;=\; \frac{\omega}{k} \;=\; c \times \frac{\omega}{\sqrt{\omega^2 - \omega_p^2}} \;>\; c, \tag{17}$$

while the group velocity is

$$v_{\text{group}} \;=\; \frac{d\omega}{dk} \;=\; \frac{d(\omega^2)}{d(k^2)} \Big/ \frac{2\omega}{2k} \;=\; c^2 \times \frac{k}{\omega} \;=\; c \times \frac{\sqrt{\omega^2 - \omega_p^2}}{\omega} \;<\; c. \tag{18}$$

At first blush, the superluminal phase velocity looks troublesome, but it's actually OK because the phase of a uniform plane wave does not carry with it any matter, energy, or information. On the other hand, the amplitude pulse of the wave does carry both energy and information, so it really should not propagate faster than light. And indeed, the group velocity (18) of this pulse's motion comes out to be slower than light.

In your next homework — set#8, problem#4 — you shall see that *for normal dispersion, the group velocity of an EM wave is always slower than c.* The first step in that direction is the relation of the group and phase velocities to the refraction index $n(\omega)$ as a function of ω and hence $ck(\omega) = \omega n(\omega)$. For the phase velocity, we immediately have

$$v_{\text{phase}}(\omega) \;=\; \frac{\omega}{k(\omega)} \;=\; \frac{c}{n(\omega)}\,, \tag{19}$$

while for the group velocity

$$\frac{c}{v_{\text{group}}(\omega)} \;=\; c\frac{dk}{d\omega} \;=\; \frac{d}{d\omega}(\omega n(\omega)) \;=\; n(\omega) \;+\; \omega\frac{dn}{d\omega}\,, \tag{20}$$

thus

$$v_{\text{group}}(\omega) \;=\; \frac{c}{n(\omega) \;+\; \omega\frac{dn}{d\omega}}\,. \tag{21}$$

Using these two formulae, you should be able to show that *for normal dispersion*

(a) $$v_{\text{group}} \;<\; v_{\text{phase}}\,, \quad \text{and}$$

(b) $$v_{\text{group}} \times v_{\text{phase}} \;<\; c^2,$$

hence $v_{\text{group}} < c$.

For the *anomalous dispersion,* the situation is more complicated. First of all, for the anomalous dispersion $n(\omega)$ becomes complex, and its imaginary part is comparable to the real part. Consequently, we have a complex $k(\omega) = k_r(\omega) + i\kappa(\omega)$, so the exact meaning of the complex $v_g = d\omega/dk(\omega)$ becomes unclear. A naive guess would be that the speed of the signal propagation should be

$$\textit{either} \quad \operatorname{Re} v_g = \operatorname{Re} \frac{d\omega}{dk(\omega)} \quad \text{or} \quad \frac{1}{\operatorname{Re}(1/v_g)} = \frac{d\omega}{dk_r(\omega)}\,, \tag{22}$$

— and near a resonance both of these quantities could become superluminal for some ω's, — but the actual signal speed is a lot more complicated than (22). Indeed, the imaginary part of $k(\omega)$ means an attenuating wave, and $\operatorname{Im} k \sim \operatorname{Re} k$ means a wave attenuating over a distance scale comparable to the wavelength λ. Because of this attenuation, we cannot form a wave packet of width a much larger than the attenuation distance, thus we are limited to $a \lesssim \lambda$. Consequently, in the k-space, the wave packet must have a rather large width

$$\Delta k = \frac{1}{2a} \sim k_0\,, \tag{23}$$

and this completely invalidates the approximation

$$\omega(k) \approx \omega_0 + \frac{d\omega}{dk} \times (k - k_0) \quad \text{for } |k - k_0| \lesssim \Delta k \tag{24}$$

we have used to calculate the wave-packet's propagation. As we shall see in the next section, going beyond this approximation makes the wave packets not only move in space but also spread out and change their shape. Consequently, the velocity of a wave packet becomes somewhat ill-defined as the front end of a packet moves faster than its rear end, and calculating the velocities of each part of the packet becomes rather complicated. In principle, we should consider a packet with an abrupt front end and verify that that front end never moves faster than c — hence no superluminal signals — but this would be a very hard exercise way beyond the scope of this class.

Dispersion

A non-linear relation between ω and k is called *dispersion* because it makes the wave packets widen — *i.e.*, disperse — as they travel. When a series of pulses carries some information — like in a telegraph line — this widening can make the pulses overlap each other and make the signal unreadable. To avoid this problem, one has to keep the pulses rather far from each other, which severely limits the rate at which they can carry information.

To see how this works, let's start with a Gaussian wave packet

$$\Psi(x,t_0=0) \;=\; \Psi_0 e^{ik_0x}\times\exp\left(-\frac{(x-x_0)^2}{4a^2}\right), \qquad \text{real } a\gg\frac{1}{k_0}. \tag{10}$$

but this time use a better approximation for the $\omega(k)$, namely

$$\omega(k) \;\approx\; \omega(k_0) \;+\; \frac{d\omega}{dk}\times(k-k_0)+\;\frac{1}{2}\frac{d^2\omega}{dk^2}\times(k-k_0)^2 \qquad \text{for } |k-k_0|\lesssim\frac{1}{a}. \tag{25}$$

Or in more compact notations

$$\omega(k) \;\approx\; \omega_0 \;+\; v_g\times(k-k_0) \;+\; \tfrac{1}{2}\omega''\times(k-k_0)^2 \tag{26}$$

where

$$\omega_0 \;=\; \omega(k_0), \quad v_g \;=\; \omega' \;=\; \frac{d\omega}{dk}, \quad \omega'' \;=\; \frac{d^2\omega}{dk^2}, \tag{27}$$

and we assume a real ω for a real k and vice verse, thus no attenuation, just dispersion.

Fourier transforming the initial wave packet (10) to the k space, we have

$$\widetilde{\Psi}(k;t_0=0) \;=\; \int dx\, e^{-ikx}\times\Psi(x;t_0=0) \;=\; 2\sqrt{\pi}a\Psi_0\times\exp\bigl(-a^2(k-k_0)^2\bigr)\times e^{-ix_0(k-k_0)}, \tag{11}$$

and hence at the future times $t>0$

$$\begin{aligned}
\widetilde{\Psi}(k,t) \;&=\; \exp\bigl(-it\omega(k)\bigr)\times\widetilde{\Psi}(k;t_0=0)\\
&\approx\; \exp\bigl(-it\omega_0 \;-\; itv_g\times(k-k_0) \;-\; \tfrac{i}{2}t\omega''\times(k-k_0)^2\bigr)\times\\
&\qquad \times 2\sqrt{\pi}a\Psi_0\times\exp\bigl(-a^2(k-k_0)^2\bigr)\times e^{-ix_0(k-k_0)}\\
&=\; 2\sqrt{\pi}a\Psi_0\times\exp\left(-\left(a^2+\frac{it\omega''}{2}\right)\times(k-k_0)^2 \;-\; i(x_0+tv_g)\times(k-k_0) \;-\; it\omega_0\right).
\end{aligned} \tag{28}$$

Note: the exponent here has a quadratic term with a complex coefficient

$$\tfrac{1}{2}A_k(t) \;=\; a^2 \;+\; \frac{it\omega''}{2}\,, \tag{29}$$

but it has a positive real part, so (28) is a kind of a Gaussian wave packet in the k space, and its Fourier transform is a similar Gaussian wave packet in the x space. Specifically, as explained in my notes on Gaussian integrals,

$$\begin{aligned}\Psi(x,t) &= \int\frac{dk}{2\pi}\,e^{ikx}\times\widetilde{\Psi}(k,t)\\ &= \frac{a\Psi_0}{\sqrt{\pi}}\,\exp(ik_0x-i\omega_0t)\times\\ &\qquad\times\int dk\,\exp\left(-\left(a^2+\tfrac{i}{2}\omega''t\right)\times(k-k_0)^2+i(x-x_0-v_gt)\times(k-k_0)\right)\end{aligned} \tag{30}$$

where

$$\begin{aligned}&-\left(a^2+\tfrac{i}{2}\omega''t\right)\times(k-k_0)^2 \;+\; i(x-x_0-v_gt)\times(k-k_0)\\ &\qquad\qquad= -\left(a^2+\tfrac{i}{2}\omega''t\right)\times\left(k\,-\,k_0\,-\,\frac{i(x-x_0-tv_g)}{2a^2+i\omega''t}\right)\\ &\qquad\qquad\quad-\;\frac{(x-x_0-tv_g)^2}{4a^2+2i\omega''t}\,,\end{aligned} \tag{31}$$

hence

$$\begin{aligned}&\int dk\,\exp\left(-\left(a^2+\tfrac{i}{2}\omega''t\right)\times(k-k_0)^2+i(x-x_0-v_gt)\times(k-k_0)\right)\\ &\quad= \exp\left(-\frac{(x-x_0-tv_g)^2}{4a^2+2i\omega''t}\right)\times\\ &\qquad\times\int dk\,\exp\left(-\left(a^2+\tfrac{i}{2}\omega''t\right)\times(k+\text{const})^2\right)\\ &\quad= \exp\left(-\frac{(x-x_0-tv_g)^2}{4a^2+2it\omega''}\right)\times\sqrt{\frac{\pi}{a^2+\frac{i}{2}\omega''t}},\end{aligned} \tag{32}$$

and therefore

$$\Psi(x,t) \;=\; \Psi_0\sqrt{\frac{a^2}{a^2+\frac{i}{2}\omega''t}}\times\exp(ik_0x-i\omega_0t)\times\exp\left(-\frac{(x-x_0-tv_g)^2}{4a^2+2it\omega''}\right). \tag{33}$$

By inspection, the wave (33) is a Gaussian wave packet centered at $x_0 + v_g \times t$ — so it indeed moves with the group velocity $v_g = \omega' = d\omega/dk$, — but the packet's width parameter

$$A \;=\; \frac{1}{2a^2 + i\omega''t} \;=\; \frac{2a^2 - i\omega''t}{4a^4 + (\omega''t)^2} \tag{34}$$

is complex rather than real, so its RMS width^2 is

$$\Delta x^2 \;=\; \frac{1}{2\,\mathrm{Re}\,A} \;=\; \frac{4a^4 + (\omega''t)^2}{4a^2} \;=\; a^2 \;+\; \frac{(\omega'')^2 t^2}{4a^2}\,, \tag{35}$$

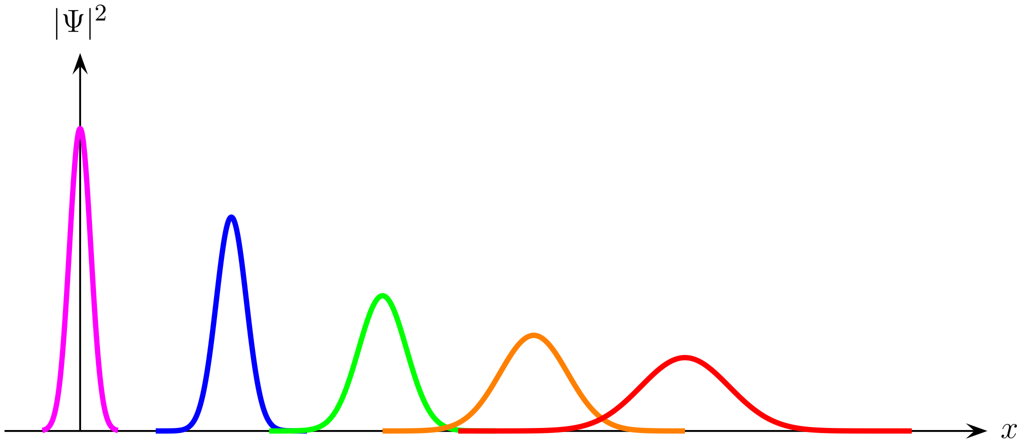
thus

$$\Delta x(t) \;=\; \sqrt{a^2 \;+\; \frac{(\omega'')^2 t^2}{4a^2}} \tag{36}$$

where $\omega'' = d^2\omega/dk^2$. At the same time, the pulse's central magnitude diminishes with time from $|\Psi_0|^2$ to

$$|\Psi_0|^2 \times \left|\frac{a^2}{a^2 + \frac{i}{2}\omega''t}\right| \;=\; |\Psi_0|^2 \times \sqrt{\frac{a^4}{a^4 + (\omega''t/2)^2}} \;=\; |\Psi_0|^2 \times \frac{a}{\Delta x(t)}\,. \tag{37}$$

To illustrate this effect, let me plot the magnitude $|\Psi|^2$ of the pulse as a function of x at several times $t = 0, 1, 2, 3, 4$ (in units of $a^2/|\omega''|$):



The dispersion limits the pulse rate — and hence the information transfer rate — in long transmission lines, from 19$^{\text{th}}$ century telegraph cables to modern fiber optic cables. Indeed, whatever the initial pulse width a, by the time the pulse reaches the end of the line at time $T = L/v_g$, its width $\Delta x(T)$ must be shorter than the space interval between the pulses, or else we would not be able to resolve them from each other. In terms of the pulse rate

$$\nu = \frac{1}{\text{times between pulses}}, \tag{38}$$

we need

$$\frac{v_g}{\nu} > \Delta x(T) \tag{39}$$

and hence

$$\frac{v_g^2}{\nu^2} > \Delta x^2(T) = a^2 + \frac{(\omega'')^2 T^2}{4a^2}. \tag{40}$$

For a given travel time T, the RHS here is minimized for $a^2 = \frac{1}{2}|\omega''|T$, thus even for this optimal width of the initial pulse, we need

$$\frac{v_g^2}{\nu^2} > |\omega''| \times T. \tag{41}$$

In other words, the pulse rate cannot be faster than

$$\nu_{\max} = \frac{v_g}{\sqrt{|\omega''| \times T}} = \sqrt{\frac{v_g^3}{|\omega''| \times L}}, \tag{42}$$

and that's why it's important to keep the dispersion ω'' in transmission lines as small as possible.

In terms of the refraction index $n(\omega)$, the group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{c}{n + \omega(dn/d\omega)}, \tag{43}$$

hence

$$\begin{aligned} \omega'' &\stackrel{\text{def}}{=} \frac{d^2\omega}{dk^2} = \frac{d\omega}{dk} \times \frac{d}{d\omega}\left(\frac{d\omega}{dk}\right) = v_g \times \frac{dv_g}{d\omega} \\ &= -\frac{v_g^3}{c} \times \frac{d}{d\omega}\left(\frac{c}{v_g} = n + \omega\frac{dn}{d\omega}\right) = -\frac{v_g^3}{c} \times \left(2\frac{dn}{d\omega} + \omega\,\frac{d^2n}{d\omega^2}\right), \end{aligned} \tag{44}$$

and therefore

$$\nu_{\max}^2 \;=\; \frac{v_g^3}{|\omega''| \times L} \;=\; \frac{c}{L} \Bigg/ \left| 2\frac{dn}{d\omega} + \omega\,\frac{d^2 n}{d\omega^2} \right| . \tag{45}$$

Thus, to maximize the pulse rate ν, we should endeavor to keep the refraction index $n(\omega)$ as ω-independent as possible.