EDDY CURRENTS, DIFFUSION, AND SKIN EFFECT

The magnetic field cannot instantly penetrate a conducting material, it has to diffuse in from the surface. The reason for this behavior are *eddy currents* due to EMF induced by the changing magnetic field; these currents in turn cause magnetic fields opposing the change of the original field.

To see how this works, consider a large piece of a uniform material of electric conductivity σ and magnetic permeability μ. There are several equations relating the electric field, the magnetic field, and the conduction current in this material:

the Ohm's law

$$\mathbf{J} = \sigma\mathbf{E}; \tag{1}$$

the Ampere's law⋆

$$\nabla\times\mathbf{B} = \mu\mu_0\nabla\times\mathbf{H} = \mu\mu_0\mathbf{J}; \tag{2}$$

and the Faraday's law of induction for the fields

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\,. \tag{3}$$

Combining these equations, we obtain

$$\nabla\times\mathbf{B} = \mu\mu_0\,\mathbf{J} = \mu\mu_0\sigma\,\mathbf{E}, \tag{4}$$

$$\nabla\times(\nabla\times\mathbf{B}) = \mu\mu_0\sigma\,\nabla\times\mathbf{E} = -\mu\mu_0\sigma\,\frac{\partial\mathbf{B}}{\partial t}\,, \tag{5}$$

$$\nabla^2\mathbf{B} = \nabla(\nabla\cdot\mathbf{B}) - \nabla\times(\nabla\times\mathbf{B}) = 0 + \mu\mu_0\sigma\,\frac{\partial\mathbf{B}}{\partial t}\,, \tag{6}$$

⋆ Strictly speaking, for the time-dependent field we should use the Maxwell–Ampere law which includes the displacement current $\partial\mathbf{D}/\partial t$ in addition to the conduction current $\mathbf{J}$. But in good conductors and less-then-optical frequencies the conduction current is so much stronger than the displacement current that the latter may be neglected.

and hence the *diffusion equation* for the magnetic field

$$\frac{\partial}{\partial t}\mathbf{B}(\mathbf{x},t) \;=\; \mathcal{D}\,\nabla^2\mathbf{B}(\mathbf{x},t) \tag{7}$$

for the *diffusion coefficient*

$$\mathcal{D} \;=\; \frac{1}{\mu\mu_0\sigma} \;=\; \frac{\rho}{\mu\mu_0}\,. \tag{8}$$

The current density $\mathbf{J}(\mathbf{x},t)$ obeys a similar diffusion equation with the same diffusion coefficient $\mathcal{D}$. Indeed, taking the curl of both sides of eq. (7), multiplying by $\mu\mu_0$, and using the Ampere's law, we obtain

$$\frac{\partial}{\partial t}\big(\mu\mu_0\nabla\times\mathbf{B}\big) \;=\; \mu\mu_0\nabla\times\frac{\partial\mathbf{B}}{\partial t} \;=\; \mu\mu_0\,\nabla\times\big(\mathcal{D}\nabla^2\mathbf{B}\big) \;=\; \mathcal{D}\,\nabla^2\big(\mu\mu_0\nabla\times\mathbf{B}\big) \tag{9}$$

and hence

$$\frac{\partial}{\partial t}\mathbf{J}(\mathbf{x},t) \;=\; \mathcal{D}\,\nabla^2\mathbf{J}(\mathbf{x},t). \tag{10}$$

Solving the Diffusion Equation: An Example

As an example of magnetic diffusion, consider a solid metal cylinder surrounded by a solenoidal coil. When we turn on the current in the coil, the surface of the metal cylinder is suddenly exposed to the coil's $\mathbf{H}$ field parallel to the cylinder. But this field cannot instantly penetrate the cylinder; instead, it has to diffuse inward from the surface according to the diffusion equation (7). This means that at the moment the coil's current I is turned on, we get an equal and opposite counter-current on the cylinder's surface,

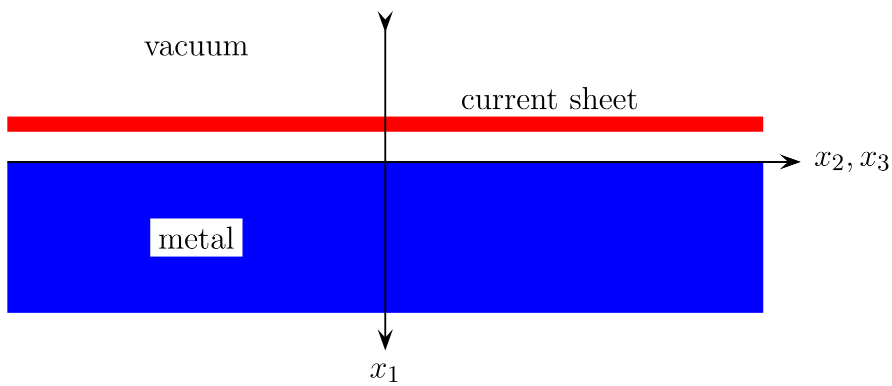
$$\mathbf{J}(z,s,\phi) \;=\; -K\delta(s-R)\,\mathbf{n}_\phi \quad \text{for } K \;=\; \frac{IN}{L}\,. \tag{11}$$

But as the time passes, this counter-current diffuses inward towards the cylinder's center, and this allows the magnetic field to penetrate the surface and also diffuse inward:

$$\text{for } t>0, \quad \mathbf{J}(\mathbf{x},t) \;=\; J(s,t)\,\mathbf{n}_\phi\,, \quad \mathbf{B}(\mathbf{x},t) \;=\; B(s,t)\,\mathbf{n}_z \tag{12}$$

for some time-dependent radial profiles $J(s,t)$ and $B(s,t)$.

Alas, solving the diffusion equation for the time dependence of these radial profiles involves Bessel functions and their relatives, so let's consider a simpler, one-dimensional example: An infinite slab of metal with a flat current sheet just above it:



In this picture the x_1 axis points down while the x_2 and x_3 axes are horizontal, the metal fills up the $x_1 > 0$ half space, and the current sheet carries a uniform current density in the x_3 direction. All by itself, this current would create a uniform magnetic field below the sheet in the x_2 direction, but the conduction current in the metal makes the problem much more interesting. Still, it's a one-dimensional problem where the magnetic field and the current in the metal depend only on the x_1 coordinate but not the x_2 or the x_3. Also, in the metal the current always flows in the $\pm x_3$ direction while the $\mathbf{H}$ field points in the $\pm x_2$ direction, thus

$$\mathbf{J}(\mathbf{x}, t) = -J(x_1, t)\,\mathbf{n}_3, \quad \mathbf{H}(\mathbf{x}, t) = H(x_1, t)\,\mathbf{n}_2\,. \tag{13}$$

With these sign/direction conventions, the Ampere law becomes

$$J(x_1, t) = -\frac{\partial H}{\partial x_1}\,, \tag{14}$$

so once we solve the 1D diffusion equation

$$\frac{\partial J}{\partial t} = \mathcal{D}\,\frac{\partial^2 J}{\partial x^2} \tag{15}$$

for the current, the solution for the magnetic field obtains by integration

$$H(x_1,t) \;=\; +\int\limits_{x_1}^{\infty} J(x',t)\,dx'. \tag{16}$$

Note: the upper limit in this formula follows from the asymptotic condition $H=0$ infinitely deep inside the metal for any finite time t.

The simplest way to solve the diffusion equation (15) is via the Fourier transform. First, to avoid problems with the abrupt discontinuity of the current at the metal's edge $x=0$, let's formally continue the $J(x_1)$ profile to negative x_1 but making it an even function of x_1 thus

$$\tilde{J}(\pm x) \;=\; J(+x). \tag{17}$$

Next we Fourier transform from x_1 to k for each time t,

$$\tilde{J}(x,t) \;=\; \int\limits_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{ikx} \times F(k,t), \tag{18}$$

$$F(k,t) \;=\; \int\limits_{-\infty}^{+\infty} dx\, e^{-ikx} \times \tilde{J}(x,t) \;=\; \int\limits_{0}^{\infty} dx\left(e^{-ikx} + e^{+ikx}\right) \times J(x,t). \tag{19}$$

In terms of the $F(k,t)$, the x derivative acts by multiplying by ik, so the diffusion equation becomes

$$\frac{\partial F}{\partial t} \;=\; -\mathcal{D}k^2 \times F(k,t). \tag{20}$$

For each k, this is a simple linear differential equation WRT time t, whose solution is

$$F(k,t) \;=\; F(k,0) \times \exp(-\mathcal{D}k^2 \times t). \tag{21}$$

Now let's apply eqs.(18), (19), and (21) to the problem at hand. Suppose the current K in the current sheet stays zero for a long while, and then abruptly turns on at $t=0$ and then stays constant. This outside current suddenly creates a magnetic field H_0 just outside the metal, and since this field cannot instantly penetrate the metal, it has to be screened by the surface current in the metal itself. This means that at the initial time $t = 0$, $H(x_1)$ inside or near metal is given by the step function $H(x_1) = H_0\Theta(-x_1)$, or rather

$$H(x_1) \;\approx\; H_0\Theta(\epsilon - x_1) \tag{22}$$

for some microscopically small depth ϵ inside the metal. Consequently, the initial current in the metal is

$$@t = 0, \quad J(x_1, 0) \;=\; -\frac{\partial H}{\partial x_1} \;=\; +H_0\delta(x_1 - \epsilon), \tag{23}$$

and according to eq. (19). the Fourier transform of this current is

$$F(k, 0) \;= \int\limits_0^\infty dx \left(e^{-ikx} + e^{+ikx}\right) \times H_0\delta(x - \epsilon) \;=\; 2H_0 \cos(k\epsilon) \;\approx\; \text{constant } 2H_0\,. \tag{24}$$

According to eq. (21), at later times $t > 0$ this transform becomes

$$F(k, t) \;=\; 2H_0 \times \exp(-\mathcal{D}k^2 \times t), \tag{25}$$

so Fourier transforming from k back to x_1, we obtain the current profile

$$\begin{aligned}
J(x_1, t) \;&=\; \int \frac{dk}{2\pi}\, e^{ikx_1} \times 2H_0 \exp(-\mathcal{D}t \times k^2) \\
&=\; \frac{H_0}{\pi} \int dk \,\exp\left(-\mathcal{D}t \times k^2 \,+\, ix_1 \times k \;=\; -\mathcal{D}t\left(k \,-\, \frac{ix_1}{2\mathcal{D}t}\right)^2 \,-\, \frac{x_1^2}{4\mathcal{D}t}\right) \\
&=\; \frac{H_0}{\pi} \times \exp\bigl(-x_1^2/(4\mathcal{D}t)\bigr) \times \int d(k + \text{const}) \,\exp\bigl(-\mathcal{D}t(k + \text{const})^2\bigr) \\
&=\; \frac{H_0}{\pi} \times \exp\left(-\frac{x_1^2}{4\mathcal{D}t}\right) \times \sqrt{\frac{\pi}{\mathcal{D}t}} \\
&=\; \frac{2H_0}{\sqrt{4\pi\mathcal{D}t}} \times \exp\left(-\frac{x_1^2}{4\mathcal{D}t}\right).
\end{aligned} \tag{26}$$

This is a *Gaussian profile* of the form

$$J(x_1, t) \;=\; \frac{2H_0}{\sqrt{\pi}\, a(t)} \times \exp\left(-\frac{x_1^2}{a^2(t)}\right) \tag{27}$$

— or rather the $x_1 > 0$ half of the Gaussian profile, — *whose width increases with time as*

$$a(t) \;=\; \sqrt{4\mathcal{D}\times t}\,. \tag{28}$$

As to the magnetic field $H(x_1,t)$, it obtains from integrating the Ampere's law according to eq. (16):

$$H(x_1,t) \;=\int\limits_{x_1}^{\infty} J(x_1,t) \;=\; \frac{2H_0}{\sqrt{\pi}\,a(t)}\times\int\limits_{x_1}^{\infty} \exp(-x'^2/a^2(t))\,dx' \;=\; H_0\times\bigl(1-\mathrm{erf}(x_1/a(t))\bigr) \tag{29}$$

where erf is the *error function* of the Gaussian distribution.

Skin Effect

Now consider an AC current

$$\mathbf{J}(\mathbf{x},t) \;=\; \mathrm{Re}\left(\mathbf{J}(\mathbf{x})\,e^{-i\omega t}\right) \tag{30}$$

flowing down a thick wire. For such an AC current, the diffusion equation (10) becomes

$$\nabla^2\mathbf{J}(\mathbf{x}) \;=\; \frac{1}{\mathcal{D}}\frac{\partial\mathbf{J}}{\partial t} \;=\; \frac{-i\omega}{\mathcal{D}}\,\mathbf{J}(\mathbf{x}). \tag{31}$$

Because of the imaginary coefficient on the RHS, all solutions to this equation are complex, which means that not only the amplitude but also the phase of the AC current vary from one point $\mathbf{x}$ to another.

For a round wire, the radial profile of an axially symmetric solution of eq. (31) involves Bessel functions of complex arguments, so its qualitative features are rather hard to extract. So let's consider a wire which is so thick that near its surface it looks like an infinite half-space worth of metal, just like we had in the previous example. Again, we assume that the current density $\mathbf{J}(\mathbf{x}) = J(x_1)\mathbf{n}_3$ depends only on the depth x_1, so eq. (31) becomes the 1D differential equation

$$\frac{d^2J}{dx_1^2} \;=\; \frac{-i\omega}{\mathcal{D}} \times J(x_1). \tag{32}$$

The two independent solutions to this equation are

$$J(x_1) \;\propto\; \exp(\pm\kappa x_1) \tag{33}$$

for $\kappa = \sqrt{-i\omega/\mathcal{D}}$; for the sake of definiteness, let's define κ as the root which has a positive real part, thus

$$\kappa \;=\; \sqrt{\frac{-i\omega}{\mathcal{D}}} \;=\; \sqrt{\frac{\omega}{\mathcal{D}}} \times \left(\sqrt{-i} \;=\; e^{-\pi i/4} \;=\; \frac{1-i}{\sqrt{2}}\right) \;=\; \frac{1-i}{\delta} \tag{34}$$

$$\text{for } \delta \;\stackrel{\text{def}}{=}\; \sqrt{\frac{2\mathcal{D}}{\omega}} \;=\; \sqrt{\frac{2\rho}{\mu\mu_0\omega}}\,, \tag{35}$$

so the two solutions become

$$J(x_1) \;\propto\; \exp\left((+1-i)\frac{x_1}{\delta}\right) \quad\text{and}\quad J(x_1) \;\propto\; \exp\left((-1+i)\frac{x_1}{\delta}\right). \tag{36}$$

But besides the equation (32) itself, we also have the asymptotic condition: the current density should diminish to zero at infinite depth rather than grow out of control, $\mathbf{J} \to 0$ for $x_1 \to +\infty$. Consequently, only the second solution is allowed, thus

$$J(x_1) \;=\; J_0 \times \exp(-\kappa x_1) \;=\; J_0 \times \exp(-x_1/\delta) \times \exp(+ix_1/\delta), \tag{37}$$

whose magnitude decreases with the depth as

$$|J(x_1)| = \; |J_0| \times \exp(-x_1/\delta). \tag{38}$$

This decrease of the current amplitude with depth is called the *skin effect:* a high-frequency AC current flows only near the surface of the conductor, and δ — the effective depth through which the current flows is called the *skin depth*.

According to eq. (35), this skin depth decreases with frequency as $1/\sqrt{\omega}$. For example, consider a copper wire; at room temperature copper has $\rho = 1.68 \cdot 10^{-8}\ \Omega/\mathrm{m}$ and $\mu \approx 1$, hence

$$\delta \;=\; \frac{65.2\ \mathrm{mm}}{\sqrt{f[\mathrm{in\ Hz}]}}\,. \tag{39}$$

Thus, for the 60 Hz AC current in the power wires, the skin depth is 8.4 mm, but for the 700 MHz frequency used by many cellphones, the skin depth in a copper wire is only 2.5 microns.

In a round wire of radius r_w much larger than the skin depth δ, the AC currents flows mostly near the surface of the wire and decreases with depth $x = r_w - r$ similarly to the current in the metal slab of our example,

$$J_z(x) \;\approx\; J_{\mathrm{surface}} \times e^{-x/\delta} \times e^{ix/\delta}. \tag{40}$$

Consequently, the net current through the wire is

$$I \;\approx\; 2\pi r_w \int\limits_0^{r_w} dx J(x) \;\approx\; 2\pi r_w J_{\mathrm{surface}} \int\limits_0^{\infty} dx\, \exp(-(1-i)x/\delta) \;=\; 2\pi r_w J_{\mathrm{surface}} \times \frac{\delta}{1-i}\,, \tag{41}$$

while the voltage — measured along the wire's surface — is

$$V \;=\; L \times E_{\mathrm{surface}} \;=\; L\rho J_{\mathrm{surface}} \tag{42}$$

where L is the wire's length. Thus, the AC impedance of the wire is

$$Z_{\mathrm{AC}} \;=\; \frac{V}{I} \;=\; \frac{L\rho}{2\pi r_w\,\delta}\,(1-i) \;=\; \frac{L\rho}{2\pi r_w\,\delta}\,(1+j), \tag{43}$$

much larger that the wire's DC resistance

$$R_{\mathrm{DC}} \;=\; \frac{\rho L}{\pi r_w^2}\,. \tag{44}$$

For example, a copper wire of diameter $2r_w = 0.5$ mm has DC resistance of only $0.086\ \Omega/\mathrm{m}$ but at 700 MHz its AC impedance becomes $4.3(1+j)\ \Omega/\mathrm{m}$.

Finally, as a cross-check on the AC impedance (43), consider the net power dissipated by the current flowing along the wire's skin:

$$\begin{aligned} P &= \frac{1}{2}\iiint d^3\mathbf{x}\,\mathrm{Re}(\mathbf{E}^*\cdot\mathbf{J}) \;=\; \frac{\rho}{2}\iiint d^3\mathbf{x}\,|\mathbf{J}|^2 \\ &\approx \frac{\rho}{2}\times 2\pi r_w L\times\int_0^\infty dx\,|J_{\mathrm{surface}}|^2\times e^{-2x/\delta} \\ &= \frac{\rho}{2}\times 2\pi r_w\times|J_{\mathrm{surface}}|^2\times\frac{\delta}{2}\,. \end{aligned} \tag{45}$$

Comparing this formula to the voltage and current amplitudes V and I, we find

$$\mathrm{Re}(Z_{\mathrm{AC}}) \;=\; \frac{2P}{|I|^2} \;=\; \frac{L\rho}{2\pi r_w\,\delta} \tag{46}$$

while

$$\mathrm{Re}\left(\frac{1}{Z_{\mathrm{AC}}}\right) \;=\; \frac{2P}{|V|^2} \;=\; \frac{2\pi r_w\,\delta}{2L\rho} \;=\; \frac{1}{2}\times\frac{1}{\mathrm{Re}(Z_{\mathrm{AC}}}\,. \tag{47}$$

On the other hand,

$$\mathrm{Re}\left(\frac{1}{Z_{\mathrm{AC}}}\right) \;=\; \frac{\mathrm{Re}(Z_{\mathrm{AC}})}{|Z_{\mathrm{AC}}|^2}\,, \tag{48}$$

so the factor $\frac{1}{2}$ in eq. (47) calls for

$$\mathrm{Im}(Z_{\mathrm{AC}}) \;=\; \pm\,\mathrm{Re}(Z_{\mathrm{AC}}) \tag{49}$$

and therefore

$$Z_{\mathrm{AC}} \;=\; \frac{L\rho}{2\pi r_w\,\delta}\times(1\pm j), \tag{50}$$

in perfect agreement with the impedance (43).