1. Consider the space outside a sphere of radius R: there are some unknown electric charges inside the sphere but no charges outside it. In the previous homework (set#1, problem 3), you (should have) found that given the boundary potential $\Phi_b(\theta,\phi)$ on the sphere's surface and $\Phi \to 0$ for $r \to \infty$, the potential everywhere outside the sphere obtains as

$$\Phi(\mathbf{y}) \;=\; \frac{\mathbf{y}^2 - R^2}{4\pi R} \iint\limits_{\text{sphere}} d^2\text{Area}(\mathbf{x})\, \frac{\Phi(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^3}\,. \tag{1}$$

In the previous homework, you (should have) derived this formula using the separation of variable method and the spherical harmonics. In this problem, your task is to re-derive this formula using the Green's theorem.

(∗) If you are not familiar with the image charges for conducting spheres, read §2.2 of the Jackson's textbook.

(a) Use the image charge method to show that the Green's function for the outside the sphere with Dirichlet boundary condition $\Phi = 0$ in the sphere's surface and $\Phi \to 0$ for $r \to \infty$ is

$$G_D(\mathbf{x},\mathbf{y}) \;=\; \frac{1}{4\pi\sqrt{x^2+y^2-2xyc}} \;-\; \frac{R}{4\pi\sqrt{x^2y^2+R^4-2R^2xyc}} \tag{2}$$

where $x = |\mathbf{x}|$, $y = |\mathbf{y}|$, and $c = \mathbf{n}_x \cdot \mathbf{n}_y$.

(b) Evaluate the normal derivative of this Green's function at the boundary.

(c) Finally, use the Green's function method (*cf.* textbook §1.10) to derive eq. (1).

2. Next, consider the electrostatic potentials of multipole moments in the tensor picture.

(a) Let's start with a quadrupole-like potential

$$\Phi(\mathbf{x}) \;=\; \frac{Q_{ij}n_in_j}{4\pi\epsilon_0\, r^3} \;=\; \frac{Q_{ij}x_ix_j}{4\pi\epsilon_0\, |\mathbf{x}|^5} \tag{3}$$

for some would-be quadrupole moment tensor Q_{ij}, which you may assume to be symmetric — $Q_{ij} = Q_{ji}$ — but not necessarily traceless.

Show that this potential obeys the Laplace equation $\nabla^2\Phi(\mathbf{x}) = 0$ for $r \neq 0$ if and only if the Q_{ij} tensor is traceless, $Q_{ii} \equiv \delta_{ij}Q_{ij} = 0$.

(b) Next, an octupole-like potential

$$\Phi(\mathbf{x}) \;=\; \frac{\mathcal{O}_{ijk}n_in_jn_k}{4\pi\epsilon_0\, r^4} \;=\; \frac{\mathcal{O}_{ijk}x_ix_jx_k}{4\pi\epsilon_0\, |\mathbf{x}|^7} \tag{4}$$

for some totally symmetric tensor $\mathcal{O}_{ijk}$,

$$\mathcal{O}_{123} \;=\; \mathcal{O}_{312} \;=\; \mathcal{O}_{231} \;=\; \mathcal{O}_{213} \;=\; \mathcal{O}_{321} \;=\; \mathcal{O}_{132}\,. \tag{5}$$

Show that the potential (4) obeys the Laplace equation (for $r \neq 0$) if an only if the would-be octupole moment tensor $\mathcal{O}_{ijk}$ has zero trace,

$$\mathcal{O}_{iik} \;=\; \delta_{ij}\mathcal{O}_{ijk} \;=\; 0 \quad \forall k = x, y, z. \tag{6}$$

In general, a trace of a tensor with $\ell \geq 2$ indices is a tensor with $\ell - 2$ indices obtained by contracting the original tensor with the Kronecker's $\delta_{2\,\text{of the indices}}$. For example, for a 3-index tensor T_{ijk}, its trace is a one-index tensor (*i.e.*, a vector) $t_k = \delta_{ij}T_{ijk}$ (implicit sum over $i = j = x, y, z$). Likewise, the trace of a 4-index tensor $T_{ijk\ell}$ is a 2-index tensor $t_{k\ell} = \delta_{ij}T_{ijk\ell}$ (implicit sum over $i = j = x, y, z$).

Note: the non-symmetric tensors have several different traces for different choice of the two indices being contracted. For example, the 3-index non-symmetric tensor T^{ijk} has 3 different traces,

$$t^{(12)}_k \;=\; \delta_{ij}T_{ijk}\,, \quad t^{(23)}_i \;=\; \delta_{jk}T_{ijk}\,, \quad t^{(13)}_j \;=\; \delta_{ik}T_{ijk}\,. \tag{7}$$

But for the totally symmetric traces like the $\mathcal{O}_{ijk}$ in part (b), all such traces are equal, $t^{(12)}_i = t^{(23)}_i = t^{(13)}_i$, so there is only one trace $t_i = \mathrm{tr}(T)_i$. Also, for $\ell \geq 4$, the trace of a totally symmetric tensor with ℓ indices is itself a totally symmetric tensor with $\ell - 2$ indices.

(c) Now let's generalize the result of the parts (a) and (b) to the higher multipoles, $\ell \geq 4$. The potential for such would-be multipole has form

$$\Phi(\mathbf{x}) \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{\mathcal{M}^{\ell}_{i_1,i_2,\ldots,i_\ell} n_{i_1} n_{i_2}\cdots n_{i_\ell}}{r^{\ell+1}} \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{\mathcal{M}^{\ell}_{i_1,i_2,\ldots,i_\ell} x_{i_1} x_{i_2}\cdots x_{i_\ell}}{|\mathbf{x}|^{2\ell+1}} \tag{8}$$

for some totally symmetric tensor

$$\mathcal{M}^{\ell}_{i_1,i_2,\ldots,i_\ell} \;=\; \mathcal{M}^{\ell}_{\text{any permutation of } i_1,i_2,\ldots,i_\ell}\,. \tag{9}$$

Show that the potential (8) obeys the Laplace equation (for $r \neq 0$) if and only if the tensor (9) has zero trace,

$$\operatorname{tr}\big(\mathcal{M}^{\ell}\big)_{i_1,\ldots,i_{\ell-2}} \;\equiv\; \mathcal{M}^{\ell}_{i_1,i_2,\ldots,i_\ell}\times\delta_{i_{\ell-1},i_\ell} \;=\; 0 \quad \forall i_1,\ldots i_{\ell-2} \;=\; x,y,z. \tag{10}$$

(d) Finally, show that a totally symmetric ℓ-index tensor with a zero trace has $2\ell+1$ independent components.

3. Now let's put a small body with a non-zero electric dipole, quadrupole, or higher multipole moment into a slowly varying external electric field. By external field I mean field created by some distant charges far away from the body in question and not including the field due to the body itself; consequently, the external field obeys

$$\nabla\times\mathbf{E} \;=\; 0, \quad \nabla\cdot\mathbf{E} \;=\; 0, \quad \mathbf{E} \;=\; -\nabla\Phi, \quad \nabla^2\Phi \;=\; 0. \tag{11}$$

Out task in this problem is to calculate the potential energy of the body in the external field and hence the net force and the net torque on the body.

As a warm-up exercise, consider an ideal dipole: a body of negligible size with zero net charge, a finite dipole moment $\mathbf{p}$, and negligible quadrupole moment, octupole moment, *etc.*, *etc.*

(a) Show that the potential energy of such ideal dipole located at point $\mathbf{x}_0$ in an external electric field is

$$U(\mathbf{x}_0, \mathbf{p}) \;=\; -\mathbf{p}\cdot\mathbf{E}(\mathbf{x}_0) \;=\; +\mathbf{p}\cdot\nabla\Phi(\mathbf{x}_0). \tag{12}$$

In classical mechanics, the net force $\mathbf{F}$ and the net torque $\vec{\tau}$ acting on a rigid body can be obtained from the variation of its potential energy under displacements and rotations of the body: For an infinitesimal displacement $\delta\mathbf{x}_0$ of the whole body and a rotation through infinitesimal angle $\delta\vec{\alpha}$, we should get

$$\delta U \;=\; -\mathbf{F}\cdot\delta\mathbf{x}_0 \;-\; \vec{\tau}\cdot\delta\vec{\alpha}. \tag{13}$$

Note: if you rotate the body relative to its own center $\mathbf{x}_0$, you get the torque $\vec{\tau}$ WRT that center rather than WRT the coordinate origin.

(b) Apply this rule to the electric dipole with potential energy (12) and show that

$$\mathbf{F} \;=\; \nabla(\mathbf{p}\cdot\mathbf{E}) \;=\; (\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{x}_0) \quad\text{and}\quad \vec{\tau} \;=\; \mathbf{p}\times\mathbf{E}(\mathbf{x}_0). \tag{14}$$

Now consider an ideal quadrupole: a body of negligible size which has zero net charge, zero dipole moment, a finite quadrupole moment $\mathcal{Q}_{ij}$, and negligibly small higher multipole moments.

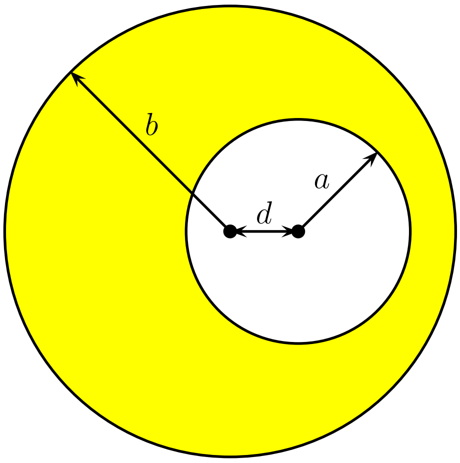
(c) Show that the potential energy of such a quadrupole in a slowly varying external potential $\Phi(\mathbf{x})$ is

$$U \;=\; \tfrac{1}{3}\mathcal{Q}_{ij}\,\nabla_i\nabla_j\Phi(\mathbf{x}_0). \tag{15}$$

Hint: the external potential $\Phi(\mathbf{x})$ obeys the Laplace equation $\nabla_i\nabla_i\Phi = 0$.

(d) Find the net force and the net torque on the quadrupole from eq. (15) for the potential energy.

4. Finally, a question on magnetostatics. Consider a conductor in the shape of a long cylinder of radius b with a hole drilled in it. The hole is also cylindrical, or radius a, but it's not coaxial with the conductor itself. Instead, the hole's axis and the outer cylinder's axis are parallel but at non-zero distance d from each other. Take $a + d < b$ so that the hole does not reach the conductor's outer surface. Here is the cross-section:



A current of uniform density $\mathbf{J}$ flows along this conductor in the direction $+z$ (out from the page into your face).

Use the Ampere's Law and the linear superposition principle to find the magnetic field $\mathbf{B}$ — both the magnitude and the direction — inside the hole.