PHY–387 K. Problem set #4. Due February 18, 2022.

1. First, a couple of reading assignments about examples of boundary problems in dielectric and magnetic materials.

   (a) First, read the examples of dielectric boundary problems in §4.4 of the Jackson's textbook, especially the dielectric sphere example (see also my notes on dielectrics and magnetic materials).

   (b) Second, read §5.12 of Jackson's textbook about magnetic shielding by a spherical shell of high-permeability material. If you have trouble following the boundary condition for the magnetic scalar potential $\Psi(\mathbf{x})$ — which Jackson calls $\Phi_M(\mathbf{x})$, — go back to part (a).

2. Now consider a wire loop $\mathcal{L}$ carrying steady current I. The loop $\mathcal{L}$ may have any size or shape, as long as it is closed. The magnetic field $\mathbf{H}$ generated by the current in this loop obtains from the scalar potential

$$\Psi(\mathbf{x}) \;=\; \frac{I}{4\pi}\,\Omega(\mathbf{x}) \tag{1}$$

   where $\Omega(\mathbf{x})$ is the solid angle spanned by the loop $\mathcal{L}$ when viewed from the point $\mathbf{x}$.

   By convention, $\Omega(\mathbf{x})$ is positive if the current in $\mathcal{L}$ viewed from point $\mathbf{x}$ appears to run clockwise, and negative if the current appears to run counterclockwise. To avoid a discontinuity when $\mathbf{x}$ is surrounded by the loop, $\Omega(\mathbf{x})$ should be analytically continued while $\mathbf{x}$ moves from one side of the loop to another. Such continuations make Ω multivalued, with different values of $\Omega(\mathbf{x})$ at the same point $\mathbf{x}$ differing by 4π, or more generally by $4\pi\times$ an integer.

   (a) Show that

$$\Omega(\mathbf{x}) \;=\; \iint\limits_{\mathcal{S}} \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3}\cdot d^2\mathbf{area}(\mathbf{y}) \tag{2}$$

   where $\mathcal{S}$ is a surface spanning the loop $\mathcal{L}$.

(b) Explain how eq. (2) leads to the sign convention for the $\Omega(\mathbf{x})$, and also how different surfaces $\mathcal{S}$ can yield values of $\Omega(\mathbf{x})$ which differ from each other by $4\pi \times$ an integer.

(c) Show that $\mathbf{H} = -\nabla\Psi$ for the scalar potential (1) agrees with the Biot–Savart–Laplace formula for the magnetic field of the current I in the loop $\mathcal{L}$.

Hint: prove and use

$$\nabla_y \times \left(\frac{(\mathbf{y}-\mathbf{x})\times\mathbf{c}}{|\mathbf{y}-\mathbf{x}|^3}\right) \;=\; (\mathbf{c}\cdot\nabla_y)\,\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} \tag{3}$$

for $\mathbf{y} \neq \mathbf{x}$ and any constant vector $\mathbf{c}$.

3. Next, a capacitor force problem. Take two large parallel vertical metal plates at small distance d between them, and immerse them part way into transformer oil with dielectric constant ϵ and *mass density* ρ. Connect the plates by wires to a battery or any other DC power supply of voltage V.

   Show that this makes the oil between the plates rise to the height

$$h \;=\; \frac{(\epsilon-1)\epsilon_0 V^2}{2\rho g d^2} \tag{4}$$

   relative to the oil outside the plates. ($g = 9.8$ N/kg is the gravitational field.)

4. Finally, an easy problem on Faraday's Induction Law. Consider a long straight vertical wire moving at constant velocity $\mathbf{v}$ in a horizontal direction. The wire carries a constant current I, which creates a magnetic field; in the quasistatic approximation (valid for $v \ll c$), $\mathbf{B}(\mathbf{x}, t)$ obtains via the Biot–Savart–Laplace equation using the wire's location at time t. But the moving wire make this quasistatic field move, so at a fixed location $\mathbf{x}$ the magnetic field changes with time.

   Find the electric field $\mathbf{E}(\mathbf{x}, t)$ *induced* by this time-dependent magnetic field.