1. In three space dimension, the retarded solution of the wave equation with an instant point source for example, a light flash at $\mathbf{x}=0$ and $t=0$ — is a spherical shell disturbance of radius $R=ct$ and zero thickness,

$$\Box\Psi(\mathbf{x},t) \;=\; \delta(t)\delta^{(3)}(\mathbf{x}) \quad\Longrightarrow\quad \Psi(\mathbf{x},t) \;=\; \frac{\delta(t-|\mathbf{x}|/c)}{4\pi|\mathbf{x}|}\,. \tag{1}$$

In spaces of other odd dimensions $d=3,5,7,\ldots$ (but not $d=1$), we have similar behavior, but in even space dimensions $d=2,4,6,\ldots$, the wave of an instant point source has a *wake* behind the light front. For example, in two space dimensions

$$\Box\Psi(\mathbf{x},t) \;=\; \delta(t)\delta^{(2)}(\mathbf{x}) \quad\Longrightarrow\quad \Psi(\mathbf{x},t) \;=\; \frac{c\Theta(ct-|\mathbf{x}|)}{2\pi\sqrt{c^2t^2-\mathbf{x}^2}} \tag{2}$$

where Θ is the step function.

(a) Derive the 2D wave (2) from the 3D wave generated by an instant *line* source.

In one space dimension, the disturbance spreads out at light speed, but then does not go away; instead,

$$\Box\Psi(x,t) \;=\; \delta(t)\delta(x) \quad\Longrightarrow\quad \Psi(x,t) \;=\; \frac{c}{2}\,\Theta(ct-|x|). \tag{3}$$

(b) Again, derive this 1D wave from the 3D wave generated by an instant source on an infinite plane.

2. In the Coulomb gauge $\nabla\cdot\mathbf{A}\equiv 0$, the potentials Φ and $\mathbf{A}$ obey

$$-\nabla^2\Phi(\mathbf{x},t) \;=\; \frac{1}{\epsilon_0}\,\rho(\mathbf{x},t), \quad \Box\mathbf{A}(\mathbf{x},t) \;=\; \mu_0\mathbf{J}_T(\mathbf{x},t), \tag{4}$$

where the transverse current $\mathbf{J}_T$ is

$$\mathbf{J}_T \;=\; \mathbf{J} \;+\; \nabla\left(\frac{-1}{\nabla^2}\,(\nabla\cdot\mathbf{J})\right), \quad i.e,, \quad \mathbf{J}_T(\mathbf{x},t) \;=\; \mathbf{J}(\mathbf{x},t) \;+\; \nabla_x\iiint d^3\mathbf{y}\,\frac{\nabla_y\cdot\mathbf{J}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\,. \tag{5}$$

As I've explained in class, the scalar potential Φ and the part of the vector potential $\mathbf{A}$ due to the second term in the transverse current (5) both respond instantaneously to the charges and currents, but these instantaneous terms cancel out from the electric and magnetic fields. In this exercise, we shall see how this works for a particularly simple source, namely an electric dipole $\mathbf{p}$ which turns up for just a moment and then turns back off,

$$\rho(\mathbf{x},t) \;=\; -\delta(t)\,(\mathbf{p}\cdot\nabla)\delta^{(3)}(\mathbf{x}), \quad \mathbf{J}(\mathbf{x},t) \;=\; \delta'(t)\,\mathbf{p}\,\delta^{(3)}(\mathbf{x}). \tag{6}$$

Note: derivatives of the delta functions are defined via integration by parts,

$$\int dt\, f(t)\times\delta'(t-t_0) \;\stackrel{\text{def}}{=}\; -\int dt\, f'(t)\times\delta(t-t_0) \;=\; -f'(t_0) \tag{7}$$

and likewise for the $\nabla\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)$.

(a) As a warm-up trivial exercise, verify the continuity equation for the dipole flash (6) and calculate the scalar potential $\Phi(\mathbf{x},t)$ in the Coulomb gauge.

(b) Calculate the transverse current $\mathbf{J}_T(\mathbf{x},t)$ and show that

$$\mathbf{J}_T(\mathbf{x},t) \;=\; \delta'(t)\left(\mathbf{p}\,\delta^{(3)}(\mathbf{x}) \;+\; \nabla(\mathbf{p}\cdot\nabla)\left(\frac{1}{4\pi r}\right)\right) \tag{8}$$

$$= \delta'(t)\left(\frac{2}{3}\,\mathbf{p}\,\delta^{(3)}(\mathbf{x}) \;+\; \frac{3(\mathbf{n}\cdot\mathbf{p})\mathbf{n} \;-\; \mathbf{p}}{4\pi r^3}\right). \tag{9}$$

(c) Next, prove a couple of lemmas you would need in the following parts.

Lemma 1: for any smooth function of position $F(\mathbf{z})$,

$$\iiint\limits_{\substack{\text{whole}\\ \text{space}}} d^3\mathbf{z}\,\frac{\delta'(t-|\mathbf{z}|/c)}{|\mathbf{z}|}\,F(\mathbf{z}) \;=\; c^2\Theta(t)\left[\left(1 \;+\; r\frac{\partial}{\partial r}\right)\oiint d^2\Omega_n\,F(\mathbf{z}=r\mathbf{n})\right]_{@r=ct} \tag{10}$$

where Θ is the step-function.

Lemma 2:

$$\frac{1}{4\pi}\oiint d^2\Omega_n\,\frac{1}{|\mathbf{x}+R\mathbf{n}|} \;=\; \frac{1}{\max(|\mathbf{x}|,R)}\,. \tag{11}$$

(d) Use the retarded Green's function to solve the wave equation $\Box\mathbf{A} = \mu_0\mathbf{J}_T$ for the vector potential under initial condition $\mathbf{A}(\mathbf{x}, t<0) \equiv 0$. Show that

$$\text{for } t < |\mathbf{x}|/c, \quad \mathbf{A}(\mathbf{x},t) = \mu_0 c^2 \Theta(t)\,\nabla(\mathbf{p}\cdot\nabla)\left(\frac{1}{4\pi|\mathbf{x}|}\right) = \Theta(t)\,\frac{3(\mathbf{n}\cdot\mathbf{p})\mathbf{n} - \mathbf{p}}{4\pi\epsilon_0\,|\mathbf{x}|^3}. \tag{12}$$

Hints: (1) use eq. (8) rather that eq. (9) for the transverse current; (2) after applying the retarded Green's function, change the integration variable from $\mathbf{y}$ to $\mathbf{z} = \mathbf{y} - \mathbf{x}$, then turn $\partial/\partial y$ derivatives into $\partial/\partial x$ at fixed $\mathbf{z}$; (3) use lemmas (10) and (11).

(e) Use vector potential (12) and the scalar potential you have computed in part (a) to verify that the electric and the magnetic fields do not propagate faster than light,

$$\text{for } t < c|\mathbf{x}|, \quad \mathbf{E}(\mathbf{x},t) = \mathbf{B}(\mathbf{x},t) = 0. \tag{13}$$

(f) Now calculate the vector potential $\mathbf{A}(\mathbf{x},t)$ for $t \geq |\mathbf{x}|/c$, including the light front $t = |\mathbf{x}|/c$ itself.

(g) Finally, use the vector potential from part (f) to find the electric and magnetic fields.

3. In class I have discussed electromagnetic energy, momentum, and related quantities for the vacuum. In a uniform linear medium we have similar formulae:

$$\text{power density:} \quad P = \mathbf{J}\cdot\mathbf{E}, \tag{14}$$

$$\text{force density:} \quad \mathbf{f} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}, \tag{15}$$

$$\text{energy density:} \quad u = \tfrac{1}{2}\mathbf{E}\cdot\mathbf{D} + \tfrac{1}{2}\mathbf{H}\cdot\mathbf{B}, \tag{16}$$

$$\text{energy flux density:} \quad \mathbf{S} = \mathbf{E}\times\mathbf{H}, \tag{17}$$

$$\text{momentum density:} \quad \mathbf{g} = \mathbf{D}\times\mathbf{B} = \frac{\epsilon\mu}{c^2}\mathbf{S}, \tag{18}$$

stress tensor: $$T_{ij} \;=\; E_i D_j \;+\; H_i B_j \;-\; \tfrac{1}{2}\delta_{ij}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}). \tag{19}$$

(a) Verify the local energy and momentum conservation laws

$$\frac{\partial u}{\partial t} \;+\; \nabla\cdot\mathbf{S} \;+\; P \;=\; 0, \tag{20}$$
$$\frac{\partial g_i}{\partial t} \;-\; \nabla_j T_{ij} \;+\; f_i \;=\; 0. \tag{21}$$

Hint: for a linear medium $E_i D_j = D_i E_j$ and $H_i B_j = B_i H_j$.

Now consider a linear but non-uniform medium where ϵ and μ vary with $\mathbf{x}$ (but not with time or frequency),

$$\mathbf{D}(\mathbf{x},t) \;=\; \epsilon(\mathbf{x})\epsilon_0\mathbf{E}(\mathbf{x},t), \quad \mathbf{B}(\mathbf{x},t) \;=\; \mu(\mathbf{x})\mu_0\mathbf{H}(\mathbf{x},t), \tag{22}$$

For such a medium, the force density equation (15) has additional terms involving gradients $\nabla\mu$ and $\nabla\epsilon$, while the remaining equations (14) and (16) through (19) remain exactly as for a uniform medium.

(b) Verify the local conservation laws (20) and (21) for a non-uniform medium and a modified force density equation. Also, write down the modified force density equation and explain the physical meaning of the extra terms.