1. This problem is about *birefringence* in anisotropic materials. The dielectric "constant" of anisotropic dielectric is a tensor ϵ_{ij} rather than a scalar, thus

$$D_i \;=\; \epsilon_{ij}\epsilon_0 E_j\,. \tag{1}$$

For simplicity, let's assume that at the optical frequencies $\epsilon_{ij}(\omega)$ is a real symmetric tensor, and that the material in question is non-conducting and non-magnetic, $\sigma = 0$ and $\mu = 1$.

Consider a plane EM wave

$$\mathbf{E}(\mathbf{x},t) \;=\; \vec{\mathcal{E}}\,\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t) \tag{2}$$

propagating through such anisotropic material.

(a) Show that the electric and magnetic fields of this wave obey

$$-\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) \;=\; \omega^2\mu_0\mathbf{D}, \qquad \mathbf{H} \;=\; \frac{\mathbf{k}}{\omega\mu_0}\times\mathbf{E}, \tag{3}$$

where the magnetic field $\mathbf{H}$ and the electric displacement field $\mathbf{D}$ are transverse to the wave direction, but the electric tension field $\mathbf{E}$ is generally not transverse.

(b) A plane EM wave in an isotropic medium has its energy moving in the same direction as the wavefront, *i.e.* the direction $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ of the wave vector. But this is generally not true in an anisotropic medium: Show that for a plane wave with $\mathbf{E} \not\perp \mathbf{k}$, the wave's energy and the wavefront move in somewhat different directions.

In an anisotropic medium, the refraction index $n = c|\mathbf{k}|/\omega$ depends on the direction $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ of the wave vector. Moreover, for a given direction $\mathbf{k}$ the two independent polarizations of the wave generally have different refraction indices $n_1 \neq n_2$.

(c) Use eq. (3) to show that the refraction indices and the polarization vectors of the two independent polarizations obtain from the generalized eigenvalue problem

$$\Big(\epsilon_{ij} \;-\; n^2\big(\delta_{ij} \;-\; \hat{k}_i\hat{k}_j\big)\Big)\mathcal{E}_j \;=\; 0. \tag{4}$$

In particular, the (squares of the) refraction indices obtain as zeros of the determinant

$$\chi(n^2) \;=\; \det\Big(\epsilon_{ij} \;-\; n^2\big(\delta_{ij} \;-\; \hat{k}_i\hat{k}_j\big)\Big). \tag{5}$$

From now on, let's work in a Cartesian coordinate system where the ϵ_{ij} tensor is diagonal, $\epsilon_{ij} = \epsilon_i\delta_{ij}$.

(d) Calculate the determinant (5) in this basis and show that

$$\chi(n^2) \;=\; \sum_{i=1}^{3} \hat{k}_i{}^2\epsilon_i \times \prod_{j\neq i}(n^2 - \epsilon_j). \tag{6}$$

If you get bogged down in algebra, use Mathematica.

(e) Suppose the three eigenvalues of the ϵ_{ij} tensor are different, say $\epsilon_1 > \epsilon_2 > \epsilon_3 > 0$. Show that in this case, the square of one of the refraction indices lies between ϵ_1 and ϵ_2 while the square of the other lies between ϵ_2 and ϵ_3,

$$\epsilon_1 \;\geq\; n_1^2 \;\geq\; \epsilon_2 \;\geq\; n_2^2 \;\geq\; \epsilon_3\,. \tag{7}$$

Moreover, all these inequalities become strict when all 3 of the $\hat{k}_1^2$, $\hat{k}_2^2$, $\hat{k}_3^2$ are positive, *i.e.* when the wave vector $\mathbf{k}$ is not parallel to any principal axis of the ϵ_{ij} tensor. Also, in this case, the $\chi(n^2) = 0$ equation is equivalent to the *Fresnel equation*

$$\sum_{i=1}^{3} \frac{\epsilon_i\hat{k}_i^2}{n^2 - \epsilon_i} \;=\; 0. \tag{8}$$

Now suppose $\epsilon_1 = \epsilon_2 \neq \epsilon_3$; birefringence like this is called *uniaxial*, and the direction of the non-degenerate eigenvector is called the *optical axis*. For the waves traveling in the direction of that optical axis, there is no birefringence — both polarizations have the same $n = \sqrt{\epsilon_1 = \epsilon_2}$.

(f) Check this statement.

(g) Show that for the waves in all other directions $\mathbf{k} \neq \pm\mathbf{m}_3$, there are two independent polarizations with different refraction indices. Specifically:

($\perp$) $\vec{\mathcal{E}}$ is normal to both the optical axis and the wave direction $\hat{\mathbf{k}}$; for this wave, $n = \sqrt{\epsilon_1}$ regardless of the ϵ_3 or the angle θ.

($\|$) $\vec{\mathcal{E}}$ lies in the same plane as $\hat{\mathbf{k}}$ and the optical axis; for this wave,

$$n \;=\; \left(\frac{\sin^2\theta}{\epsilon_3} \,+\, \frac{\cos^2\theta}{\epsilon_1}\right)^{-1/2} \tag{9}$$

where θ is the angle between the wave direction $\mathbf{k}$ and the optical axis.

(h) Finally, show that for the ($\perp$) polarization, the wave's energy moves in the same direction $\hat{\mathbf{k}}$ as the wavefront; but for the ($\|$) polarization, the energy moves in a different direction from $\hat{\mathbf{k}}$. Also, calculate the angle between the directions of the energy's and the wave-front's motion for the ($\|$) polarization.

2. Now consider plasma in a uniform magnetic field $\mathbf{B}$. For simplicity, ignore the ions in the plasma and focus on the effect of the free electrons.

(a) Show that for a radio wave of frequency ω propagating through this plasma, the effective permittivity tensor is

$$\epsilon_{ij} \;=\; \delta_{ij} \;-\; \frac{\omega_p^2}{\omega^2(\omega^2-\Omega^2)}\left(\omega^2\delta_{ij} \;-\; \Omega^2\hat{b}_i\hat{b}_j \;-\; i\omega\Omega\epsilon_{ijk}\hat{b}_k\right) \tag{10}$$

where $\omega_p \;=\; \sqrt{e^2 n_e/\epsilon_0 m_e}$ is the plasma frequency, $\Omega \;=\; (e/m_e)B$ is the cyclotron frequency of an electron in the magnetic field B, and $\hat{\mathbf{b}} = (\hat{b}_x, \hat{b}_y, \hat{b}_z)$ is the unit vector in the magnetic field's direction.

The tensor (10) is complex rather than real, but its matrix is Hermitian, $\epsilon_{ij}^* = \epsilon_{ji}$, so it has real eigenvalues $\epsilon_1, \epsilon_2, \epsilon_3$, although the corresponding eigenvectors $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ are complex rather than real.

(b) Find these eigenvalues and eigenvectors. For simplicity, work in the coordinate system where the z axis points in the direction of the magnetic field, thus $\hat{\mathbf{b}} = (0, 0, 1)$.

(c) Go back to the previous problem and show that for a complex orthonormal basis $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ for vectors' components,

$$\mathbf{m}_i^* \cdot \mathbf{m}_j \;=\; \delta_{ij} \quad \forall i, j = 1, 2, 3, \tag{11}$$

$$\text{any vector } \mathbf{v} \;=\; \sum_i v_i \mathbf{m}_i \quad \text{for } v_i \;=\; m_i^* \cdot \mathbf{v}, \tag{12}$$

eq. (4) becomes

$$\Big(\epsilon_{ij} \,-\, n^2\big(\delta_{ij} \,-\, \hat{k}_i \hat{k}_j^*\big)\Big)\mathcal{E}_j \;=\; 0. \tag{13}$$

Also show that for the $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ being complex eigenvectors of an Hermitian permittivity tensor, the Fresnel equation for the refraction indices[2] becomes

$$\sum_{i=1}^{3} \frac{\epsilon_i \times \big(|\hat{k}_i|^2 = |\mathbf{m}_i^* \cdot \hat{\mathbf{k}}|^2\big)}{n^2 - \epsilon_i} \;=\; 0. \tag{14}$$

Now return to the plasma in a magnetic field, and consider a wave propagating in a direction at angle θ from the direction of $\mathbf{B}$.

(d) Solve the Fresnel equation (14) for the plasma in the the high-frequency limit $\omega \gg \omega_p$.

- For simplicity, you may assume that $\omega \gg \Omega$ as well as $\omega \gg \omega_p$; in this limit, you should get

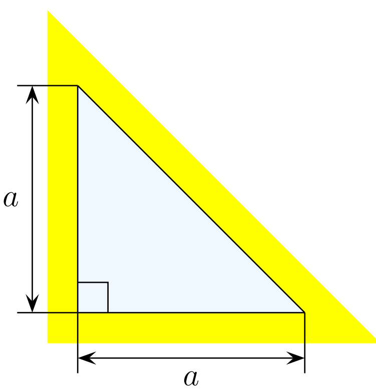
$$n_{1,2}^2 \;=\; 1 \;-\; \frac{\omega_p^2}{\omega^2} \;\pm\; \frac{\omega_p^2 \Omega \cos\theta}{\omega^3} \,+\; O(1/\omega^4). \tag{15}$$

⋆ For extra credit, assume $\omega(\omega - \Omega) \gg \omega_p^2$ but do not assume that $\omega \gg \Omega$.

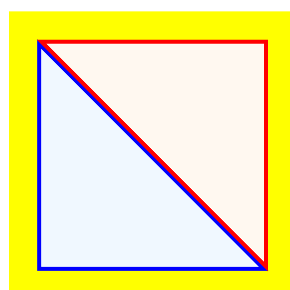
- **Updated 3/25 at 19:30:**

  Problem 3 below is postponed to the next homework assignment.

3. Finally, a problem on a very different subject, namely the TE and TM waves in a triangular waveguide. Specifically, consider a waveguide whose cross-section is a right isosceles triangle



   (a) Let's mirror-reflect the triangle off its long side, so the original triangle plus its image form a square,



(16)

Suppose $\psi(x,y)$ obeys the eigenstate equation $(\nabla^2+\Gamma)^2\psi=0$ and the Neumann or Dirichlet boundary conditions on all 3 sides of the triangle. Let's continue this ψ to the whole square by mirror reflection off the diagonal side,

$$\psi(\mathbf{x}_{\text{mirror}}) \;=\; \pm\psi(x_{\text{orig}}) \tag{17}$$

where the sign is $+$ for the Neumann boundary conditions and $-$ for Dirichlet.

Show that the extended ψ obeys the eigenstate equation and the Neumann/Dirichlet boundary conditions for the whole square. In particular, show that $\psi(x,y)$ and all its derivatives are continuous across the diagonal line separating the original triangle from its image.

(b) Using the results of part (a), find all the eigenstates and the eigenvalues for the original triangle.

(c) Finally, using the results of part (b), describe all the TM and TE waves of the triangular waveguide. For simplicity, assume that the waveguide is filled with air or vacuum and let $\epsilon = \mu = 1$.