⋆ Problems 1 and 2 of this set concern radiation by a compact dipole with a time-dependent dipole moment $\mathbf{p}(t)$. In a pure dipole approximation — meaning, vanishing geometric size but finite $\mathbf{p}(t)$, the electric charge density and the current density of this system become

$$\rho(\mathbf{y},t) \;=\; -\big(\mathbf{p}(t)\cdot\nabla_y\big)\delta^{(3)}(\mathbf{y}),\quad \mathbf{J}(\mathbf{y},t) \;=\; \frac{d\,\mathbf{p}}{dt}\,\delta^{(3)}(\mathbf{y}). \tag{1}$$

1. First, consider a harmonically oscillating dipole moment $\mathbf{p}(t)=\mathbf{p}_0 e^{-i\omega t}$.

   (a) Calculate the electric and the magnetic fields produced by this oscillating dipole and show that

$$\mathbf{H}(\mathbf{x},t) \;=\; \frac{k\omega}{4\pi}\,\frac{e^{ikr-i\omega t}}{r}\left(1\,+\,\frac{i}{kr}\right)(\mathbf{n}\times\mathbf{p}), \tag{2}$$

$$\mathbf{E}(\mathbf{x},t) \;=\; \frac{k\omega Z_0}{4\pi}\,\frac{e^{ikr-i\omega t}}{r}\left[\frac{i}{kr}\left(1+\frac{i}{kr}\right)\big(\mathbf{p}\,-\,3(\mathbf{n}\cdot\mathbf{p})\mathbf{n}\big)\,-\,\mathbf{n}\times(\mathbf{n}\times\mathbf{p})\right]. \tag{3}$$

   (b) Explain the long-distance and the short-distance limits of these EM fields.

   (c) Calculate the Poynting vector of the dipole radiation and show that for a non-linear dipole with $\mathbf{p}\times\mathbf{p}^*\neq 0$ this Poynting vector has a non-radial component in the direction of

$$\mathbf{n}\times\operatorname{Im}\big(\mathbf{p}\times\mathbf{p}^*\big). \tag{4}$$

   (d) Calculate the angular momentum carried away by the EM radiation due to this non-radial component of the Poynting vector, or rather the rate at which this angular momentum is carried away.

   (e) Now consider a specific non-linear dipole moment, namely the classical Rutherford atom with an electron on a circular orbit around the nucleus. This atom looses both energy U and angular momentum $\mathbf{L}$ to the radiation.

Show that the rates at which they are lost are related as

$$\frac{dU}{dt} = \vec{\omega}\cdot\frac{d\mathbf{L}}{dt} \tag{5}$$

and also check that the $d\mathbf{L}/dt$ has precisely opposite direction from the $\mathbf{L}$.

(f) Finally, use the results of part (e) to argue that the electron's orbital plane remains fixed, and if the orbit was initially circular that it would remain circular until the ultimate collapse.

Hint: an orbit in a Coulomb potential is circular if and only if its energy and angular momentum are related as

$$U\mathbf{L}^2 = -\frac{m\alpha^2}{2}, \quad \text{for} \quad \alpha = \frac{e^2}{4\pi\epsilon_0}. \tag{6}$$

2. Now consider a dipole moment with a non-harmonic time dependence but completely general $\mathbf{p}(t)$.

(a) Use Efimenko equations to show that in this case

$$\mathbf{H}(\mathbf{x},t) = \frac{1}{4\pi}\left[\frac{\dot{\mathbf{p}}\times\mathbf{n}}{r^2} + \frac{\ddot{\mathbf{p}}\times\mathbf{n}}{rc}\right]_{\text{ret}}, \tag{7}$$

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\mathbf{n}\cdot\mathbf{p})\mathbf{n}-\mathbf{p}}{r^3} + \frac{3(\mathbf{n}\cdot\dot{\mathbf{p}})\mathbf{n}-\dot{\mathbf{p}}}{r^2c} + \frac{(\mathbf{n}\cdot\ddot{\mathbf{p}})\mathbf{n}-\ddot{\mathbf{p}}}{rc^2}\right]_{\text{ret}}, \tag{8}$$

where the subscript 'ret' means evaluated at the retarded time

$$t_{\text{ret}} = t - \frac{r}{c}. \tag{9}$$

Hint: remember that the retarded time — and hence $\mathbf{p}(t_{\text{ret}})$ and its time derivtives — depend on $\mathbf{x}$.

(b) Calculate the long-distance limit of the Poynting vector and show that the net power emitted by the dipole moment is

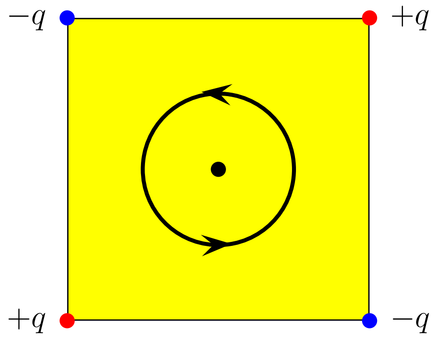
$$P(t) \;=\; \frac{Z_0}{6\pi c^2}\,\|\ddot{\mathbf{p}}_{\rm ret}\|^2. \tag{10}$$

(c) As an example, consider a parallel-plate capacitor with plates of area A at distance b from each other. The capacitor is slowly charged to charge Q_0 and then is allowed to discharge through a resistor R, thus $Q(t) = Q_0 \exp(-t/RC)$.

Find the net energy radiated by the capacitor while it discharges as a fraction of its initial energy $U_0 = Q_0^2/2C$.

(d) Calculate the actual numeric ratio $U_{\rm rad}/U_0$ for $A = 10$ cm $\times$ 10 cm, $b = 1$ mm, and $R = 10\ \Omega$.

3. Finally, a simple problem about the electric quadrupole radiation.
Four charges $\pm q$ sit at corners of a square of size $a \times a$, which rotates with frequency ω around the $\perp$ axis through the square's center.



(a) Find the electric quadrupole moment tensor of this system. With what frequency does it rotate?

(b) Find the angular distribution of the EM power radiated by the rotating quadrupole.

(c) Find the net EM power radiated by the rotating quadrupole.