⋆ The first 3 problems of this set concern emission of photons by atoms and nuclei. After that, there are 2 problems (or rather a reading assignment and a problem) about spherical electromagnetic waves.

1. A quantum hydrogen atom initially in the excited 2p state drops to the ground 1s state while emitting a photon. Calculate the matrix element of the electric dipole operator between these two states and hence the transition rate (in the electric dipole approximation).

   For the sake of definiteness, let the initial 2p state be $|n=2,\ell=1,m=0\rangle$ with the wave function

$$\Psi_{2p}(\mathbf{x}) \;=\; \frac{1}{\sqrt{32\pi a^5}}\, re^{r/2a}\,\cos\theta \tag{1}$$

   while the final 1s state $|n=1,\ell=0,m=0\rangle$ has wavefunction

$$\Psi_{1s}(\mathbf{x}) \;=\; \frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}, \tag{2}$$

   where

$$a \;=\; \frac{4\pi\epsilon_0\,\hbar^2}{e^2 m_e} \;=\; \frac{\hbar}{\alpha m_e c} \;\approx\; 0.53\ \text{Å} \tag{3}$$

   is the Bohr radius. For simplicity, ignore the electron's spin.

2. In a hydrogen-like atom or ion, the closest quantum analogues of circular electron orbits are the states $|n,\ell,m\rangle$ with $m=\ell=n-1$. For $n\gg 1$, the wave function of such a state is strongly peaked in a thin torus around the classical circular orbit. (This is FYI, you do not need to prove this.)

   Show that the only allowed transition from such a state to a lower-energy state is to a similar state $|n',\ell',m'\rangle$ with $m'=\ell'=n'-1$ for $n'=n-1$. Consequently, if an atom is initially in such a state with large n, then it de-excites down to the ground state through a cascade of transitions

$$|n,n-1,n-1\rangle \to |n-1,n-2,n-2\rangle \to |n-2,n-3,n-3\rangle \to \\ \to \cdots \to |3,2,2\rangle \to |2,1,1\rangle \to |1,0,0\rangle\,. \tag{4}$$

This cascade is the quantum analogy of the classical circular orbit spiraling down to the nucleus.

3. A few metastable nuclear isomers have extraordinarily long lifetimes. I have discussed the technetium $\mathrm{Tc}^{99\mathrm{m}}$, $\mathrm{Tc}^{97\mathrm{m}}$, and $\mathrm{Tc}^{95\mathrm{m}}$ isomers in class, but there is also cobalt $\mathrm{Co}^{58\mathrm{m1}}$ (half-life of 9 hours), hafnium $\mathrm{Hf}_{72}^{178\mathrm{m2}}$ (half-life of 31 years), holmium $\mathrm{Ho}_{67}^{166\mathrm{m1}}$ (half-life of 1200 years), and and most remarkably tantalum $\mathrm{Ta}_{73}^{182\mathrm{m}}$ — its half-life is estimated to be so longer than 10^{16} years but nobody have actually seen it decay! Most of these metastable nuclei decays by emission of γ rays or by internal conversion, — the nucleus emits a γ-ray photon but the atomic electrons immediately absorb it and use its energy to kick an electron out. The exception is holmium $\mathrm{Ho}_{67}^{166\mathrm{m1}}$ whose γ-decay rate is so slow that it's pre-empted by the β-decay to erbium Er^{166}.

   The extraordinarily slow γ-decay rate by all these metastable nuclear states are due to high-order multipoles involved in the transitions. Your task is to find which multipole order is responsible for the γ decays (or internal conversions) of cobalt $\mathrm{Co}^{58\mathrm{m1}}$, hafnium $\mathrm{Hf}_{72}^{178\mathrm{m2}}$, and tantalum $\mathrm{Ta}_{73}^{182\mathrm{m}}$.

   Google up the angular momenta and parities of the 3 isomers in questions as well as the angular momenta and parities of the states they decay to. Then use the selection rules to find the lowest multipole order — electric or magnetic — allowed for each transition and hence the power of the small (kR) factor suppressing the transition rate.

4. Next, a reading assignments — §9.6–7 of the Jackson's textbook about the spherical waves, scalar and electromagnetic. I should explain this issue in class, but the textbook goes into more details.

5. Finally, spell out the **E** and **H** fields of the divergent TE and TM spherical waves for $\ell = 1$ and $\ell = 2$.

   Hint: for the dipole $\ell = 1$ waves, you can re-express the $\ell = 1$ spherical harmonics as

$$Y_{1,m}(\theta,\phi) \;=\; \mathbf{n}(\theta,\phi)\cdot\mathbf{d}_m \quad \text{for} \quad d_0 \;=\; \sqrt{\frac{3}{4\pi}}\begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{and} \quad d_{\pm1} \;=\; \sqrt{\frac{3}{8\pi}}\begin{pmatrix}\mp1\\-i\\0\end{pmatrix}. \tag{5}$$

   Likewise, for the qudrupole $\ell = 2$ waves, you can re-express the $\ell = 1$ spherical harmonics as

$$Y_{2,m}(\theta,\phi) \;=\; \mathbf{n}(\theta,\phi)\cdot\overleftrightarrow{Q}_m\cdot\mathbf{n}(\theta,\phi) \tag{6}$$

   for

$$\overleftrightarrow{Q}_0 \;=\; \sqrt{\frac{5}{16\pi}}\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & +2\end{pmatrix}, \tag{7}$$

$$\overleftrightarrow{Q}_{\pm1} \;=\; \sqrt{\frac{15}{32\pi}}\begin{pmatrix}0 & 0 & \mp1\\ 0 & 0 & -i\\ \mp1 & -i & 0\end{pmatrix}, \tag{8}$$

$$\overleftrightarrow{Q}_{\pm2} \;=\; \sqrt{\frac{15}{32\pi}}\begin{pmatrix}+1 & \pm i & 0\\ \pm i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{9}$$