1. Consider a microwave resonator cavity in the shape of a sphere of radius R. The spherical TM waves in this cavity have form:

$$\mathbf{H}(r,\theta,\phi) = H_0\, j_\ell(kr)\, \mathbf{L}Y_{\ell,m}(\theta,\phi), \tag{1.a}$$
$$\mathbf{E}(r,\theta,\phi) = \frac{iZ_0}{k}\,\nabla\times\mathbf{H}(r,\theta,\phi), \tag{1.b}$$

while the spherical TE waves have form

$$\mathbf{E}(r,\theta,\phi) = E_0\, j_\ell(kr)\, \mathbf{L}Y_{\ell,m}(\theta,\phi), \tag{2.a}$$
$$\mathbf{H}(r,\theta,\phi) = \frac{1}{ikZ_0}\,\nabla\times\mathbf{E}(r,\theta,\phi). \tag{2.b}$$

Note: for both kinds of waves, the $j_\ell(kr)$ is the spherical Bessel function that is regular at the center $kr = 0$.

(a) Separate the radial components of the EM fields from their transverse components (in the directions of θ and ϕ) and show that:
for a TM wave,

$$H_r = 0, \tag{3.a}$$
$$\mathbf{H}_t = H_0\, j_\ell(kr)\, \mathbf{L}Y_{\ell,m}(\theta,\phi), \tag{3.b}$$
$$E_r = -Z_0 H_0\, \frac{\ell(\ell+1)}{kr}\, j_\ell(kr)\, Y_{\ell,m}(\theta,\phi), \tag{3.c}$$
$$\mathbf{E}_t = iZ_0 H_0 \left(\frac{j_\ell(kr)}{kr} + j'_\ell(kr)\right) \mathbf{n}(\theta,\phi)\times\mathbf{L}Y_{\ell,m}(\theta,\phi)\,, \tag{3.d}$$

while for a TE wave

$$E_r = 0, \tag{4.a}$$
$$\mathbf{E}_t = E_0\, j_\ell(kr)\, \mathbf{L}Y_{\ell,m}(\theta,\phi), \tag{4.b}$$
$$H_r = \frac{E_0}{Z_0}\,\frac{\ell(\ell+1)}{kr}\, j_\ell(kr)\, Y_{\ell,m}(\theta,\phi), \tag{4.c}$$
$$\mathbf{H}_t = -i\frac{E_0}{Z_0}\left(\frac{j_\ell(kr)}{kr} + j'_\ell(kr)\right) \mathbf{n}(\theta,\phi)\times\mathbf{L}Y_{\ell,m}(\theta,\phi)\,, \tag{4.d}$$

where $j'_\ell(x) = dj_\ell(x)/dx$.

(b) Suppose the surface of the spherical cavity is perfectly conducting. Apply the boundary conditions at that surface to the TE and the TM waves and show that they resonate at frequencies for which

$$\begin{aligned} j_\ell(x) &= 0 \text{ @ } x = kR \quad \text{for a TE}_\ell \text{ wave,} && (5.a)\\ y j_\ell'(y) + j_\ell(y) &= 0 \text{ @ } y = kR \quad \text{for a TM}_\ell \text{ wave.} && (5.b) \end{aligned}$$

In other words, the resonant frequencies are

$$\omega_n(\mathrm{TE}_\ell) = \frac{c}{R} \times x_{\ell,n}\,, \qquad \omega_n(\mathrm{TM}_\ell) = \frac{c}{R} \times y_{\ell,n}\,, \tag{6}$$

where $x_{\ell,n}$ is the n^{th} positive zero of $j_\ell(x)$ while $y_{\ell,n}$ is the n^{th} positive zero of

$$f_\ell(y) = y j_\ell'(y) + j_\ell(y) = \frac{d}{dy}\big(y j_\ell(y)\big). \tag{7}$$

(c) Use Mathematica to find the 4 lowest frequencies numerically (in units of c/R). Also, state which modes these frequencies belong to.

(d) Now suppose the outer wall of the spherical cavity has a small surface resistivity R_s. Calculate the quality factor Q of the spherical resonator for all the modes and show that all the TE modes have

$$Q = \frac{Z_0}{2R_s} \times (x_{\ell,n} = \omega R/c), \tag{8}$$

while the TE modes have

$$Q = \frac{Z_0}{2R_s} \times \left(y_{\ell,n} - \frac{\ell(\ell+1)}{y_{\ell,n}}\right) = \frac{Z_0}{2R_s} \times \left((\omega R/c) - \frac{\ell(\ell+1)}{(\omega R/c)}\right). \tag{9}$$

Math help: spherical Bessel functions obey all kinds of rather obscure identities. In particular, here are a couple of integral identities you need for this problem:

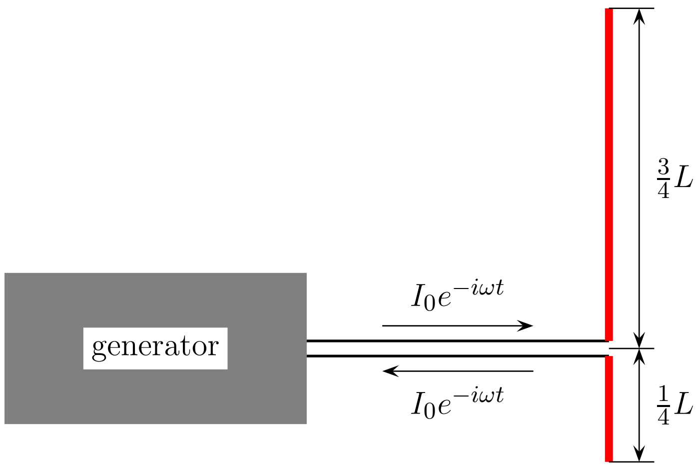
(i) For any $X = x_{\ell,n}$ such that $j_\ell(X) = 0$,

$$\int_0^X dx\, x^2 \left(j_\ell(x)\right)^2 \;=\; \frac{X^3}{2} \times \left(j'_\ell(X)\right)^2. \tag{10}$$

(ii) For any $Y = y_{\ell,n}$ such that $Y j'_\ell(Y) + j_\ell(Y) = 0$,

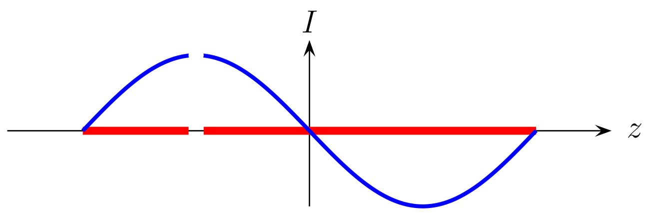
$$\int_0^Y dx\, x^2 \left(j_\ell(x)\right)^2 \;=\; \frac{Y}{2}\left(Y^2 - \ell(\ell+1)\right) \times \left(j_\ell(Y)\right)^2. \tag{11}$$

2. Consider a linear antenna that's precisely one wavelength long $L = \lambda$. The antenna is fed at a point at distance $L/4$ from one end rather than at in the middle,



For simplicity, approximate the current in the antenna by a sine wave with nodes at both ends, thus

$$I(z) \;=\; -I_0 \sin\frac{2\pi z}{L = \lambda} \tag{12}$$



Note: this sine wave is different from the current waves in the center-fed antennas, so the radiation pattern of this antenna is quite different from the $L = \lambda$ center-fed antanna discussed in class.

(a) Calculate the $\mathbf{f}(\mathbf{n})$ for this antenna without using the multipole expansion.

(b) Plot the angular dependence of the power (per solid angle) emitted by the antenna in question in the direction $\mathbf{n}$ as a function of the angle θ between that direction and the antenna's axis.

(c) Calculate the net power emitted by the antenna and hence the antenna's radiation resistance. Note: the integral here requires special functions or numeric integration. Don't try to do it by hand but use Mathematica or equivalent software.

Although the antenna in question is too long to trust the multipole expansion, let's use it anyway and see how far off the mark we would get by using just the leading multipoles. In terms of the multipole expansion,

$$\mathbf{f}(\mathbf{n}) = \sum_{m=0}^{\infty} \mathbf{f}_m(\mathbf{n}), \tag{13}$$

$$\mathbf{f}_m(\mathbf{n}) = \frac{(-ik)^m}{4\pi m!} \iiint\limits_{\text{antenna}} d^3\mathbf{y}\, \mathbf{J}(\mathbf{y})\, (\mathbf{n}\cdot\mathbf{y})^m. \tag{14}$$

(d) Use symmetries of the antenna in question to argue that it nas zero magnetic multipole moments for all ℓ, while the electric multipole moments vanish for all odd ℓ. Thus, the only multipole moments for this antenna are the electric quadrupole, electric 16-pole, electric 64-pole, *etc.*.

To avoid the messy indexologies of the higher multipole moments, it is easier to directly calculate the $\mathbf{f}_m(\mathbf{n})$ for the antenna in question. In light of part (c), the $\mathbf{f}_m$ should vanish for all even $m = 0, 2, 4, 6, \ldots$.

(e) Verify this, then calculate the three leading non-zero terms $\mathbf{f}_m(\mathbf{n})$ for the odd $m = 1, 3, 5$.

(f) Use successive approximations

$$\begin{aligned}
\mathbf{f}_a(\mathbf{n}) &= \mathbf{f}_1(\mathbf{n}),\\
\mathbf{f}_b(\mathbf{n}) &= \mathbf{f}_1(\mathbf{n}) + \mathbf{f}_3(\mathbf{n}),\\
\mathbf{f}_c(\mathbf{n}) &= \mathbf{f}_1(\mathbf{n}) + \mathbf{f}_3(\mathbf{n}) + \mathbf{f}_5(\mathbf{n}),
\end{aligned} \tag{15}$$

to calculate the $dP/d\Omega$ and the net power emitted by the antenna. Compare the angular distributions of the power to the 'exact' result from part (a) — and plot them all on the same graph — and also compare the net power to the result from part (b).