

POLARIZATION AND MAGNETIZATION

Neutral matter is made of atoms and molecules. The polar molecules have built-in electric dipole moments \mathbf{p} ; normally, they are randomly oriented, but in presence of an external electric field this randomness is biased in favor of $\mathbf{p} \uparrow \uparrow \mathbf{E}$, hence non-zero average dipole moment vector $\langle \mathbf{p} \rangle$. The non-polar molecules normally have $\mathbf{p} = 0$, but an external electric field *induces* a non-zero electric dipole moment by perturbing the electron's quantum states. One way or another, matter is chock-full of electric dipoles, and their *macroscopic* effect stems from the dipole moment density

$$\mathbf{P} = \frac{\text{net electric dipole moment}}{\text{volume}} \quad (1)$$

called the *polarization*. For matter in a non-uniform electric field $\mathbf{E}(\mathbf{x})$ — or for a non-uniform matter — the polarization $\mathbf{P}(\mathbf{x})$ varies from place to place on a macroscopic scale, so it acts as a *macroscopic field*.

Likewise, many atoms and molecules have built-in magnetic dipole moments \mathbf{m} , and in presence of an external magnetic field \mathbf{B} the directions of atomic dipoles are biased in favor of $\mathbf{m} \uparrow \uparrow \mathbf{B}$, hence non-zero average magnetic moment vector $\langle \mathbf{m} \rangle$. Also, the external magnetic field can induce a magnetic moment in a molecule that otherwise would not have it. So one way or the other, matter is full of magnetic dipole moments, and their *macroscopic* effect stems from the magnetic moment density

$$\mathbf{M} = \frac{\text{net magnetic dipole moment}}{\text{volume}} \quad (2)$$

called the *magnetization*. And just like the electric polarization $\mathbf{P}(\mathbf{x})$, the magnetization $\mathbf{M}(\mathbf{x})$ could be non-uniform and act as a macroscopic field.

Now consider the electric field $\mathbf{E}(\mathbf{x})$ generated by a given polarization $\mathbf{P}(\mathbf{y})$, or the magnetic field $\mathbf{B}(\mathbf{x})$ generated by a given magnetization $\mathbf{M}(\mathbf{y})$. To be precise, let's focus on the *macroscopic* electric and magnetic fields — that, the fields which would be measured by macroscopic probes much larger than atoms — rather than microscopic fields which rapidly

change from one place inside an atom or molecule to another place inside the same atom or molecule. Technically, the macroscopic fields are *space averages* of the microscopic fields,

$$\begin{aligned}\mathbf{E}^{\text{macro}}(\mathbf{x}) &= \iiint \mathbf{E}^{\text{micro}}(\mathbf{x} + \delta\mathbf{x}) f(\delta\mathbf{x}) d^3\delta\mathbf{x}, \\ \mathbf{B}^{\text{macro}}(\mathbf{x}) &= \iiint \mathbf{B}^{\text{micro}}(\mathbf{x} + \delta\mathbf{x}) f(\delta\mathbf{x}) d^3\delta\mathbf{x},\end{aligned}\tag{3}$$

for some averaging function $f(\delta\mathbf{x})$ such as Gaussian

$$f(\delta\mathbf{x}) = \frac{1}{\pi^{3/2}a^3} \exp(-\delta\mathbf{x}^2/a^2), \quad \int f(\delta\mathbf{x}) d^3\mathbf{x} = 1,\tag{4}$$

with an averaging scale a much larger than the atom's or molecule's size but smaller than any macroscopic probe we may use to measure the electric or magnetic field. The specific form of the averaging function $f(\delta\mathbf{x})$ does not matter as long as it's non-negative, smooth, integrates to 1, and the volume where it's non-negligible is large on atomic scale but small on the macroscopic scale.

The macroscopic potentials $\Phi(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ also obtain from space-averaging with the same measure $f(\delta\mathbf{x})$ as the macroscopic fields (3),

$$\begin{aligned}\Phi^{\text{macro}}(\mathbf{x}) &= \iiint \Phi^{\text{micro}}(\mathbf{x} + \delta\mathbf{x}) f(\delta\mathbf{x}) d^3\delta\mathbf{x}, \\ \mathbf{A}^{\text{macro}}(\mathbf{x}) &= \iiint \mathbf{A}^{\text{micro}}(\mathbf{x} + \delta\mathbf{x}) f(\delta\mathbf{x}) d^3\delta\mathbf{x},\end{aligned}\tag{5}$$

Now that I've reminded you about the macroscopic fields, let's calculate the macroscopic electric field $\mathbf{E}(\mathbf{x})$ generated by a given polarization $\mathbf{P}(\mathbf{y})$. The potential $\Phi(\mathbf{x})$ generated at point \mathbf{x} by a single electric dipole \mathbf{p} located at some other point \mathbf{y} is

$$\Phi(\mathbf{x}) = \frac{\mathbf{p}}{4\pi\epsilon_0} \cdot \left(\frac{\mathbf{n}_{xy}}{r_{xy}^2} = \frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} \right) = \frac{\mathbf{p}}{4\pi\epsilon_0} \cdot \nabla_{\mathbf{x}} \left(\frac{-1}{|\mathbf{x} - \mathbf{y}|} \right),\tag{6}$$

hence the potential of a continuous density $\mathbf{P}(\mathbf{y})$ of dipole moments is

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \iiint d^3\mathbf{y} \mathbf{P}(\mathbf{y}) \cdot \nabla_{\mathbf{x}} \left(\frac{-1}{|\mathbf{x} - \mathbf{y}|} \right) \\
&= \frac{1}{4\pi\epsilon_0} \iiint d^3\mathbf{y} \mathbf{P}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \left(\frac{+1}{|\mathbf{x} - \mathbf{y}|} \right) \\
&\quad \langle\langle \text{integrating by parts} \rangle\rangle \\
&= -\frac{1}{4\pi\epsilon_0} \iiint_{\substack{\text{whole} \\ \text{space}}} d^3\mathbf{y} \frac{(\nabla \cdot \mathbf{P})(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}.
\end{aligned} \tag{7}$$

On the last line here, the electric potential $\Phi(\mathbf{x})$ looks exactly like the Coulomb potential of an effective charge density — called the *bound charge density* —

$$\rho_b(\mathbf{y}) = -\nabla \cdot \mathbf{P}(\mathbf{y}). \tag{8}$$

Consequently, the macroscopic Gauss Law in presence of polarization $\mathbf{P}(\mathbf{x})$ becomes

$$\epsilon_0 \nabla \cdot \mathbf{E}(\mathbf{x}) = \rho_b(\mathbf{x}) = -\nabla \cdot \mathbf{P}(\mathbf{x}), \tag{9}$$

or if we allow for a macroscopic charge density $\rho(\mathbf{x})$ in addition to the polarization $\mathbf{P}(\mathbf{x})$, then

$$\epsilon_0 \nabla \cdot \mathbf{E}(\mathbf{x}) = \rho(\mathbf{x}) - \nabla \cdot \mathbf{P}(\mathbf{x}). \tag{10}$$

A point of terminology: the macroscopic charge density $\rho(\mathbf{x})$ in this formula is often called the *free charge density* to distinguish it from the bound charge density (8). Notation-wise, some books use ρ_f for the free charge density and ρ for the net charge density $\rho_f + \rho_b$, while other books — as well as most of my notes — use ρ for the free charge density and ρ_{net} for the net charge density.

Note: to integrate by parts in eq. (7) without a surface term, we must integrate over the whole space, even if the polarization $\mathbf{P}(\mathbf{x})$ is limited to a finite piece of some dielectric. Consequently, the bound charge $\rho_b = -\nabla \cdot \mathbf{P}$ appears not only in the bulk of the dielectric but

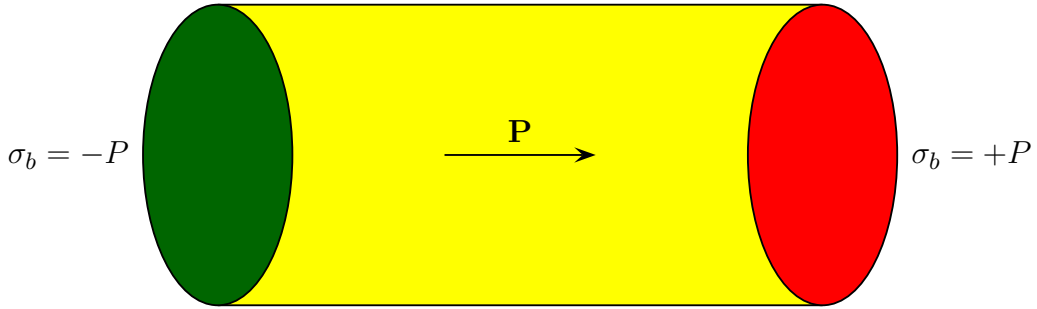
also on its surface, where abrupt cessation of the polarization \mathbf{P} produces a delta-function term

$$\rho_b(\mathbf{x}) \supset \mathbf{P}(\mathbf{x}_{\text{just inside}}) \cdot \mathbf{n}(\mathbf{x}) \delta(x_{\text{normal}}). \quad (11)$$

In other words, at the surface of a dielectric we have a surface density of bound charges

$$\sigma_b = \mathbf{P}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}), \quad (12)$$

for example



Anyway, in the macroscopic Gauss Law equation (10) one usually moves the $\nabla \cdot \mathbf{P}$ term to the left hand side: we define the *electric displacement field* as

$$\mathbf{D}(\mathbf{x}) = \epsilon_0 \mathbf{E}(\mathbf{x}) + \mathbf{P}(\mathbf{x}), \quad (13)$$

then it obeys

$$\nabla \cdot \mathbf{D}(\mathbf{x}) = \rho(\mathbf{x}) \quad (14)$$

where $\rho(\mathbf{x})$ is the free charge density, *i.e.* the the macroscopic charge density unrelated to the polarization.

Note that in presence of dielectrics, the key equations of macroscopic electrostatics — the Gauss law and the zero-curl law — are obeyed by two different electric fields,

$$\nabla \cdot \mathbf{D} = \rho \quad \text{but} \quad \nabla \times \mathbf{E} = 0. \quad (15)$$

To relate the electric tension field \mathbf{E} and the electric displacement field \mathbf{D} to each other, we need the equation of state of the dielectric in question. (Or if we are dealing with several

dielectric materials, we need the equation of state for each dielectric, including the trivial $\mathbf{D} = \epsilon_0 \mathbf{E}$ for the vacuum). In many dielectrics the equation of state is approximately linear,

$$\mathbf{D} = \epsilon_{\text{abs}} \mathbf{E} = \epsilon_{\text{rel}} \epsilon_0 \mathbf{E} \quad (16)$$

where ϵ_{abs} is the (absolute) *permittivity* of the dielectric in question and $\epsilon_{\text{rel}} = \epsilon_{\text{abs}}/\epsilon_0$ is the *relative permittivity* AKA the dielectric constant. A point of notation: depending on a book, ϵ may denote either absolute or relative permittivity, so beware. In particular, in Jackson's textbook $\epsilon = \epsilon_{\text{abs}}$ thus $\mathbf{D} = \epsilon \mathbf{E}$, while in my notes $\epsilon = \epsilon_{\text{rel}}$ thus $\mathbf{D} = \epsilon \epsilon_0 \mathbf{E}$.

In any case, eq. (16) does not govern all the dielectrics. Some dielectrics are linear but unisotropic, so their dielectric constants are tensors ϵ_{ij} rather than scalars, thus

$$D_i = \epsilon_{ij} \epsilon_0 E_j. \quad (17)$$

In some other dielectrics, the polarization \mathbf{P} — and hence the displacement \mathbf{D} — are non-linear functions of the electric tension field \mathbf{E} ; and in some materials they depend not only on the current electric field but also on its prior history (hysteresis). The electrostatic problems in such materials are much more complicated than in the linear dielectrics!

The macroscopic magnetic field $\mathbf{B}(\mathbf{x})$ of a given magnetization $\mathbf{M}(\mathbf{y})$ of some magnetic material obtains in a rather similar way. The vector potential \mathbf{A} at point \mathbf{x} of a single magnetic moment \mathbf{m} located at some other point \mathbf{y} is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 \mathbf{m}}{4\pi} \times \left(\frac{\mathbf{n}_{xy}}{r_{xy}^2} = \frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} \right) = \frac{\mu_0 \mathbf{m}}{4\pi} \times \nabla_{\mathbf{x}} \left(\frac{-1}{|\mathbf{x} - \mathbf{y}|} \right), \quad (18)$$

hence the vector potential of a continuous density $\mathbf{M}(\mathbf{y})$ of magnetic moments is

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \iiint d^3\mathbf{y} \mathbf{M}(\mathbf{y}) \times \nabla_{\mathbf{x}} \left(\frac{-1}{|\mathbf{x} - \mathbf{y}|} \right) \\ &= \frac{\mu_0}{4\pi} \iiint d^3\mathbf{y} \mathbf{M}(\mathbf{y}) \times \nabla_{\mathbf{y}} \left(\frac{+1}{|\mathbf{x} - \mathbf{y}|} \right) \\ &\quad \langle\langle \text{integrating by parts} \rangle\rangle \\ &= -\frac{\mu_0}{4\pi} \iiint_{\text{whole space}} \frac{\mathbf{M}(\mathbf{y}) \times \overleftarrow{\nabla}_{\mathbf{y}}}{|\mathbf{x} - \mathbf{y}|} = +\frac{\mu_0}{4\pi} \iiint_{\text{whole space}} \frac{(\nabla \times \mathbf{M})(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}. \end{aligned} \quad (19)$$

The last line here looks exactly like the vector potential of an effective current density —

called the bound current density —

$$\mathbf{J}_b(\mathbf{y}) = \nabla \times \mathbf{M}(\mathbf{y}), \quad (20)$$

so the macroscopic Ampere Law becomes

$$\nabla \times \mathbf{B}(\mathbf{x}) = \mu_0 \nabla \times \mathbf{M}(\mathbf{x}). \quad (21)$$

Or if we have both the magnetization and the ordinary macroscopic *conduction current* $\mathbf{J}(\mathbf{x})$, then

$$\nabla \times \mathbf{B}(\mathbf{x}) = \mu_0 \nabla \times \mathbf{M}(\mathbf{x}) + \mu_0 \mathbf{J}(\mathbf{x}). \quad (22)$$

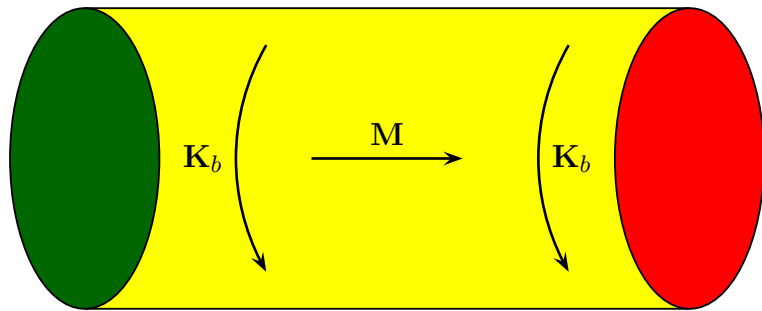
Just as in the dielectric case, integration by parts in eq. (19) without the surface term means integrating over the whole space, even if the magnetization is limited to a finite piece of magnetic material. Consequently, the abrupt cessation of magnetization at the boundary of the magnetic material gives rise to the delta-function term in the bound current density (20):

$$\mathbf{J}_b(\mathbf{x}) \supset -\delta(x_{\text{normal}})\mathbf{n}(\mathbf{x}) \times \mathbf{M}(\mathbf{x}) \quad (23)$$

In other words, there is a surface density of the bound current

$$\mathbf{K}_b(\mathbf{x}) = -\mathbf{n}(\mathbf{x}) \times \mathbf{M}(\mathbf{x}) = +\mathbf{M}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}). \quad (24)$$

For example:



Similar to the macroscopic Gauss Law, the macroscopic Ampere Law (22) is usually written with the $\nabla \times \mathbf{M}$ term on the other side of the equation. That is, in addition to the

macroscopic *magnetic induction field* $\mathbf{B}(\mathbf{x})$, we define the *magnetic intensity field*

$$\mathbf{H}(\mathbf{x}) = \frac{1}{\mu_0} \mathbf{B}(\mathbf{x}) - \mathbf{M}(\mathbf{x}), \quad (25)$$

then for this \mathbf{H} field, the Ampere's Law become simply

$$\nabla \times \mathbf{H}(\mathbf{x}) = \mathbf{J}(\mathbf{x}) \quad (26)$$

where $\mathbf{J}(\mathbf{x})$ is just the macroscopic conduction current, unrelated to the bound currents due to magnetization.

Note that in the macroscopic magnetostatics — just like in the macroscopic electrostatics — the two key equations are obeyed by two different macroscopic fields:

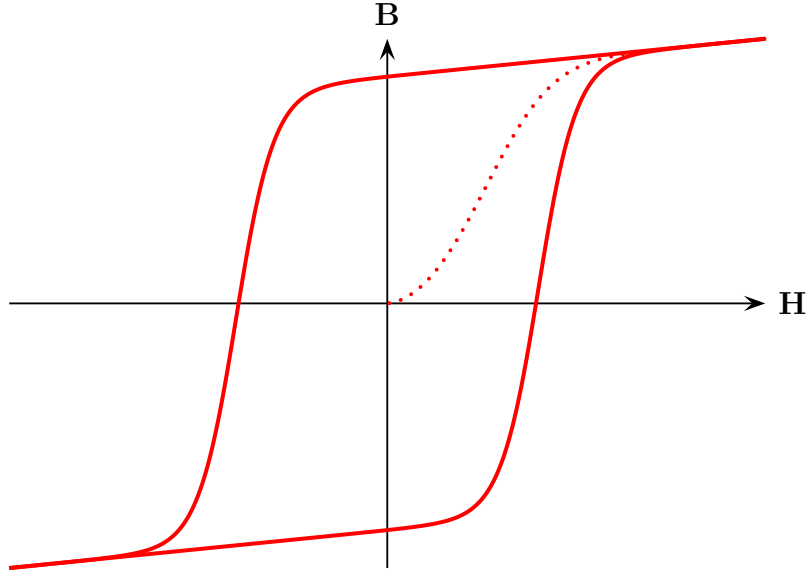
$$\nabla \cdot \mathbf{B} = 0 \quad \text{but} \quad \nabla \times \mathbf{H} = \mathbf{J}. \quad (27)$$

Thus, we need a relation between the magnetic induction field \mathbf{B} and the magnetic intensity field \mathbf{H} — the equation(s) of state for the magnetic material(s) in question. For the diamagnetic and paramagnetic materials, the equation of state is approximately linear,

$$\mathbf{B} = \mu\mu_0\mathbf{H} \quad (28)$$

where μ is the relative permeability of the material in question. ($\mu = (1 - \text{small})$ for the diamagnetics and $\mu = (1 + \text{small})$ for the paramagnetics.) For very soft ferromagnetics like pure iron, the equation of state is also approximately linear in weak magnetic fields, but with $\mu \gg 1$. But for other ferromagnetics, the relation between the \mathbf{H} and the \mathbf{B} fields is non-linear and depends on the prior history of the magnetic field. The best way to describe

such a relation is by drawing the hysteresis loop like



Equations of state and the differential conditions like (15) or (27) are good for the bulk of a dielectric or a magnetic material. But macroscopically, the boundaries between different materials (or a material and a vacuum) appear discontinuous, so what happens at such a boundary?

Mathematically, if a field is discontinuous across a boundary, then its normal derivative has a delta-function spike

$$\frac{\partial f(\mathbf{x})}{\partial x_{\text{normal}}} = \text{disc}(f) \times \delta(x_{\text{normal}}) + \text{finite} \quad (29)$$

while the derivatives in the tangential directions are finite. For a vector field $\mathbf{V}(\mathbf{x})$, this means

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \text{disc}(\mathbf{V}_{\text{normal}}) \delta(x_{\text{normal}}) + \text{finite}, \\ \nabla \times \mathbf{V} &= \mathbf{n} \times \text{disc}(\mathbf{V}_{\text{tangential}}) \delta(x_{\text{normal}}) + \text{finite}. \end{aligned} \quad (30)$$

In particular, for the electric \mathbf{E} and \mathbf{D} fields at the edge of a dielectric, this means

$$\nabla \times \mathbf{E} = 0 \implies \text{disc}(\mathbf{E}_{\text{tangential}}) = 0, \quad (31)$$

and if there are no free electric charges at the surface, then also

$$\nabla \cdot \mathbf{D} = \rho = \text{finite} \implies \text{disc}(\mathbf{D}_{\text{normal}}) = 0. \quad (32)$$

In other words:

- The \mathbf{E}^{\parallel} and the \mathbf{D}^{\perp} fields must be continuous across the boundary.
- But the \mathbf{E}^{\perp} and the \mathbf{D}^{\parallel} fields may be discontinuous.

For the outer boundary of a *linear* dielectric — or for a boundary between two linear dielectrics — these conditions become

$$\mathbf{E}_{\text{side 1}}^{\parallel} = \mathbf{E}_{\text{side 2}}^{\parallel} \quad \text{but} \quad \epsilon_1 \mathbf{E}_{\text{side 1}}^{\perp} = \epsilon_2 \mathbf{E}_{\text{side 2}}^{\perp}. \quad (33)$$

Or in terms of the electric potential $\Phi(\mathbf{x})$, the potential itself is continuous across the boundary, but its normal derivative changes by the ϵ_1/ϵ_2 factor, thus

$$\Phi(\mathbf{x}, \text{side\#1}) = \Phi(\mathbf{x}, \text{side\#2}) \quad \text{but} \quad \epsilon_1 \times \left. \frac{\partial \Phi}{\partial x_{\text{normal}}} \right|_{\text{side\#1}} = \epsilon_2 \times \left. \frac{\partial \Phi}{\partial x_{\text{normal}}} \right|_{\text{side\#2}}. \quad (34)$$

Likewise, for the magnetic fields at the boundary of magnetic materials eqs. (30) tells us that

$$\nabla \cdot \mathbf{B} = 0 \implies \text{disc}(\mathbf{B}_{\text{normal}}) = 0, \quad (35)$$

and if there are no surface conduction currents, then also

$$\nabla \times \mathbf{H} = \mathbf{J} = \text{finite} \implies \text{disc}(\mathbf{H}_{\text{tangent}}) = 0. \quad (36)$$

In other words:

- The \mathbf{H}^{\parallel} and the \mathbf{B}^{\perp} fields must be continuous across the boundary.
- But the \mathbf{H}^{\perp} and the \mathbf{B}^{\parallel} fields may be discontinuous.

For the outer boundary of a *linear* magnetic material — or for a boundary between two such materials — these conditions become

$$\mathbf{H}_{\text{side 1}}^{\parallel} = \mathbf{H}_{\text{side 2}}^{\parallel} \quad \text{but} \quad \mu_1 \mathbf{H}_{\text{side 1}}^{\perp} = \mu_2 \mathbf{H}_{\text{side 2}}^{\perp}, \quad (37)$$

or equivalently

$$\mathbf{B}_{\text{side 1}}^{\perp} = \mathbf{B}_{\text{side 2}}^{\perp} \quad \text{but} \quad \frac{1}{\mu_1} \mathbf{B}_{\text{side 1}}^{\parallel} = \frac{1}{\mu_2} \mathbf{B}_{\text{side 2}}^{\parallel}. \quad (38)$$

Note that the boundary conditions for the \mathbf{H} and \mathbf{B} fields at the surface of a magnetic material look completely similar to the boundary conditions for the \mathbf{E} and \mathbf{D} fields at the surface of a dielectric. However, there are differences affecting the *applications* of these boundary conditions.

- Most dielectric materials are linear. On the other hand, most ferromagnetic materials are non-linear. In the extreme case, the permanent magnets are rather common while the permanently polarized dielectrics are fairly rare.
- The boundary conditions for the electric fields can be easily rephrased in terms of the electric potential $\Phi(\mathbf{x})$. Combining these conditions with the appropriate Laplace or Poisson equations for the $\Phi(\mathbf{x})$ in the bulk of a dielectric or in the vacuum outside it, we get a mathematical problem which may be solved by the usual tools of electrostatics: mirror charges, separation of variables, Green's functions, multipole expansion, *etc.*, *etc.* On other hand, rephrasing the boundary conditions for the magnetic fields in terms of the vector potential $\mathbf{A}(\mathbf{x})$ is not so easy, and the resulting mathematical problem is harder than in the electric case.

EXAMPLE OF A DIELECTRIC BOUNDARY PROBLEM

Consider a solid ball of radius R made from some uniform dielectric of relative permittivity ϵ and surrounded by vacuum. There are no free charges anywhere in the system except infinitely far away, and those far-away charges create a uniform external electric field \mathbf{E}_0 . That is, the electric field $\mathbf{E}(\mathbf{x})$ asymptotes to uniform \mathbf{E}_0 for $|\mathbf{x}| \rightarrow \infty$.

Inside the ball, the Gauss Law $\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \epsilon_0 \mathbf{E}) = \rho = 0$ leads to $\nabla \cdot \mathbf{E} = 0$ (since ϵ is constant inside the ball) and hence to the Laplace equation $\nabla^2 \Phi = 0$ for the potential.

Outside the ball, the potential also obeys the Laplace equation. However, on the surface of the ball Φ obeys

$$\nabla^2\Phi(r, \theta, \phi) = \delta(r - R) \times \text{some_function_of}(\theta, \phi) \neq 0 \quad (39)$$

rather than $\nabla^2\Phi = 0$. Thanks to the delta-function on the RHS here, the potential itself is continuous across the surface, but its radial derivative is discontinuous according to eq. (34).

To solve the Laplace equation for the Φ inside and outside the ball, let's separate the variables in spherical coordinates (r, θ, ϕ) . Thanks to the axial symmetry of the system, Φ depends only on the r and θ coordinates but not the ϕ coordinate. Consequently, inside the ball

$$\Phi_{\text{inside}}(r, \theta) = \sum_{\ell=0}^{\infty} A_{\ell} \times r^{\ell} \times P_{\ell}(\cos \theta) \quad (40)$$

for some constants A_{ℓ} , while outside the ball

$$\Phi_{\text{inside}}(r, \theta) = \sum_{\ell=0}^{\infty} \left(B_{\ell} \times r^{\ell} + \frac{C_{\ell}}{r^{\ell+1}} \right) \times P_{\ell}(\cos \theta) \quad (41)$$

for some other constants B_{ℓ} and C_{ℓ} . The asymptotic behavior of the potential at $r \rightarrow \infty$ follows from the positive powers of r and hence the B_{ℓ} coefficients. Matching this behavior to the given asymptotics

$$\Phi_{\text{far away}} = -E_0 x_3 = -E_0 \times r \times \cos \theta = -E_0 \times r^1 \times P_1(\cos \theta) \quad (42)$$

tells us that $B_1 = -E_0$ while all other $B_{\ell} = 0$.

To find the A_{ℓ} and the C_{ℓ} constants, we use the boundary conditions (34) at $R = 0$. Matching the potentials (40) and (41) at $r = R$ but all θ means matching the coefficients of $P_{\ell}(\cos \theta)$ in both series at $r = R$, thus

$$A_{\ell} \times R^{\ell} = B_{\ell} \times R^{\ell} + \frac{C_{\ell}}{R^{\ell+1}}. \quad (43)$$

Likewise, matching $\epsilon \times (\partial\Phi^{\text{inside}}/\partial r) = (\partial\Phi^{\text{outside}}/\partial r)$ at $r = R$ and any θ leads to

$$\epsilon \times \ell A_\ell R^{\ell-1} = \ell B_\ell R^{\ell-1} - (\ell+1) \frac{C_\ell}{R^{\ell+2}}. \quad (44)$$

Solving these linear equations for the A_ℓ and C_ℓ constants in terms of the B_ℓ gives us

$$A_\ell = \frac{2\ell+1}{(\epsilon+1)\ell+1} \times B_\ell, \quad C_\ell = -\frac{\ell(\epsilon-1)}{(\epsilon+1)\ell+1} R^{2\ell+1} \times B_\ell. \quad (45)$$

In light of the asymptotic conditions at $r \rightarrow \infty$, this means

$$A_\ell = B_\ell = C_\ell = 0 \quad \text{for any } \ell \neq 1 \quad (46)$$

while

$$B_1 = -E_0, \quad A_1 = -\frac{3}{\epsilon+2} E_0, \quad C_1 = +\frac{\epsilon-1}{\epsilon+2} R^3 E_0. \quad (47)$$

Therefore,

$$\Phi_{\text{inside}} = -\frac{3}{\epsilon+2} E_0 r \cos \theta \quad (48)$$

while

$$\Phi_{\text{outside}} = -E_0 r \cos \theta + \frac{\epsilon-1}{\epsilon+2} R^3 \frac{\cos \theta}{r^2}. \quad (49)$$

In other words, the electric field inside the dielectric ball is *uniform*

$$\mathbf{E}_{\text{inside}} = +\frac{3}{\epsilon+2} \mathbf{E}_0 \quad (50)$$

while outside the ball we have a superposition of the external field \mathbf{E}_0 and a pure dipole field of dipole moment

$$\mathbf{p} = 4\pi\epsilon_0 \frac{\epsilon-1}{\epsilon+2} R^3 \mathbf{E}_0. \quad (51)$$

Scalar Magnetic Potential

Many boundary problems for the magnetic fields can be solved using the scalar magnetic potential $\Psi(\mathbf{x})$. (In Jackson's textbook it's called $\Phi_M(\mathbf{x})$.) Unlike the vector magnetic potential $\mathbf{A}(\mathbf{x})$ which can be defined in any situation involving magnetic fields, static or time-dependent, the scalar magnetic potential $\Psi(\mathbf{x})$ is limited to static problems without any macroscopic currents \mathbf{J} , just the permanent magnets and perhaps other magnetic materials. In such a situation, the macroscopic Ampere law $\nabla \times \mathbf{H} = \mathbf{J}$ becomes simply $\nabla \times \mathbf{H} = 0$, and if this equations holds everywhere, then the magnetic intensity field $\mathbf{H}(\mathbf{x})$ must be a gradient of some scalar field. So by analogy with the electrostatic material, let

$$\mathbf{H}(\mathbf{x}) = -\nabla\Psi(\mathbf{x}). \quad (52)$$

Now consider the magnetic Gauss law $\nabla \cdot \mathbf{B} = 0$. In terms of the \mathbf{H} field, this means

$$\nabla \cdot \mathbf{H} = \frac{1}{\mu_0} \nabla \cdot \mathbf{B} - \nabla \cdot \mathbf{M} = -\nabla \cdot \mathbf{M} \quad (53)$$

and hence

$$\nabla^2\Psi(\mathbf{x}) = +\nabla \cdot \mathbf{M}(\mathbf{x}). \quad (54)$$

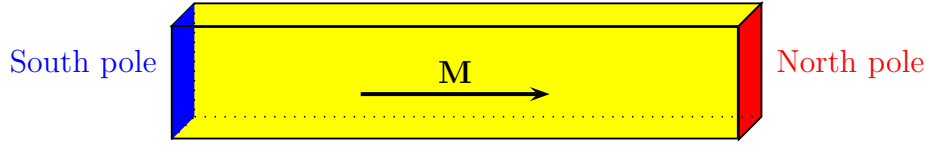
This is the Poisson equation for the scalar magnetic potential where $-\nabla \cdot \mathbf{M}(\mathbf{x})$ acts as a charge density. By analogy with the Coulomb potential, the formal solution to this equation is

$$\Psi(\mathbf{x}) = -\iiint \frac{(\nabla \cdot \mathbf{M})(\mathbf{y}) d^3\mathbf{y}}{4\pi|\mathbf{x} - \mathbf{y}|}. \quad (55)$$

But please note that the divergence $\nabla \cdot \mathbf{M}$ includes the delta-function spike at the surface of a magnet due to abrupt cessation of the magnetization $\mathbf{M}(\mathbf{x})$, so eq. (55) should better be written as

$$\Psi(\mathbf{x}) = -\iiint_{\text{inside of a magnet}} \frac{(\nabla \cdot \mathbf{M})(\mathbf{y}) d^3\mathbf{y}}{4\pi|\mathbf{x} - \mathbf{y}|} + \iint_{\text{magnet's surface}} \frac{\mathbf{M}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) d^2\mathbf{y}}{4\pi|\mathbf{x} - \mathbf{y}|}. \quad (56)$$

For an example, consider a permanent magnet shaped like a long thin bar:



Assume the magnetization \mathbf{M} is uniform inside the magnet and its direction is parallel to the magnet's long dimension. In this case, $\nabla \cdot \mathbf{M} = 0$ inside the magnet, so only the surface integral in eq. (56) contributes to the $\Psi(\mathbf{x})$ and hence to the magnetic field. As to the magnet's surface, $\mathbf{M} \cdot \mathbf{n} = +M$ at the North pole, $\mathbf{M} \cdot \mathbf{n} = -M$ at the South pole, and $\mathbf{M} \cdot \mathbf{n} = 0$ along the long sides of the magnet. Consequently, only the poles of the magnet contribute to the surface integral, which means

$$\Psi(\mathbf{x}) = \frac{Ma}{4\pi} \left(\frac{1}{|\mathbf{x} - \mathbf{y}_{N,\text{pole}}|} - \frac{1}{|\mathbf{x} - \mathbf{y}_{S,\text{pole}}|} \right) \quad (57)$$

where a is the cross-sectional area of the magnet, and hence

$$\mathbf{H}(\mathbf{x}) = \frac{Ma}{4\pi} \left(\frac{\mathbf{x} - \mathbf{y}_N}{|\mathbf{x} - \mathbf{y}_N|^3} - \frac{\mathbf{x} - \mathbf{y}_S}{|\mathbf{x} - \mathbf{y}_S|^3} \right). \quad (58)$$

Thus, the magnetic field of a long thin bar magnet looks just like the electric field of a physical dipole, with a positive charge at the North pole and the negative charge at the South pole. Very far from the bar magnet, we may approximate this field as a pure dipole field

$$\mathbf{H}(\mathbf{x}) \approx \frac{3(\mathbf{m} \cdot \mathbf{n})\mathbf{n} - \mathbf{m}}{4\pi r^3} \quad (59)$$

of net magnetic moment

$$\mathbf{m} = La\mathbf{M} = (\text{magnet's volume}) \mathbf{M}. \quad (60)$$

Closer to the magnet — but not too close to either pole — we should use eq. (58). This formula is valid as long as the distance to the nearest pole is much longer than the magnet's

width; note width rather than length! In particular, for a long thin magnet, eq. (58) is valid inside the bar magnet itself, where it yields an \mathbf{H} field pointing in the opposite direction from the magnetization \mathbf{M} . However, inside the magnet — and not too close to the poles —

$$|\mathbf{x} - \mathbf{y}_N| \sim |\mathbf{x} - \mathbf{y}_S| \sim L \implies H \sim \frac{MA}{L^2} \ll M, \quad (61)$$

so the $\mathbf{B} = \mu_0(\mathbf{M} + \mathbf{H})$ field inside the magnet points in the same direction as the magnetization \mathbf{M} despite the \mathbf{H} field pointing in the opposite direction.

For a more interesting example, consider a permanent magnet in the shape of a solid ball. Again, we assume uniform magnetization \mathbf{M} inside this magnet, so only the surface integral term in eq. (56) contributes to the scalar magnetic potential $\Psi(\mathbf{x})$. But instead of evaluating this surface integral, let's use a different method for solving the Poisson equation

$$\nabla^2 \Psi(\mathbf{x}) = \nabla \cdot \mathbf{M}(\mathbf{x}) = -M \cos \theta \delta(r - R) \quad [\text{in spherical coordinates } (r, \theta, \phi)]. \quad (62)$$

Separating the variables, we note that the angular dependence of the RHS here has the form of the spherical harmonic $Y_{1,0}(\theta, \phi) \propto \cos \theta$, so the scalar potential Ψ should have similar angular dependence,

$$\Psi(r, \theta, \phi) = f(r) \times \cos \theta \quad (63)$$

for some radial function $f(r)$. In terms of this function, the Poisson equation becomes

$$f''(r) + \frac{2}{r} f'(r) - \frac{\ell(\ell+1)}{r^2} f(r) = -M \delta(r - R). \quad (64)$$

Solving this equation subject to boundary conditions of finite Ψ at both $r = 0$ and $r = \infty$, we have

$$\begin{aligned} f(r) &= A \times r \quad \text{for } r < R, \\ f(r) &= \frac{B}{r^2} \quad \text{for } r > R, \end{aligned} \quad (65)$$

where A and B are some constants which solve the conditions

$$\text{disc}(f) = 0, \quad \text{disc}(f') = -M \quad @r = R, \quad (66)$$

which amount to

$$\frac{B}{R^2} - A \times R = 0 \quad \text{but} \quad \frac{-2B}{R^3} - A = -M. \quad (67)$$

Solving these linear equations gives us

$$A = \frac{M}{3}, \quad B = \frac{MR^3}{3}. \quad (68)$$

and hence

$$\Psi_{\text{inside}} = +\frac{M}{3} r \cos \theta, \quad \Psi_{\text{outside}} = \frac{MR^3}{3} \frac{\cos \theta}{r^2}. \quad (69)$$

Consequently, inside the magnet the magnetic intensity field is *uniform*

$$\mathbf{H} = -\nabla \Psi = -\frac{\mathbf{M}}{3} \quad (70)$$

and hence the magnetic induction field

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \frac{2\mu_0}{3} \mathbf{M}. \quad (71)$$

And outside the magnet we have a pure dipole field

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = -\nabla \Psi = \frac{3(\mathbf{m} \cdot \mathbf{n})\mathbf{n} - \mathbf{m}}{4\pi r^3} \quad (72)$$

for the net magnetic moment

$$\mathbf{m} = 4\pi \frac{MR^3}{3} = (\text{magnet's volume}) \mathbf{M}. \quad (73)$$

Besides the permanent magnets, the scalar magnetic potential $\Psi(\mathbf{x})$ is very useful for calculating the magnetic fields in and around linear magnetic materials subject to external fields. In terms of Ψ , such problems are mathematically similar to the dielectric boundary problems in terms of the electric potential $\Phi(\mathbf{x})$, so one may employ all the usual electrostatic techniques. Your homework includes reading in Jackson's textbook (§5.12) about using the scalar magnetic potential to understand magnetic shielding by a spherical shell of a high-permeability material.

In presence of macroscopic currents, the scalar magnetic potential has a much more limited use. Obviously, we cannot possibly use $\Psi(\mathbf{x})$ inside a conductor where $\nabla \times \mathbf{H} = \mathbf{J} \neq 0$. But outside the conductors — which is most of the space when the current flows in thin wires — we have $\nabla \times \mathbf{H} = 0$, so *locally* $\mathbf{H}(\mathbf{x})$ is a gradient of some scalar. But *globally*, any such scalar $-\Psi(\mathbf{x})$ would be a multivalued function of \mathbf{x} .

For example, consider the magnetic field of a long thin wire. In cylindrical coordinates (z, s, ϕ) ,

$$\mathbf{H}(z, s, \phi) = \frac{I}{2\pi} \frac{\mathbf{n}_\phi}{s} = \frac{I}{2\pi} \nabla \phi, \quad (74)$$

which obviously corresponds to

$$\Psi(z, s, \phi) = -\frac{I}{2\pi} \phi. \quad (75)$$

But the polar angle ϕ is a multivalued coordinate because $\phi + 2\pi \times \text{any integer}$ corresponds to exactly the same direction as ϕ . Moreover, if we track the ϕ coordinate along some closed loop circling around the z axis and make sure ϕ changes in a continuous fashion, then by the end of the loop ϕ does not come back to its starting value but changes by 2π . (Or -2π , depending on the direction of the loop.) Technically, we can make ϕ single-valued by shoehorning it into the interval $0 \leq \phi < 2\pi$, but then we would sacrifice the continuity, and a smooth motion in space might cause a sudden jump of ϕ from 2π to zero or vice versa, and that would introduce all kinds of spurious delta-functions into derivatives. To avoid such spurious delta-functions, we have to treat ϕ as multi-valued: the values of this coordinate are not numbers but rather numbers modulo 2π .

Consequently, the scalar magnetic potential (75) is multivalued: At any point \mathbf{x} , the value of $\Psi(\mathbf{x})$ is a number modulo I rather than a regular number. This is general behavior of Ψ in presence of any current-carrying wires, straight or curved: $\Psi(\mathbf{x})$ is multivalued, and if we follow any of its values as \mathbf{x} moves around the wire in a closed loop, then

$$\Psi(\text{end of the loop}) - \Psi(\text{beginning of the loop}) = \mp I \quad (76)$$

even though the loop begins and ends at the same point \mathbf{x} . (The sign of $\mp I$ on the RHS here depends on the sense of the loop, counterclockwise versus clockwise.) Indeed by the

Ampere's Law

$$\Delta\Psi = \oint d\Psi = \oint \nabla\Psi(\mathbf{x}) \cdot d\mathbf{x} = - \oint \mathbf{H} \cdot d\mathbf{x} = -I[\text{through the loop}]. \quad (77)$$

In case of the electric potential $\Phi(\mathbf{x})$, such multivaluedness would not be allowed since voltages are physically measurable, and the voltage between any point and itself must be zero. But unlike the electric potential Φ , the magnetic scalar potential Ψ is not physically measurable; indeed, it does not have any direct physical meaning, it's just a tool for calculating the magnetic intensity field $\mathbf{H}(\mathbf{x})$. So there is no harm in $\Psi(\mathbf{x})$ being multivalued, as long as its gradient $\nabla\Psi = -\mathbf{H}$ is single-valued.

In the homework, I let you work out another example, namely a closed wire loop \mathcal{L} (of any shape, as long as it's closed) carrying current I . Your task is to show that for any such loop

$$\Psi(\mathbf{x}) = \frac{I}{4\pi} \Omega(\mathbf{x}) \quad (78)$$

where $\Omega(\mathbf{x})$ is the solid angle spanned by the cone through the loop \mathcal{L} with a vertex at \mathbf{x} . You shall also see that the properly defined $\Omega(\mathbf{x})$ is multivalued, with $\Omega + 4\pi \times$ an integer having the same physical meaning as Ω .