Symmetries of Mechanics and Electromagnetism

Rotations

In Physics, a vector is more than just an array of 3 variables, it should also transform like a vector under space rotations. Indeed, consider a rotation through some angle ϕ around some axis the coordinate origin. Under such rotation, a point A with coordinates (x_1, x_2, x_3) moves to point A' with coordinates (x'_1, x'_2, x'_3), specifically

$$x'_i \;=\; R_{ij}x_j \quad \text{or in matrix notations} \quad \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{1}$$

for

$$R_{ij} \;=\; \cos(\phi)\delta_{ij} \;+\; \sin(\phi)\epsilon_{ijk}n_k \;+\; (1-\cos(\phi))n_i n_j \tag{2}$$

where $\mathbf{n}$ is a unit vector in the direction of the rotation axis. For example, for a rotation around the z axis $\mathbf{n} = (0, 0, 1)$ and hence

$$\|R_{ij}\| \;=\; \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

A variable — or an array of variables — can be called a scalar, a vector, a tensor, *etc.*, *only* if transforms in a right way under rotations (2):

- A scalar must be invariant under all rotations, $S' = S$.
- The components of a vector must transform exactly like the coordinates of a point: $V'_i = R_{ij}V_j$, for the same R_{ij} as $x'_i = R_{ij}x_j$.
- The components of a two-index tensor must transform according to $T'_{ij} = R_{ik}R_{j\ell}T_{k\ell}$ — a rotation matrix R acting on each index of T.
- Similarly, for a 3–index tensor we should have $T'_{ijk} = R_{i\ell}R_{jm}R_{kn}T_{\ell mn}$, *etc.*, *etc.*

As a matrix, any rotation matrix $R(\phi, \mathbf{n})$ is a *special orthogonal* 3×3 *matrix,* meaning that $R^\top R = 1_{3\times 3}$ and $\det(R) = +1$. Such matrices form a group called $SO(3)$, and every matrix in this group is a rotation matrix (2) for some axis and some angle.

Physically, orthogonality of the rotation matrices — $R^\top R = 1$, or in index notations $R_{ij}R_{ik} = \delta_{jk}$ — makes a dot product of two vectors a scalar, $\vec{a} \cdot \vec{b} \to \vec{a}' \cdot \vec{b}' = \vec{a} \cdot \vec{b}$. Indeed,

$$\vec{a}' \cdot \vec{b}' \;=\; a'_i b'_i \;=\; R_{ij} a_j R_{ik} b_k \;=\; \delta_{jk} a_j b_k \;=\; \vec{a} \cdot \vec{b}. \tag{4}$$

Also, orthogonality combined with $\det(R) = 1$ makes sure that a cross product of two vectors transforms like a vector. Indeed, for any 3×3 matrix

$$\epsilon_{ijk} R_{i\ell} R_{jm} R_{kn} \;=\; \det(R) \times \epsilon_{\ell mn}\,, \tag{5}$$

hence for an $SO(3)$ matrix

$$\begin{aligned} \epsilon_{ijk} R_{j\ell} R_{km} \;&=\; \epsilon_{njk} R_{j\ell} R_{km} \times \big(\delta_{ni} = R_{np} R_{ip}\big) \;=\; \epsilon_{njk} R_{np} R_{j\ell} R_{km} \times R_{ip} \\ &=\; \big(\det(R) = 1\big) \times \epsilon_{p\ell m} R_{ip}\,, \end{aligned} \tag{6}$$

and therefore

$$\begin{aligned} (\vec{a}' \times \vec{b}')_i \;&=\; \epsilon_{ijk} (a'_j = R_{j\ell} a_\ell)(b'_k = R_{km} b_m) \\ &=\; \big(\epsilon_{ijk} R_{j\ell} R_{km} \;=\; \epsilon_{p\ell m} R_{ip}\big)\, a_\ell b_m \\ &=\; R_{ip} \big(\epsilon_{p\ell m} a_\ell b_m\big) \\ &=\; R_{ip} (\vec{a} \times \vec{b})_p\,. \end{aligned} \tag{7}$$

Mechanical examples of scalars, vectors, and tensors:

- Time, energy, and power are scalars.
- Velocity, acceleration, force, and momentum are vectors. The angular velocity $\vec{\omega}$, the torque $\vec{\tau}$, and the angular momentum $\mathbf{L}$ are also vectors.
- The stresses — compression, tension, and shear — form a two-index stress tensor T_{ij}. The strains causing these stresses also form a two-index tensor $S_{ij} = \partial \delta x_i / \partial x_j$.

Electromagnetic examples:

- The electrostatic potential $\Phi(\mathbf{x})$ is a scalar, or rather a *scalar field*: under rotations, it transforms as $\Phi'(\mathbf{x}') \;=\; \Phi(\mathbf{x})$. Note same old $\Phi' = \Phi$ but at a new location, $\mathbf{x}'$ instead of $\mathbf{x}$.

- The electric field $\mathbf{E}(\mathbf{x})$ and the magnetic field $\mathbf{B}(\mathbf{x})$ are *vector fields,* which transform according to

$$E_i'(\mathbf{x}') \;=\; R_{ij}E_j(\mathbf{x}), \quad B_i'(\mathbf{x}') \;=\; R_{ij}B_j(\mathbf{x}). \tag{8}$$

  The vector potential $\mathbf{A}(\mathbf{x})$ for the magnetic field is also a vector field.

- Maxwell's stress tensor

$$T_{ij} \;=\; \tfrac{1}{2}\big(E_iD_j \,+\, D_iE_j \,-\, \delta_{ij}(\mathbf{E}\cdot\mathbf{D})\big) + \;\tfrac{1}{2}\big(H_iB_j \,+\, B_iH_j \,-\, \delta_{ij}(\mathbf{H}\cdot\mathbf{B})\big) \tag{9}$$

  is a tensor field, which transforms according to

$$T_{ij}'(\mathbf{x}') \;=\; R_{ik}R_{j\ell}T_{k\ell}(\mathbf{x}). \tag{10}$$

- $\star$ The ∇ operator is a vector when it acts on properly transforming scalar, vector, or tensor fields. For example, for a scalar field $\Phi(\mathbf{x})$, its gradient $\nabla\Phi(\mathbf{x})$ is a vector field. Indeed, under rotations it transforms as

$$\nabla_i\Phi(\mathbf{x}) \;\to\; \nabla_j'\Phi'(\mathbf{x}') \;=\; \frac{\partial x_j}{\partial x_i'} \times \nabla_j\Big(\Phi'(\mathbf{x}') = \Phi(\mathbf{x})\Big) \;=\; R_{ij}\nabla_i\Phi(\mathbf{x}) \tag{11}$$

  because

$$\frac{\partial x_k'}{\partial x_\ell} \;=\; R_{k\ell} \quad\Longrightarrow\quad \frac{\partial x_j}{\partial x_i'} \;=\; \big(R^{-1}\big)_{ji} \;=\; \big(R^{\top}\big)_{ji} \;=\; R_{ij}\,. \tag{12}$$

Space Reflection

Geometric symmetries of the 3D space include not only pure rotations but also mirror reflections. For example, consider a reflection $(x_1, x_2, x_3) \to (+x_1, +x_2, -x_3)$ off a mirror in the (x_1, x_2) plane. If we combine this reflection with a 180° rotation around the x_3 axis, we would get

$$(x_1', x_2', x_3') \;=\; (-x_1, -x_2, -x_3), \quad \text{or in vector notations} \quad \mathbf{x}' \;=\; -\mathbf{x}. \tag{13}$$

This combined transform is called *the space reflection* or *parity symmetry*.

Different vectors transforms differently under space reflection: the *polar vectors* reverse directions while the *axial vectors* remain unchanged:

$$\mathbf{x}' \;=\; -\mathbf{x}, \quad \mathbf{V}'_{\text{polar}} \;=\; -\mathbf{V}_{\text{polar}}\,, \quad \mathbf{V}'_{\text{axial}} \;=\; +\mathbf{V}_{\text{axial}}\,. \tag{14}$$

Physically, the polar vectors have unambiguous direction (*i.e.*, from point A to point B) while the axial vectors define an axis, but the particular direction along that axis is a matter of convention. For example, the angular velocity $\vec{\omega}$ has a definite magnitude and a definite axis — the axis of rotations — but its direction from the South pole to the North pole is a matter of convention; we could have just as well defined as pointing in the opposite direction.

Mechanical examples:

- The velocity, the acceleration, the force, and the momentum are polar vectors.
- But the angular velocity $\vec{\omega}$, the torque $\vec{\tau}$, and the angular momentum $\mathbf{L}$ are axial vectors. Their direction follows from the right hand rule; had we used the left-hand rule instead of the right-hand rule, the $\vec{\omega}$, the $\vec{\tau}$, and the $\mathbf{L}$ vectors would had pointed in the opposite directions.

Algebraically, the key to distinguishing polar vectors from axial vectors is the cross product:

$$\text{polar vector} \times \text{polar vector} = \text{ axial vector}, \tag{15}$$

$$\text{axial vector} \times \text{polar vector} = \text{ polar vector}, \tag{16}$$

$$\text{polar vector} \times \text{axial vector} = \text{ polar vector}, \tag{17}$$

$$\text{axial vector} \times \text{axial vector} = \text{ axial vector}. \tag{18}$$

For example, the torque $\vec{\tau} = \mathbf{x} \times \mathbf{F}$ is a cross product of two polar vectors, so it's an axial vector. Likewise, the angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{F}$ is a cross product of two polar vectors, so it's an axial vector. As to the angular velocity vector $\vec{\omega}$, it governs the linear velocity of rigid rotations as $\mathbf{v} = \vec{\omega} \times \mathbf{x}$, and since $\mathbf{x}$ and $\mathbf{v}$ are both polar vectors, the $\vec{\omega}$ must be an axial vector.

Electromagnetic examples:

- The force on a charge q subject to electric and magnetic fields is

$$\mathbf{F} \ = \ q\mathbf{E} \ + \ q\mathbf{v} \times \mathbf{B}. \tag{19}$$

  Since the force $\mathbf{F}$ is a true vector, *the electric field* $\mathbf{E}$ *is a true vector.* Also, the cross product of a true vector $\mathbf{v}$ with the magnetic field must be a true vector, so *the magnetic field* $\mathbf{B}$ *is an axial vector.*

- In continuous media, the polarization $\mathbf{P}$ and the electric displacement field $\mathbf{D}$ are true vectors, just like the electric tension field $\mathbf{E}$. OOH, the magnetization $\mathbf{M}$ and the magnetic tension field $\mathbf{H}$ are axial vectors just like the magnetic induction field $\mathbf{B}$.

- The vector potential $\mathbf{A}$ is a true vector, so that its curl $\mathbf{B} = \nabla \times \mathbf{A}$ is an axial vector.

- The electric current $\mathbf{J}$ is a polar vector.

- The Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is a polar vector.

(Pseudo)scalars and Pseudo(tensors):

Besides distinguishing between the polar and the axial vectors, we may also classify the scalars and the tensors according to how they transform under space reflections.

- A *true scalar* is invariant under both rotations and reflections of space.

- A *pseudoscalar* is invariant under pure rotations but changes sign under reflections.

⋆ Algebraically,

$$\text{polar vector} \cdot \text{polar vector} = \text{true scalar}, \tag{20}$$

$$\text{axial vector} \cdot \text{polar vector} = \text{pseudoscalar}, \tag{21}$$

$$\text{polar vector} \cdot \text{axial vector} = \text{pseudoscalar}, \tag{22}$$

$$\text{axial vector} \cdot \text{axial vector} = \text{true scalar}. \tag{23}$$

Examples:

- Time, energy, power, energy density, *etc.*, are true scalars.
- $\mathbf{E}^2$, $\mathbf{B}^2$, scalar potential Φ, electric charge density ρ, are true scalars,
  - but $\mathbf{E} \cdot \mathbf{B}$ is a pseudoscalar.
  - In quantum mechanics, the helicity $(\mathbf{p}/|\mathbf{p}|) \cdot \mathbf{Spin}$ is a pseudoscalar.
  - In magnetostatics, the scalar magnetic potential $\Psi(\mathbf{x})$ (whose gradient is $\nabla\Psi = -\mathbf{H}$) is a pseudoscalar.

Finally, a tensor can transform under space reflection $\mathbf{x} \to -\mathbf{x}$ as $T'_{ij\ldots n} = \pm T_{ij\ldots n}$ where the sign $\pm$ is the tensor's parity. For a *true tensor* parity $= (-1)^{\#\text{indices}}$ while for a *pseudotensor* parity $= -(-1)^{\#\text{indices}}$. For example, the stress tensor T_{ij} — mechanical or electromagnetic — has two indices and positive parity, so it is a true tensor.

The space reflection symmetry is important in electromagnetism because it's an exact symmetry of the microscopic Maxwell equations as well as related equations for the forces, energies, *etc.* Macroscopically, the electromagnetic equation are invariant under space reflection in media which are left-right symmetric — which includes most inorganic and synthetic materials. But many organic molecules of biological origin — proteins, sugars, DNA, *etc.* — are *chiral*, which means distinct from their mirror reflections. Consequently, crystals or water solutions of such molecules "know left from right" — they are distinct from their mirror images, which breaks the space reflection symmetry.

Electromagnetically, broken space reflection symmetry means that the EM wave velocity could be different for the left and right circular polarization. For a linearly polarized EM

waves, this means that the polarization plane rotates as the wave propagates through a chiral medium. This is called *optical activity* of the medium, and I shall explain it later in these notes.

Time Reversal

Microscopically, classical mechanics is invariant under yet another symmetry, the *time reversal* T or rather the *motion reversal*,

$$\mathbf{x}(t) \;\rightarrow\; \mathbf{x}'(t) \;=\; \mathbf{x}(-t). \tag{24}$$

Variables which flip sign under motion reversal are called T-odd while the variables which remain invariant are called T-even. Here are some mechanical examples:

- The velocity $\mathbf{v}$ is T-odd.
- The acceleration $\mathbf{a}$ is T-even.
- The force $\mathbf{F}$ is T-even.
- The momentum $\mathbf{p} = m\mathbf{v}$ is T-odd.
- The energy is T-even while the power is T-odd.
- The torque $\vec{\tau} = \mathbf{x} \times \mathbf{F}$ is T-even.
- The angular momentum $\mathbf{L} = \mathbf{x} \times m\mathbf{v}$ is T-odd.

The microscopic electromagnetic equations also have the time-reversal symmetry, and even the macroscopic equation have this symmetry in many kinds of media. To find which EM variables are T-even and which are T-odd, we again start with the force on a moving charge,

$$\mathbf{F} \;=\; q\mathbf{E} \;+\; q\mathbf{v} \times \mathbf{B}. \tag{25}$$

Since the force $\mathbf{F}$ is T-even, *the electric field is T-even.* Also, the product $\mathbf{v} \times \mathbf{B}$ is T-even, but since the velocity $\mathbf{v}$ is T-odd, *the magnetic field* $\mathbf{B}$ *is T-odd.* As to the other EM variables,

- The electric charge density ρ is T-even while the electric current density $\mathbf{J}$ is T-odd.

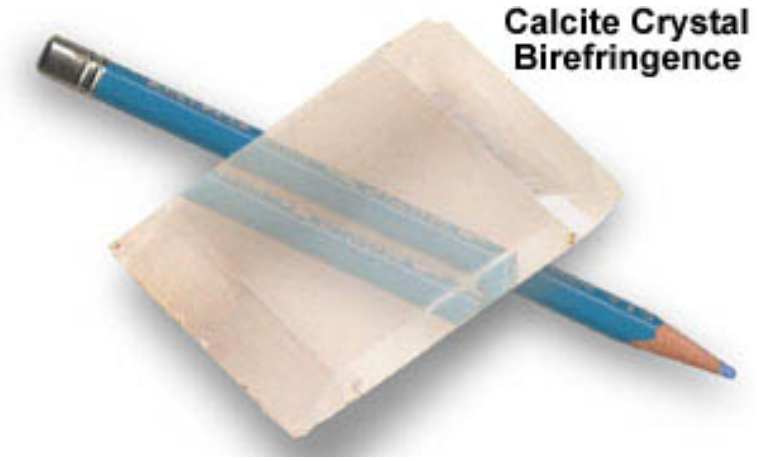
- The EM energy density U is T-even while the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is T-odd.
- Maxwell's stress tensor T_{ij} is T-even.

Macroscopically, time reversal is a symmetry only in non-dissipative media. That is, media without electric conductivity, magnetic hysteresis, or imaginary parts of $\epsilon(\omega)$ and $\mu(\omega)$. In dissipative media, the loss of EM energy to heat is irreversible, so there is no time-reversal symmetry — just like the macroscopic mechanics involving friction forces does not have time-reversal symmetry.

In non-dissipative media, the polarization $\mathbf{P}$ and the electric displacement field $\mathbf{D}$ are T-even — just like the $\mathbf{E}$ field, — while the magnetization $\mathbf{M}$ and the magnetic tension field $\mathbf{H}$ are T-odd — just like the $\mathbf{B}$ field. Likewise, the macroscopic 'free' charge density ρ is T-even, the conduction current is T-odd, the energy density U is T-even, the Poynting vector is T-odd, *etc.*, *etc.*

Optical Activity and Faraday Effect

In some materials, the refraction index n depends on the EM wave's polarization. This is called *birefringence* because light rays of two polarizations refract differently on the material's boundary and consequently follow different paths. Thus, when we look at some object in un-polarized light through a birefringent crystal like calcite, the image looks double:



For more information on the subject, please see this wikipedia article and also homework set#9, problems#1–2.

Optical activity is a special type of birefringence in which the eigenstates of the refraction index are circular rather than linear polarizations: The left circular polarization has refraction index n_L which is different from the refractions index n_R of the right circular polarization. Optical activity breaks the parity symmetry, so it happens only in materials which 'know left from right', for example crystals or solutions of biological materials such as proteins, sugars, or DNA. In general, one needs molecules which are distinct from their mirror images, just like the right hand is distinct from the left hand, for example organic molecules containing a carbon atom connected to 4 different radicals. In a synthetic organic material, one often has equal numbers of left-handed and right-handed molecules but biology often prefers one chirality over another; that's why synthetic materials often have no optical activity while organic materials of biological origin are optically active.

When a linearly polarized light travels through an optically active material, the polarization plane is rotated through angle Θ proportional to the distance traveled L. To see how this works, consider an EM wave traveling in z direction, with a linear polarization in x direction at the starting point $z = 0$. Thus,

$$@z = 0, \quad \mathbf{E}(t) \;=\; (\mathcal{E}, 0, 0)e^{-i\omega t}, \tag{26}$$

which we may re-express as a superposition of two circularly polarized waves,

$$@z = 0, \quad \mathbf{E}(t) \;=\; \frac{\mathcal{E}}{2}\,(1, +i, 0)\,e^{-i\omega t} \;+\; \frac{\mathcal{E}}{2}\,(1, -i, 0)\,e^{-i\omega t}\,. \tag{27}$$

Due to optical activity, the two polarizations travel at different velocities, hence different wave vectors

$$k_L \;=\; n_L\,\frac{\omega}{c} \quad \text{and} \quad k_R \;=\; n_R\,\frac{\omega}{c}\,. \tag{28}$$

Consequently, as the wave travels inside the material, it becomes

$$\mathbf{E}(t, z) \;=\; \frac{\mathcal{E}}{2}\,(1, +i, 0)\,\exp(ik_L z \,-\, i\omega t) \;+\; \frac{\mathcal{E}}{2}\,(1, -i, 0)\,\exp(ik_R z \,-\, i\omega t). \tag{29}$$

Let's denote

$$k_{\text{av}} \;=\; \frac{k_L + k_R}{2}\,, \quad \Delta k \;=\; k_L - k_R \quad \Longrightarrow \quad k_L \;=\; k_{\text{av}} + \tfrac{1}{2}\Delta k, \quad k_R \;=\; k_{\text{av}} - \tfrac{1}{2}\Delta k, \tag{30}$$

so we may rewrite the EM wave (29) as

$$\mathbf{E}(t,z) \;=\; \mathcal{E}\,\exp(ik_{\rm av}z \,-\, i\omega t)\,\Big(\tfrac{1}{2}(1,+i,0)\exp(+i\Delta k z/2) \,+\, \tfrac{1}{2}(1,-i,0)\exp(-i\Delta k z/2)\Big). \tag{31}$$

At point $z = L$, the polarization vector in $(\cdots)$ here has components

$$\begin{aligned} (\cdots)_x \;&=\; \tfrac{1}{2}\exp(+i\Delta k L/2) \;+\; \tfrac{1}{2}\exp(-i\Delta k L/2) \;=\; \cos(\Delta k L/2), \\ (\cdots)_y \;&=\; \tfrac{i}{2}\exp(+i\Delta k L/2) \;-\; \tfrac{i}{2}\exp(-i\Delta k L/2) \;=\; -\sin(\Delta k L/2), \end{aligned} \tag{32}$$

thus

$$\mathbf{E}(t,L) \;=\; \mathcal{E}\,\exp(ik_{\rm av}L \,-\, i\omega t)\,(\cos\Theta, -\sin\Theta, 0) \quad \text{for } \Theta \;=\; \tfrac{1}{2}\Delta k L, \tag{33}$$

— the wave at $z = L$ has its polarization plane rotated through angle

$$\Theta \;=\; \tfrac{1}{2}\Delta k L \;=\; (n_L - n_R) \times \frac{\omega L}{2c}\,. \tag{34}$$

- For $n_L > n_R$, the polarization plane is rotated to the right (clockwise); such optical activity is called *D–activity*, where D stands for 'dextro' — Latin for 'right'.
- For $n_L < n_R$, the polarization plane is rotated to the left (counterclockwise); such optical activity is called *L–activity*, where L stands for 'levo' — Latin for 'left'.
- ⋆ In water solutions of organic molecules, the optical activity is proportional to the concentration of the solution. In organic chemistry, the rotation angle Θ (in degrees) for light passing through $L = 10$ cm of 1g/mL solution is called the *specific rotation*. For other length and concentrations, the polarization plane rotates through angle

$$\Theta \;=\; \Theta_{\rm specific} \times \frac{L}{10\ {\rm cm}} \times \frac{\text{concentration}}{1\ {\rm g/mL}}\,. \tag{35}$$

  For example sucrose (the table sugar) has $\Theta_{\rm specific} = +66.5^\circ$, where + sign denotes the D-activity.

Faraday Effect

In some materials, optical activity can be induced by the magnetic field; this is called the *Faraday effect*. To see how this works from the parity symmetry point of view, note that the magnetic field $\mathbf{B}$ is an axial vector while the wave vector $\mathbf{k}$ is a polar vector, so their dot product $\mathbf{B} \cdot \mathbf{k}$ is a pseudoscalar. Consequently, when a wave propagates along the magnetic field, or at some angle to it — but not perpendicularly — we have a pseudoscalar $\mathbf{B} \cdot \mathbf{k}$ whose non-zero value breaks the parity symmetry. It is this parity breaking which allows for optical activity $n_L \neq n_R$.

Phenomenologically, the Faraday effect is parametrized as

$$\Theta \;=\; \mathcal{V} L B^{\parallel} \tag{36}$$

where Θ is the angle through which the EM wave's polarization is rotated after moving through distance L, $B^{\parallel}$ is the component of magnetic field in the direction of the wave vector, $B^{\parallel} = (\mathbf{B} \cdot \mathbf{k})/|\mathbf{k}|$, and $\mathcal{V}$ is the *Verdet constant* named after Emile Verdet. The constant depends on the material and its condition (temperature, pressure, *etc.*) and also on the light's frequency.

- For example, a crystal of terbium gallium garnet (TGG) has very large Verdet constant $\mathcal{V} = -134$ rad/(T $\cdot$ m) for red light of $\lambda = 633$ nm, but for the near-IR light of $\lambda = 1064$ nm, the Verdet constant drops to $\mathcal{V} = -40$ rad/(T $\cdot$ m).
- Everyday materials like table salt, quartz, or water also show Faraday effect, but with a much smaller Verdet constants. For example, water has $\mathcal{V} \approx 1.2 \cdot 10^{-6}$ rad/(T $\cdot$ m).

Faraday effect in plasma.
Understanding Faraday effect in condensed matter is rather complicated, so let's consider a much simple medium — the plasma with free electron density n_e. For simplicity, let me assume the magnetic field points in the same direction as the wave, which I take to be z direction. Thus, $\mathbf{B} = (0, 0, B)$ while the circularly polarized wave has

$$\mathbf{E}(x, y, z, t) \;=\; \mathcal{E}(1, \pm i, 0)\, \exp(ikz - i\omega t). \tag{37}$$

A free electron subject to this electric field moves according to

$$m_e\ddot{\mathbf{x}} \;+\; e\,\dot{\mathbf{x}}\times\mathbf{B} \;=\; -e\mathbf{E}_{\rm wave}\,; \tag{38}$$

for a harmonic circularly polarized wave like (37), this motion is with frequency ω,

$$\mathbf{x}(t) \;=\; r(1,\pm i,0)\exp(ikz-i\omega t), \tag{39}$$

whose radius obtains from eq. (38) as

$$m_e(-\omega^2 r)(1,\pm i,0) \;+\; e(-i\omega r)(1,\pm i,0)\times\mathbf{B} \;=\; -e\mathcal{E}(1,\pm i,0). \tag{40}$$

For $\mathbf{B}$ in z direction,

$$(1,\pm i,0)\times\mathbf{B} \;=\; (\pm i,-1,0)B \;=\; \pm i(1,\pm i,0)B, \tag{41}$$

hence

$$(-m_e\omega^2 \mp eB\omega)\,r(1,\pm i,0) \;=\; -e\mathcal{E}(1,\pm i,0) \tag{42}$$

and therefore

$$r \;=\; \frac{e\mathcal{E}}{m_e\omega^2 \pm eB\omega}\,. \tag{43}$$

The dipole moment $\mathbf{p}=-e\mathbf{x}$ induced by this rotating electron is

$$\mathbf{p} \;=\; \alpha\mathbf{E}_{\rm wave} \quad\text{for}\quad \alpha \;=\; \frac{-e^2}{m_e\omega^2 \pm eB\omega}\,. \tag{44}$$

Note different polarizabilities for the two circular polarizations of the wave.

The polarizability (44) of an individual free electrons translates into polarization $\mathbf{P} = n_e\mathbf{p} = n_e\alpha\mathbf{E}$ of the plasma, and hence effective dielectric constant

$$\epsilon(\omega) \;=\; 1 \;-\; \frac{e^2 n_e}{\epsilon_0} \times \frac{1}{m_e\omega^2 \pm eB\omega}\,. \tag{45}$$

It's convenient to rewrite this dielectric constant — and hence the square of the refraction index — as

$$n^2(\omega) \;=\; \epsilon(\omega) \;=\; 1 \;-\; \frac{\omega_p^2}{\omega(\omega \pm \Omega_L)} \tag{46}$$

where

$$\omega_p^2 \;=\; \frac{n_e e^2}{\epsilon_0 m_e} \tag{47}$$

is the plasma frequency, and

$$\Omega_L \;=\; \frac{eB}{m_e} \tag{48}$$

is the Larmor frequency of a free electron circling the magnetic field in the absence of any EM waves.

At high wave frequencies, $\omega \gg \omega_p, \Omega_L$, we may approximate

$$n^2 \;\approx\; 1 \;-\; \frac{\omega_p^2}{\omega^2} \;\pm\; \frac{\omega_p^2\Omega_L}{\omega^3}\,, \tag{49}$$

hence optical activity

$$n_L \;-\; n_R \;\approx\; \frac{\omega_p^2\Omega_L}{\omega^3}\,. \tag{50}$$

In terms of the rotation of a linear polarization this means

$$\Theta \;=\; \frac{\omega L}{2c} \times \Delta n \;=\; \frac{L\Omega_L}{2c} \times \frac{\omega_p^2}{\omega^2}\,. \tag{51}.$$

For example, Earth ionosphere at 500 miles above the ground has electron density $n_e \approx 10^{12}$ electrons/m^3 (at night), hence plasma frequency $\omega_p \approx 2\pi \times 10$ MHz. The geomagnetic field at that height is about 0.4 Gauss, hence Larmor frequency $\Omega_l \approx 2\pi \times 1.2$ MHz.

Therefore, we may use eq. (51) for waves of frequency $\gg 10$ MHz. For example, take a beam of 100 MHz radio waves (wavelength $\lambda = 3$ meters): its polarization would be optically rotated at the rate

$$\frac{\Theta}{L} \;=\; 9^\circ/\text{km} \tag{52}$$

— after traveling just 10 km through the plasma, the wave's polarization would be rotated by 90°. At higher frequencies, the optical rotation would be weaker. For example for a microwave wave of frequency 1500 MHz used by satellite phones, the rotation rate is only

$$\frac{\theta}{L} \;=\; 0.04^\circ/\text{km}, \tag{53}$$

so after traveling through the 500 km effective depth of the ionosphere, the wave's polarization would be rotated through only 20°.

For another example, consider the interstellar plasma between us and some distant star we are looking at through a telescope. Along the line of site, the plasma density and the magnetic field vary from place to place, so the net polarization rotation obtains as the integral

$$\Theta \;=\; \frac{\omega}{2c}\int (n_L - n_R)dz. \tag{54}$$

In terms of the electron density and the magnetic field (or rather its z component),

$$n_L \;-\; n_R \;=\; \frac{\omega_p^2(\vec{\Omega}_L)_z}{\omega^3} \;=\; \frac{e^3 n_e B_z}{\epsilon_0 m_e^2} \times \frac{1}{\omega^3} \tag{55}$$

hence

$$\Theta \;=\; \frac{e^3}{2\epsilon_0 c m_e^2} \times \frac{1}{\omega^2} \times \int\limits_{\text{line of sight}} n_e B_z \, dz \,. \tag{56}$$