Bose–Einstein Condensation and Superfluidity

Field–Particle Duality

Our brains are classical and we have trouble understanding quantum system as such. That is, once we understand what a quantum system describes, we can calculate all kinds of interesting things; but to understand the nature of the system in the first place we need to take a classical (or semiclassical) limit. Some quantum theories have two (or more) very different classical limits, and these two (or more) limits act as *dual descriptions* of the same quantum system.

For example, consider light: Is it a stream of photons or an electromagnetic wave? Turns out, these are two classical limits of exactly same quantum theory. That is, we may start with the classical electric and magnetic fields, quantize them, and get a quantum field theory. But when we look for the Hamiltonian eigenstates of that quantum theory, we find arbitrary numbers of identical bosons, each boson being a massless relativistic particle with two transverse polarization states — a photon. On the other hand, we may build a quantum theory of arbitrary number of photons — and mind the Bose statistics. But then in the Hilbert space of that theory we find coherent states which behave exactly like the classical EM fields, and even operators which act exactly like the quantum EM fields. Thus, the quantum theory of photons and the quantum theory of EM fields are exactly the same — the same Hilbert space and the same Hamiltonian.

The same field-particle duality applies to other kinds of fields and particles: some were first discovered as fields — like the EM fields — while others as particles — like the electrons — but the quantum theory always contains both the fields and the particles, and we may take whichever classical limit is more convenient for the problem at hand. The same duality works for the non-relativistic particles and fields in the condensed matter setting, and it's often very useful. For example, the superfluidity of liquid helium can be described by the Landau–Ginzburg field theory, which is the classical limit of the quantum field theory whose quanta are helium atoms. Similar Landau–Ginzburg descriptions work for the Bose–Einstein condensates of cold heavy atoms, and even for the Cooper pairs in superconductors. On the other hand, the quantized sound waves in crystals are often described in terms of quasiparticles — the phonons; similarly, other kinds of waves in condensed matter are also described in terms of quasiparticles.

In my previous extra lecture I have explained constructed the non-relativistic quantum field theory of an arbitrary number of identical bosons such as helium atoms, *cf.* my notes on the subject. In this lecture — and in these notes — we shall start by taking a classical field theory limit and seeing how it describes the Bose–Einstein condensate (BEC) of the helium atoms at zero temperature, both at rest and in superfluid motion. We shall then go back to the quantum field theory, get a better picture of the BEC ground state, and get the spectrum of its quasi-particle excitations. In particular, we shall see that all the quasiparticles move faster than some minimal velocity $v_m > 0$, and then we shall use this minimal velocity to explain how the superfluid can flow without any resistance.

Non-Relativistic QFT and the Landau–Ginzburg Theory

Let's start with the non-relativistic QFT of an arbitrary number of bosonic atoms. For simplicity, let's assume spinless atoms like ^{4}He, so the atom-annihilation field $\hat{\psi}(\mathbf{x})$ and the atom-creation field $\hat{\psi}^\dagger(\mathbf{x})$ do not carry any spin indices, and their (equal-time) commutation relations are simply

$$[\hat{\psi}(\mathbf{x}),\hat{\psi}(\mathbf{y})] \;=\; 0, \quad [\hat{\psi}^\dagger(\mathbf{x}),\hat{\psi}^\dagger(\mathbf{y})] \;=\; 0, \quad [\hat{\psi}(\mathbf{x}),\hat{\psi}^\dagger(\mathbf{y})] \;=\; \delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{1}$$

We also assume that the interactions between the atoms are dominated by the 2-body potential $V_2(\mathbf{x}-\mathbf{y})$, thus in QFT terms

$$\hat{V}_{\rm net} \;=\; \frac{1}{2}\int\! d^3\mathbf{x}\!\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y})\hat{\psi}(\mathbf{x}). \tag{2}$$

Furthermore, the 2-body potential V_2 is dominated by the short-range repulsion, so I am making a zero-range approximation $V_2(\mathbf{x}-\mathbf{y}) = \lambda\delta^{(3)}(\mathbf{x}-\mathbf{y})$: It is a good approximation to reality for a low-density ultra-cold BEC of heavy atoms, and its a good starting point for qualitative understanding of the superfluid liquid helium. Thus,

$$\hat{V}_{\rm net} \;=\; \frac{\lambda}{2}\int\! d^3\mathbf{x}\, \hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi} \quad \langle\!\langle \text{where all fields are at } \mathbf{x} \rangle\!\rangle. \tag{3}$$

To get the complete QFT Hamiltonian, we add the net kinetic energy operator

$$\hat{K}_{\rm net} \;=\; \frac{\hbar^2}{2M}\int\! d^3\mathbf{x}\, \nabla\hat{\psi}^\dagger\cdot\nabla\hat{\psi}. \tag{4}$$

Also, for a system of a thermodynamically large number of atoms, we should put it in a chemical equilibrium with an infinite atom reservoir rather than *precisely* fix the number of atoms, so the Hamiltonian should measure $E - \mu N$ — where μ is the chemical potential — rather than the internal energy E. Thus altogether,

$$\hat{H} = \hat{K}_{\text{net}} + \hat{V}_{\text{net}} - \mu\hat{N} = \int d^3\mathbf{x} \left(\frac{\hbar^2}{2M} \nabla\hat{\psi}^\dagger \cdot \nabla\hat{\psi} + \frac{\lambda}{2} \hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi} - \mu\hat{\psi}^\dagger\hat{\psi} \right). \tag{5}$$

Since I have derived this non-relativistic QFT as a theory of an arbitrary number of quantum atoms, it has an obvious classical limit as a theory of classical atoms. But it has a different classical limit as the Landau–Ginzburg theory in which the quantum field $\hat{\psi}(x)$ becomes a classical complex field $\phi(\mathbf{x})$. In the Hamiltonian formulation, the canonical conjugate 'momentum' to $\phi(\mathbf{x})$ at each $\mathbf{x}$ is $\frac{1}{i\hbar}\phi^*(\mathbf{x})$, and the Hamiltonian is the classical analogue of the quantum Hamiltonian (5), namely

$$H[\phi, \phi^*] = \int d^3\mathbf{x} \left(\frac{\hbar^2}{2M} \nabla\phi^* \cdot \nabla\phi + \frac{\lambda}{2} |\phi|^4 - \mu |\phi|^2 \right). \tag{6}$$

Or in the Lagrangian formulation,

$$L[\phi, \phi^*, \dot{\phi}, \dot{\phi}^*] = \int d^3\mathbf{x} \left(-\hbar\,\mathrm{Im}(\phi^*\dot{\phi}) + \frac{\hbar^2}{2M} \nabla\phi^* \cdot \nabla\phi + \frac{\lambda}{2} |\phi|^4 - \mu |\phi|^2 \right). \tag{7}$$

As we shall see in a moment, the Landau–Ginzburg field $\phi(x)$ describes the superfluid motion of the Bose–Einstein condensate (BEC), which the atoms form at very low temperatures.

In the undergraduate StatMech textbooks, Bose–Einstein condensation of a gas or liquid at zero temperature is often described as all the atoms being in exactly the same quantum state, namely the zero momentum state $|\mathbf{k} = 0\rangle$. In terms of the occupation numbers $n_{\mathbf{k}}$ for the momentum modes k, such naive BEC has $n_0 = N$ while all the other $n_{\mathbf{k}} = 0$,

$$|\text{naive BEC}\rangle = \frac{\left(\hat{a}_0^\dagger\right)^N}{\sqrt{N!}} |\text{vac}\rangle . \tag{8}$$

However, this naive BEC state has unphysical long-distance correlations between the quantum fields at arbitrarily distant points in space:

$$\begin{aligned}
G(\mathbf{x}-\mathbf{y}) \;&=\; \langle\text{naive BEC}|\,\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y})\,|\text{naive BEC}\rangle \\
&\qquad -\;\langle\text{naive BEC}|\,\hat{\psi}^\dagger(\mathbf{x})\,|\text{naive BEC}\rangle\times\langle\text{naive BEC}|\,\hat{\psi}(\mathbf{y})\,|\text{naive BEC}\rangle \\
&=\; n\;-\;0\times 0\;=\;n\quad\text{for any }\mathbf{x}-\mathbf{y}, \\
\text{where } n \;&=\; \text{density} \;=\; \frac{N}{\text{volume}}\,.
\end{aligned} \tag{9}$$

In real life, such long-distance correlations can never happen in any macroscopic tank of liquid helium, so we need a better model to the ground state of the BEC. For simplicity, let's keep all the atoms in the $\mathbf{k}=0$ mode, thus $n_{\mathbf{k}}=0$ for all $\mathbf{k}\neq 0$, but for the $\mathbf{k}=0$ mode we replace the $|n_0=N\rangle$ state with a *coherent state* with a similar average number of atoms, thus

$$|\text{coherent}\rangle \;=\; e^{-N/2}\exp\bigl(\sqrt{N}\hat{a}_0^\dagger\bigr)\,|0\rangle\,. \tag{10}$$

Note: not having a definite value of N is OK in the thermodynamic limit as long as $\Delta N\sim\sqrt{N}\ll N$, which is indeed what we have in the coherent state. In QFT terms, the coherent state (10) obeys

$$\hat{\psi}(\mathbf{x})\,|\text{coherent}\rangle \;=\; \sqrt{n}\,|\text{coherent}\rangle \quad \text{at all }\mathbf{x}, \tag{11}$$

hence

$$\langle\text{coherent}|\,\hat{\psi}(\mathbf{y})\,|\text{coherent}\rangle \;=\; \sqrt{n}, \tag{12}$$

$$\langle\text{coherent}|\,\hat{\psi}^\dagger(\mathbf{x})\,|\text{coherent}\rangle \;=\; \sqrt{n}, \tag{13}$$

$$\langle\text{coherent}|\,\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y})\,|\text{coherent}\rangle \;=\; n, \tag{14}$$

and therefore

$$G(\mathbf{x}-\mathbf{y}) \;=\; n\;-\;\sqrt{n}\times\sqrt{n}\;=\;0, \tag{15}$$

no long-distance correlations.

A moving BEC — such as a flowing superfluid helium — is also naively described as all the atoms being in the same quantum state, but this time its not the $|\mathbf{k}=0\rangle$ state but rather some moving state with a wave-function $\Psi(x)$. Again, to avoid un-physical long-distance correlations, it's better to approximate the moving BEC state by a coherent state with

$$\langle\text{coh}|\,\hat{\psi}(\mathbf{x})\,|\text{coh}\rangle \;=\; \sqrt{N}\times\Psi(\mathbf{x}) \;=\; \phi(\mathbf{x}), \tag{16}$$

where $\phi(\mathbf{x})$ can be identified as the classical Landau–Ginzburg field. Specifically, this coherent state obtains as

$$|\text{coh}\rangle \;=\; e^{-N/2}\exp\left(\int\!d^3\mathbf{x}\,\phi(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x})\right)|0\rangle\,, \tag{17}$$

and it obeys

$$\forall\mathbf{x}:\quad \hat{\psi}(\mathbf{x})\,|\text{coh}\rangle \;=\; \phi(\mathbf{x})\,|\text{coh}\rangle\,. \tag{18}$$

The Landau–Ginzburg field $\phi(\mathbf{x})$ encodes both the density and the velocity of the flowing BEC. To see how this works, let's start with the net atom number the net momentum operators

$$\hat{N} \;=\int\!d^3\mathbf{x}\,\hat{\psi}^\dagger\hat{\psi}, \tag{19}$$

$$\hat{\mathbf{P}} \;=\int\!d^3\mathbf{x}\left(-\tfrac{i}{2}\hbar\hat{\psi}^\dagger(\nabla\hat{\psi}) \;+\; \tfrac{i}{2}\hbar(\nabla\hat{\psi}^\dagger)\hat{\psi}\right), \tag{20}$$

which in QFT terms both look like integrals of local operators

$$\text{number density }\hat{n}(\mathbf{x}) = \;\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}), \tag{21}$$

$$\text{momentum density }\hat{\vec{\mathcal{P}}}(\mathbf{x}) \;=\; -\tfrac{i}{2}\hbar\hat{\psi}^\dagger(\mathbf{x})\nabla\hat{\psi}(\mathbf{x}) \;+\; \tfrac{i}{2}\hbar\nabla\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}). \tag{22}$$

For the coherent state (17), the expectation values of these local operators evaluate to

$$n(\mathbf{x}) \;=\; \langle\text{coh}|\,\hat{n}(\mathbf{x})\,|\text{coh}\rangle \;=\; \phi^*(\mathbf{x})\phi(\mathbf{x}) \;=\; |\phi(\mathbf{x})|^2, \tag{23}$$

$$\begin{aligned}\vec{\mathcal{P}}(\mathbf{x}) \;&=\; \langle\text{coh}|\,\hat{\vec{\mathcal{P}}}(\mathbf{x})\,|\text{coh}\rangle \;=\; -\tfrac{i}{2}\hbar\phi^*(\mathbf{x})\nabla\phi(\mathbf{x}) \;+\; \tfrac{i}{2}\hbar\phi(\mathbf{x})\nabla\phi^*(\mathbf{x})\\ &=\; \hbar\,\mathrm{Im}\big(\phi^*(\mathbf{x})\nabla\phi(\mathbf{x})\big) \;=\; \hbar|\phi(\mathbf{x})|^2\nabla\big(\text{phase}(\phi(\mathbf{x}))\big).\end{aligned} \tag{24}$$

Thus, we see that *the magnitude of the Landau–Ginzburg field governs the local density of the BEC.* As to the velocity of the BEC flow, it follows from the phase of the LG field, or rather from the gradient of this phase. Indeed, the momentum density of any fluid is related to it mass density $Mn(\mathbf{x})$ and velocity $\mathbf{v}(\mathbf{x})$ as

$$\vec{\mathcal{P}}(\mathbf{x}) \;=\; Mn(\mathbf{x})\mathbf{v}(\mathbf{x}), \tag{25}$$

hence for the flowing BEC

$$\mathbf{v}(\mathbf{x}) \;=\; \frac{\vec{\mathcal{P}}(\mathbf{x})}{Mn(\mathbf{x})} \;=\; \frac{\hbar|\phi(\mathbf{x})|^2\nabla\big(\mathrm{phase}(\phi(\mathbf{x}))\big)}{M|\phi(\mathbf{x})|^2} \;=\; \frac{\hbar}{M}\,\nabla\big(\mathrm{phase}(\phi(\mathbf{x}))\big). \tag{26}$$

Note: As written, eqs. (23) and (26) apply only at zero temperature, and only to the approximation that the quantum state of the flowing BEC is the coherent state (17). In a real superfluid at a finite (albeit rather low) temperature, the situation is a bit more complicated. The best description of the liquid helium II (below 2.17 K) is the two fluid theory: the superfluid and the normal fluid coexisting in the same space. The superfluid is comprised of the BEC, while the normal fluid is comprised of the quasiparticle excitations of the BEC ground state. For the superfluid component — and only for the superfluid component — its density and velocity are encoded in the Landau–Ginzburg field

$$\phi(\mathbf{x}) \;=\; \langle\mathrm{helium}|\,\hat{\psi}(\mathbf{x})\,|\mathrm{helium}\rangle\,, \tag{27}$$

$$n_s(\mathbf{x}) \;=\; |\phi(\mathbf{x})|^2, \tag{28}$$

$$\mathbf{v}_s(\mathbf{x}) \;=\; \frac{\hbar}{M}\,\nabla\big(\mathrm{phase}(\phi(\mathbf{x}))\big), \tag{29}$$

but the density and the velocity of the normal fluid are unrelated to the LG field.

BEC Ground State and Excitations

Let's go back to the BEC at zero temperature and at rest. In the coherent state approximation, the expectation value of its energy obtains from the classical Landau–Ginzburg Hamiltonian

$$H[\phi,\phi^*] \;=\; \langle\mathrm{coh}|\,\hat{H}\,|\mathrm{coh}\rangle \;=\; \int\!d^3\mathbf{x}\left(\frac{1}{2m}\,|\nabla\phi|^2 \;+\; \frac{\lambda}{2}\,|\phi|^4 \;-\; \mu|\phi|^2\right), \tag{30}$$

and the $\phi(\mathbf{x})$ which minimizes this classical energy gives us the best coherent-state approximation to the BEC ground state. In particular, for any negative value of the chemical potential μ, the energy (30) is minimized for $\phi(\mathbf{x}) \equiv 0$, which corresponds to the vacuum state of the quantum field theory. OOH, for a positive chemical potential, the energy (30) is minimized for

$$|\phi|^2 = \bar{n}_s = \frac{\mu}{\lambda} \quad \text{same at all } \mathbf{x}. \tag{31}$$

The phase of ϕ is arbitrary, as long as it is the same at all $\mathbf{x}$ (in the non-moving BEC), so without loss of generality we assume $\phi_{\text{ground}} = \sqrt{\bar{n}_s}$. In the quantum theory, this corresponds to a constant non-zero ground-state expectation value

$$\left\langle \hat{\psi}(\mathbf{x}) \right\rangle = \sqrt{\bar{n}_s} = \text{const.} \tag{32}$$

The simplest quantum state with this expectation value of the atom-annihilation field $\hat{\psi}(\mathbf{x})$ is the coherent state (17). However, interactions of this expectation value with the fluctuations of the quantum fields around this coherent state change the ground states of the $\mathbf{k} \neq 0$ modes and they no longer correspond to $n_{\mathbf{k}} = 0$. To see how this works, consider the *shifted quantum fields*

$$\delta\hat{\psi}(x) = \hat{\psi}(x) - \left\langle \hat{\psi} \right\rangle = \hat{\psi} - \sqrt{\bar{n}_s}, \quad \delta\hat{\psi}^\dagger(x) = \hat{\psi}^\dagger(x) - \left\langle \hat{\psi} \right\rangle^* = \hat{\psi}^\dagger - \sqrt{\bar{n}_s}, \tag{33}$$

and let's rewrite the Hamiltonian operator

$$\hat{H} = \hat{K}_{\text{net}} + \hat{V}_{\text{net}} - \mu\hat{N} = \int d^3\mathbf{x} \left(\frac{\hbar^2}{2M} \nabla\hat{\psi}^\dagger \cdot \nabla\hat{\psi} + \frac{\lambda}{2} \hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi} - \mu\hat{\psi}^\dagger\hat{\psi} \right) \tag{5}$$

in terms of these shifted fields. Term by term in eq. (5), we have

$$\nabla\hat{\psi}^\dagger \cdot \nabla\hat{\psi} = \nabla\delta\hat{\psi}^\dagger \cdot \nabla\delta\hat{\psi}, \tag{34}$$

$$\hat{\psi}^\dagger\hat{\psi} = (\sqrt{\bar{n}_s} + \delta\hat{\psi}^\dagger)(\sqrt{\bar{n}_s} + \delta\hat{\psi}) = \bar{n}_s + \sqrt{\bar{n}_s}\big(\delta\hat{\psi}^\dagger + \delta\hat{\psi}\big) + (\delta\hat{\psi}^\dagger)(\delta\hat{\psi}), \tag{35}$$

$$\begin{aligned} \hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi} &= (\sqrt{\bar{n}_s} + \delta\hat{\psi}^\dagger)^2(\sqrt{\bar{n}_s} + \delta\hat{\psi})^2 \\ &= \bar{n}_s^2 + 2\bar{n}_s^{3/2}\big(\delta\hat{\psi}^\dagger + \delta\hat{\psi}\big) + \bar{n}_s\big((\delta\hat{\psi}^\dagger)^2 + 4(\delta\hat{\psi}^\dagger)(\delta\hat{\psi}) + (\delta\hat{\psi})^2\big) \\ &\quad + 2\sqrt{\bar{n}_s}(\delta\hat{\psi}^\dagger)(\delta\hat{\psi}^\dagger + \delta\hat{\psi})(\delta\hat{\psi}) + (\delta\hat{\psi}^\dagger)^2(\delta\hat{\psi})^2, \end{aligned} \tag{36}$$

hence for $\lambda\bar{n}_s = \mu$,

$$\begin{aligned}\frac{\lambda}{2}(\hat{\psi}^\dagger)^2(\hat{\psi})^2 \; - \; \mu\hat{\psi}^\dagger\hat{\psi} \;&=\; -\frac{\lambda n_s^2}{2} \;+\; 0\times(\delta\hat{\psi}^\dagger + \delta\hat{\psi}) \\ &\quad +\; \lambda\bar{n}_s\bigl(\tfrac{1}{2}(\delta\hat{\psi}^\dagger)^2 \;+\; (\delta\hat{\psi}^\dagger)(\delta\hat{\psi}) \;+\; \tfrac{1}{2}(\delta\hat{\psi})^2\bigr) \\ &\quad +\; \lambda\sqrt{\bar{n}_s}(\delta\hat{\psi}^\dagger)(\delta\hat{\psi}^\dagger + \delta\hat{\psi})(\delta\hat{\psi}) \;+\; \tfrac{1}{2}\lambda(\delta\hat{\psi}^\dagger)^2(\delta\hat{\psi})^2.\end{aligned} \tag{37}$$

Note the organization of the RHS here according to net powers of the *shifted* fields $\delta\hat{\psi}^\dagger$ and $\delta\hat{\psi}$. Reorganizing the whole Landau–Ginzburg Hamiltonian along the similar lines, we get

$$\hat{H} \;-\; \mu\hat{N} \;=\; \text{const} \;+\; \hat{H}_{\text{free}} \;+\; \hat{H}_{\text{int}} \tag{38}$$

where

$$\hat{H}_{\text{free}} \;=\; \int\! d^3\mathbf{x}\left(\frac{\hbar^2}{2m}\nabla\delta\hat{\psi}^\dagger\cdot\nabla\delta\hat{\psi} \;+\; \lambda\bar{n}_s\bigl((\delta\hat{\psi}^\dagger)(\delta\hat{\psi}) \;+\; \tfrac{1}{2}(\delta\hat{\psi})^2 \;+\; \tfrac{1}{2}(\delta\hat{\psi}^\dagger)^2\bigr)\right) \tag{39}$$

comprises the quadratic (and bilinear) terms in the shifted fields, while

$$\hat{H}_{\text{int}} \;=\; \int\! d^3\mathbf{x}\left(\lambda\sqrt{\bar{n}_s}(\delta\hat{\psi}^\dagger)(\delta\hat{\psi}^\dagger + \delta\hat{\psi})(\delta\hat{\psi}) \;+\; \tfrac{1}{2}\lambda(\delta\hat{\psi}^\dagger)^2(\delta\hat{\psi})^2\right) \tag{40}$$

comprises the cubic and the quartic terms. Physically, the $\hat{H}_{\text{free}}$ describes the free quanta of the shifted fields — *i.e.*, of the quantum fields' fluctuations around their ground-state expectation values, — while the $\hat{H}_{\text{int}}$ describes the interactions between such quanta.

Our next task is to diagonalize the $\hat{H}_{\text{free}}$; this should give us the leading approximation to the excitation spectrum as well as the next approximation to the ground state (next after the coherent state). The better approximations after that would obtain by perturbation theory in $\hat{H}_{\text{int}}$, but I won't do it in these notes. Instead, diagonalizing just the free Hamiltonian for the fluctuations would be interesting enough.

In terms of the operators $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ creating and annihilating atoms with definite momenta, the shifted fields (33) are

$$\begin{aligned} \delta\hat{\psi}(\mathbf{x}) &= \hat{\psi}(\mathbf{x}) - \left\langle\hat{\psi}\right\rangle = L^{-3/2}\sum_{\mathbf{k}\neq 0} e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}} + \text{shifted zero mode}, \\ \delta\hat{\psi}^{\dagger}(\mathbf{x}) &= \hat{\psi}^{\dagger}(\mathbf{x}) - \left\langle\hat{\psi}\right\rangle^{*} = L^{-3/2}\sum_{\mathbf{k}\neq 0} e^{+i\mathbf{k}\cdot\mathbf{x}}\hat{a}^{\dagger}_{\mathbf{k}} + \text{shifted zero mode}. \end{aligned} \tag{41}$$

Plugging these expansions into eq. (39) and ignoring the shifted zero modes, we get

$$\hat{H}_{\text{free}} = \sum_{\mathbf{k}\neq 0}\left(\left(\frac{\hbar^2\mathbf{k}^2}{2m} + \lambda\bar{n}_s\right)\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \tfrac{1}{2}\lambda\bar{n}_s\big(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} + \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}}\big)\right). \tag{42}$$

This Hamiltonian has general form

$$\hat{H} = \sum_{\mathbf{k}}\Big(A_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \tfrac{1}{2}B_{\mathbf{k}}\big(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} + \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}}\big)\Big), \tag{43}$$

for real $A_{\mathbf{k}} = A_{-\mathbf{k}}$ and $B_{\mathbf{k}} = B_{-\mathbf{k}}$, so it may be diagonalized via the *Bogolyubov transform of the creation and annihilation operators:*

$$\begin{aligned} \hat{b}_{\mathbf{k}} &= \cosh(t_{\mathbf{k}}) \times \hat{a}_{\mathbf{k}} + \sinh(t_{\mathbf{k}}) \times \hat{a}^{\dagger}_{-\mathbf{k}}\,, \\ \hat{b}^{\dagger}_{\mathbf{k}} &= \cosh(t_{\mathbf{k}}) \times \hat{a}^{\dagger}_{\mathbf{k}} + \sinh(t_{\mathbf{k}}) \times \hat{a}_{-\mathbf{k}}\,. \end{aligned} \tag{44}$$

for appropriate real parameters $t_{\mathbf{k}} = t_{-\mathbf{k}}$. To save class time, let me summarize this Bogolyubov transform in a few lemmas that I shall prove in the Appendix to these notes instead of right here.

Lemma 1: For any real $t_{\mathbf{k}} = t_{-\mathbf{k}}$, the $\hat{b}_{\mathbf{k}}$ and $\hat{b}^{\dagger}_{\mathbf{k}}$ operators obey the same bosonic commutation relations as the $\hat{a}_{\mathbf{k}}$ and $\hat{a}^{\dagger}_{\mathbf{k}}$ operators,

$$\big[\hat{b}_{\mathbf{k}},\hat{b}_{\mathbf{k}'}\big] = 0, \quad \big[\hat{b}^{\dagger}_{\mathbf{k}},\hat{b}^{\dagger}_{\mathbf{k}'}\big] = 0, \quad \big[\hat{b}_{\mathbf{k}},\hat{b}^{\dagger}_{\mathbf{k}'}\big] = \delta_{\mathbf{k},\mathbf{k}'}\,. \tag{45}$$

Lemma 2: For any Hamiltonian of the form (43) with real $A_{\mathbf{k}} = A_{-\mathbf{k}}$, real $B_{\mathbf{k}} = B_{-\mathbf{k}}$ and $|B_{\mathbf{k}}| < A_{\mathbf{k}}$, there is a Bogolyubov transform (44) with

$$t_{\mathbf{k}} = \frac{1}{2}\,\text{artanh}\,\frac{B_{\mathbf{k}}}{A_{\mathbf{k}}}\,, \tag{46}$$

which leads to

$$\hat{H} = \sum_{\mathbf{k}}\hbar\omega(\mathbf{k})\,\hat{b}^{\dagger}_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \text{constant} \tag{47}$$

$$\text{for}\quad \hbar\omega(\mathbf{k}) \;=\; \sqrt{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2}\,. \tag{48}$$

Clearly, the ground state of the Hamiltonian (47) is the state annihilated by all the $\hat{b}_{\mathbf{k}}$ operators,

$$\forall\mathbf{k},\quad \hat{b}_{\mathbf{k}}\,|\text{ground}\rangle \;=\; 0, \tag{49}$$

while the excited states obtain by acting with the $\hat{b}^\dagger_{\mathbf{k}}$ operators on the ground state,

$$|\text{excited}\rangle \;=\; \hat{b}^\dagger_{\mathbf{k}_1}\cdots\hat{b}^\dagger_{\mathbf{k}_n}\,|\text{ground}\rangle\,,\quad E_{\text{excited}} \,-\, E_{\text{ground}} \;=\; \omega(\mathbf{k}_1) \,+\, \cdots \,+\, \omega(\mathbf{k}_n). \tag{50}$$

Physically, we may interpret such excitations as containing n *quasiparticles* of respective energies $\omega(\mathbf{k}_1),\ldots\omega(\mathbf{k}_n)$. Thus, the operators $\hat{b}^\dagger_{\mathbf{k}}$ create quasiparticles, the operators $\hat{b}_{\mathbf{k}}$ annihilate those quasiparticles, and the ground state defined by eq. (49) is the *quasiparticle vacuum.*

Lemma 3: The quasiparticle creation and annihilation operators $\hat{b}^\dagger_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ are related to the atomic creation and annihilation operators $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ by a unitary operator transform,

$$\hat{b}^\dagger_{\mathbf{k}} \;=\; e^{+\hat{F}}\times\hat{a}^\dagger_{\mathbf{k}}\times e^{-\hat{F}},\qquad \hat{b}_{\mathbf{k}} \;=\; e^{+\hat{F}}\times\hat{a}_{\mathbf{k}}\times e^{-\hat{F}}, \tag{51}$$

for the antihermitian operator

$$\hat{F} \;=\; \frac{1}{2}\sum_{\mathbf{k}} t_{\mathbf{k}}\big(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} \,-\, \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\big). \tag{52}$$

Consequently, the state

$$|\text{ground}\rangle \;=\; e^{+\hat{F}}\,|\text{coherent}\rangle \tag{53}$$

is annihilated by all the quasiparticle annihilation operators $\hat{b}_{\mathbf{k}}$, so it's the ground state of the Hamiltonian (47).

Unlike the $|\text{coherent}\rangle$ state of the BEC which has all the atoms in the $\mathbf{k}=0$ mode, the ground state (53) also has a lot of atoms paired in $(+\mathbf{k},-\mathbf{k})$ modes. In fact, experiments with the Bose–Einstein condensates of ultra-cold atoms show more atoms in such $\pm\mathbf{k}$ pairs than the atoms in the $\mathbf{k}=0$ mode itself.

Lemma 4: For the state (53), the net number of atoms in $\mathbf{k} \neq 0$ modes is

$$N_{\mathbf{k}\neq 0} \;=\; \langle \text{ground}|\, \hat{N}_{\mathbf{k}\neq 0}\, |\text{ground}\rangle \;=\; \sum_{\mathbf{k}\neq 0} \sinh^2(t_{\mathbf{k}}). \tag{54}$$

Lemma 5: The quasiparticle vacuum state (49) has zero net mechanical momentum, while the quasiparticles have definite momenta $\hbar\mathbf{k}$, thus

$$\hat{\mathbf{P}}_{\text{net}} \;=\; \sum_{\mathbf{k}} \hbar\mathbf{k}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;=\; \sum_{\mathbf{k}} \hbar\mathbf{k}\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,. \tag{55}$$

I shall prove the Lemmas 1–5 in the Appendix to these notes. Meanwhile, let me put them in the specific context of the Bose–Einstein condensate, so let's go back to the Landau–Ginzburg theory and the Hamiltonian (38) for the fluctuation fields. — or rather the free part of that Hamiltonian,

$$\hat{H}_{\text{free}} \;=\; \sum_{\mathbf{k}\neq 0} \left(\left(\frac{\hbar^2\mathbf{k}^2}{2m} \;+\; \lambda\bar{n}_s \right) \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;+\; \tfrac{1}{2}\lambda\bar{n}_s\big(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} \;+\; \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\big) \right). \tag{42}$$

Clearly, this free Hamiltonian is a special case of (43) with

$$A_{\mathbf{k}} \;=\; \frac{\hbar^2\mathbf{k}^2}{2m} \;+\; \lambda\bar{n}_s\,, \qquad B_{\mathbf{k}} \;=\; \lambda\bar{n}_s, \tag{56}$$

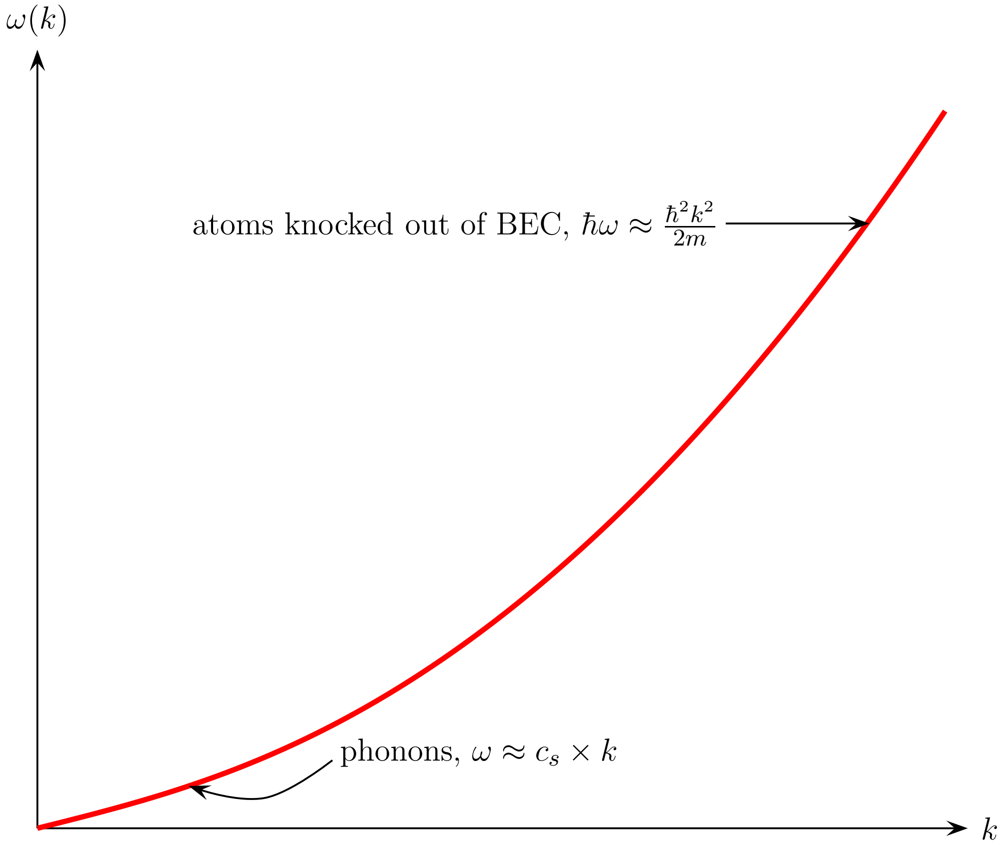
hence

$$t_{\mathbf{k}} \;=\; \frac{1}{2}\,\text{artanh}\,\frac{2\lambda\bar{n}_s m}{2\lambda\bar{n}_2 m \;+\; \hbar^2 k^2} \;\longrightarrow\; \begin{cases} \infty & \text{for small } k, \\ 0 & \text{for large } k. \end{cases} \tag{57}$$

while

$$\hbar\omega(\mathbf{k}) \;=\; \sqrt{\left(\frac{\hbar^2\mathbf{k}^2}{2m} + \lambda\bar{n}_s\right)^2 - (\lambda\bar{n}_s)^2} \;=\; \hbar k \times \sqrt{\frac{k^2}{4m^2} \;+\; \frac{\lambda\bar{n}_s}{m}}\,. \tag{58}$$

Graphically,



(59)

- Note that at high quasiparticle momenta k we have $\hbar\omega(k) \approx \hbar^2 k^2/2m$ while $t(k) \ll 1$ and hence $\hat{b}_{\mathbf{k}} \approx \hat{a}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}} \approx \hat{a}^\dagger_{\mathbf{k}}$. In other words, the quasiparticle created by the $\hat{b}^\dagger_{\mathbf{k}}$ and annihilated by the $\hat{b}_{\mathbf{k}}$ is approximately a free atom, or rather *an atom kicked out of the BEC condensate* and given a high momentum $\mathbf{k}$.

- On the other hand, for low (but non-zero) quasiparticles momenta

$$\omega(k) \;\approx\; k \times \sqrt{\lambda \bar{n}_s/m} \;\equiv\; k \times c_s \tag{60}$$

while $t(k)$ is large and hence $\hat{b}^\dagger_{\mathbf{k}} \propto (\hat{a}^\dagger_{\mathbf{k}} + \hat{a}_{-\mathbf{k}})$. This means that the $\hat{b}^\dagger_{\mathbf{k}}$ operator creates a quantum of the $\delta\phi^*(\mathbf{x}) + \delta\phi(\mathbf{x}) \propto \delta n_s(\mathbf{x})$, *i.e.*, a quantum of the condensate density wave. In other words, the quasiparticle created by the $\hat{b}^\dagger_{\mathbf{k}}$ (and annihilated by the $\hat{b}_{\mathbf{k}}$) is *a phonon*; indeed, the quasiparticle's velocity $c_s = \sqrt{\lambda \bar{n}_s/m}$ is the speed of sound in the BEC.

- Finally, at the intermediate momenta k, the quasiparticles *interpolate between the phonons and the atoms kicked out of the BEC.*

Thus far, we have ignored the interactions between the fluctuation fields $\delta\hat{\psi}(\mathbf{x})$ and $\delta\hat{\psi}^\dagger(\mathbf{x})$ and hence between the quasiparticles. In quasiparticle terms, the interaction Hamiltonian

$$\hat{H}_{\rm int} \;=\; \int\! d^3\mathbf{x} \left(\lambda\sqrt{\bar{n}_s}(\delta\hat{\psi}^\dagger)(\delta\hat{\psi}^\dagger + \delta\hat{\psi})(\delta\hat{\psi}) \;+\; \tfrac{1}{2}\lambda(\delta\hat{\psi}^\dagger)^2(\delta\hat{\psi})^2\right) \tag{40}$$

comprises terms of the form

$$\hat{b}\hat{b}\hat{b},\quad \hat{b}^\dagger\hat{b}\hat{b},\quad \hat{b}^\dagger\hat{b}^\dagger\hat{b},\quad \hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger,\quad \text{and}\quad \hat{b}\hat{b}\hat{b}\hat{b},\quad \hat{b}^\dagger\hat{b}\hat{b}\hat{b},\quad \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b},\quad \hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b},\quad \hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger. \tag{61}$$

In particular, the $\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger$ and the $\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger$ terms do not annihilate the $|\text{ground}\rangle$ state of the free Hamiltonian, so it suffers perturbative corrections. Fortunately, there is a way to recast all such corrections in terms of the unitary operator transform,

$$|\text{true ground state}\rangle \;=\; e^{\hat{F}}\,|\text{coherent}\rangle \tag{62}$$

for

$$\begin{aligned} \hat{F} \;=\; & \tfrac{1}{2}\sum_{\mathbf{k}\neq 0} t_{\mathbf{k}}(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}) \\ & + \text{ perturbative corrections involving terms of the form} \\ & \qquad (\hat{a}\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger),\quad (\hat{a}\hat{a}\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger),\quad \ldots \end{aligned} \tag{63}$$

Consequently, we may redefine the quasiparticle creation and annihilation operators as

$$\hat{b}^\dagger_{\mathbf{k}} \;=\; e^{+\hat{F}}\times\hat{a}^\dagger_{\mathbf{k}}\times e^{-\hat{F}},\qquad \hat{b}_{\mathbf{k}} \;=\; e^{+\hat{F}}\times\hat{a}_{\mathbf{k}}\times e^{-\hat{F}}, \tag{51}$$

in terms of the perturbatively corrected unitary transform $e^{\hat{F}}$, so that the $\hat{b}_{\mathbf{k}}$ and the $\hat{b}^\dagger_{\mathbf{k}}$ obey the bosonic commutation relations and all the $\hat{b}_{\mathbf{k}}$ annihilate the true ground state. Thus, we still have the quasiparticle picture of the excited states of the complete Hamiltonian, although the relation $\omega(k)$ between quasiparticles energy and momentum suffers perturbative corrections. Nevertheless, *qualitatively* the $\omega(k)$ function remains as on the diagram (59), so the low-momentum quasiparticles are phonons, the high-momentum quasiparticles are atoms kicked out of the BEC, and the intermediate-momentum quasiparticles interpolate between the two.

Superfluid Helium

Thus far we have used the Landau–Ginzburg theory which is good for the BEC of ultra-cold atoms but becomes inaccurate for the superfluid liquid helium. The problem stems from higher average momenta of the helium atoms and hence shorter De Broglie wavelength which becomes comparable to the range of the inter-atomic forces. Consequently, we may no longer approximate the two-body potential as $V_2(\mathbf{x}-\mathbf{y}) = \lambda\delta^{(3)}(\mathbf{x}-\mathbf{y})$, so instead of the local Landau–Ginzburg Hamiltonian we should use

$$\hat{H} - \mu\hat{N} \;=\; \int\! d^3\mathbf{x}\left(\frac{1}{2m}\,\hat{\nabla}\psi^\dagger\cdot\nabla\hat{\psi} \;-\; \mu\hat{\psi}^\dagger\hat{\psi}\right) + \frac{1}{2}\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y})\times\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y})\hat{\psi}(\mathbf{x}). \tag{64}$$

To find the ground state of this Hamiltonian, we start with the classical field limit and minimize the classical Hamiltonian

$$H[\phi,\phi^*] \;=\; \int\! d^3\mathbf{x}\left(\frac{1}{2m}\,|\nabla\phi|^2 \;-\; \mu|\phi|^2\right) \;+\; \frac{1}{2}\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y})\times|\phi(\mathbf{x})|^2\times|\phi(\mathbf{y})|^2. \tag{65}$$

Again, the minimum obtains for $\phi(\mathbf{x}) = \text{const}$, specifically

$$|\phi|^2 \;=\; \bar{n}_s \;=\; \frac{\mu}{\lambda}\,, \quad \text{any constant phase of } \phi, \tag{66}$$

where

$$\lambda \;\stackrel{\text{def}}{=}\; \int\! d^3\mathbf{x}\, V_2(\mathbf{x}) \;>\; 0, \tag{67}$$

or in terms of the Fourier transform of the two-atom potential

$$W(\mathbf{k}) \;=\; \int\! d^3\mathbf{x}\, V_2(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{68}$$

$$\lambda \;=\; W(0). \tag{69}$$

Given the classical ground-state expectation value of the condensate field ϕ, we go back to the quantum field theory and shift the quantum field just as we did before,

$$\delta\hat{\psi}(x) \;=\; \hat{\psi}(x) \;-\; \left\langle\hat{\psi}\right\rangle \;=\; \hat{\psi} \;-\; \sqrt{\bar{n}_s}\,, \quad \delta\hat{\psi}^\dagger(x) \;=\; \hat{\psi}^\dagger(x) \;-\; \left\langle\hat{\psi}\right\rangle^* \;=\; \hat{\psi}^\dagger \;-\; \sqrt{\bar{n}_s}\,, \tag{33}$$

and then we re-express the Hamiltonian (64) in terms of the shifted quantum fields and re-arrange the terms according to the net power of the shifted fields. Just as we had earlier for the Landau–Ginzburg theory, we end up with

$$\hat{H} - \mu\hat{N} = \text{const} + \hat{H}_{\text{free}} + \hat{H}_{\text{int}} \tag{38}$$

where $\hat{H}_{\text{free}}$ comprises the quadratic (and bilinear) terms while $\hat{H}_{\text{int}}$ comprises the cubic and the quartic terms which we treat as perturbations.

Lemma 6:

$$\begin{aligned}\hat{H}_{\text{free}} &= \frac{\hbar^2}{2m}\int d^3\mathbf{x}\,\nabla\delta\hat{\psi}^\dagger\cdot\nabla\delta\hat{\psi}\\ &\quad+\int d^3\mathbf{x}\int d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times\Big(\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y}) + \tfrac{1}{2}\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{y}) + \tfrac{1}{2}\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\Big)\\ &= \sum_{\mathbf{k}\neq 0}\left(\left(\frac{\hbar^2\mathbf{k}^2}{2m} + \bar{n}_s W(\mathbf{k})\right)\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \tfrac{1}{2}\bar{n}_s W(\mathbf{k})\big(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\big)\right).\end{aligned} \tag{70}$$

Again, this Hamiltonian can be diagonalized by a Bogolyubov transform, exactly as we did it for the LG theory in lemmas 1–5, except for the new values of

$$A_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} + \bar{n}_s W(\mathbf{k}) \quad\text{and}\quad B_k = \bar{n}_s W(\mathbf{k}). \tag{71}$$

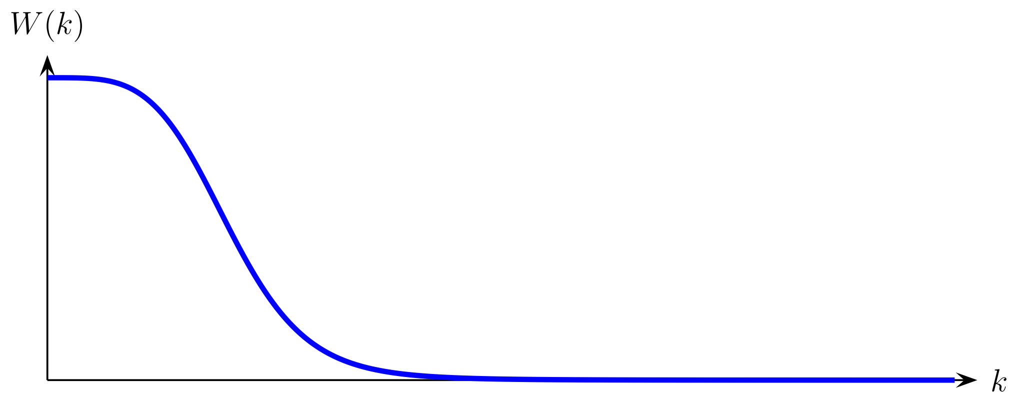
Consequently, we end up with

$$\hat{H}_{\text{free}} = \sum_{\mathbf{k}\neq 0}\hbar\omega(\mathbf{k})\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \text{const} \tag{72}$$

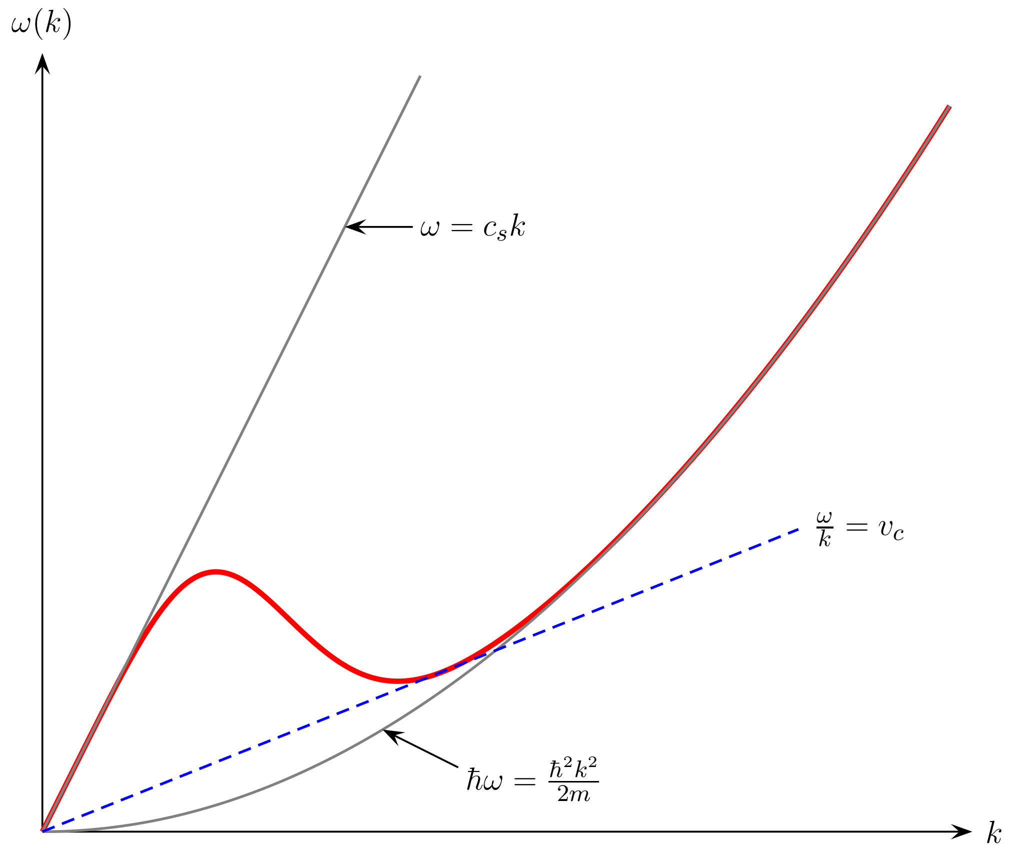
so the ground state is the quasiparticle vacuum, while the quasiparticles have definite momenta $\mathbf{k}$ and energies

$$\hbar\omega(\mathbf{k}) = \sqrt{\left(\frac{\hbar^2\mathbf{k}^2}{2m} + \bar{n}_s W(\mathbf{k})\right)^2 - (\lambda\bar{n}_s)^2} = \hbar k \times \sqrt{\frac{k^2}{4m^2} + \frac{\bar{n}_s}{m}W(k)}\,. \tag{73}$$

For the helium atoms, the $W(k)$ drops off at large momenta,



hence the energy-momentum relation $\omega(k)$ for the quasiparticles — or equivalently, the wavenumber-frequency *dispersion relation* for the waves of small fluctuations — has a dip:



(74)

Again, this curve shows that the low-momenta quasiparticles are phonons while the high momenta quasiparticles are helium atoms knocked out from the BEC. But now we also have unexpectedly-low energy quasiparticles at intermediate momenta; they are called the *rotons* for historical reason. The rotons have much larger phase space than the phonons, so at temperatures $T \sim 1$ K the rotons dominate the quasiparticle gas, which is the normal-fluid component of the finite-temperature liquid Helium II.

The most important feature of the $\omega(k)$ curve (74) is the positive lower bound on the energy-to-momentum ratio,

$$\forall \mathbf{k}: \quad \omega(\mathbf{k}) \ > v_c \times |\mathbf{k}| \quad \text{for a positive } v_c. \tag{75}$$

We shall see momentarily that it is this lower bound which gives rise to the superfluidity.

Superfluidity

Consider a flowing superfluid; for simplicity, let it flow with a uniform velocity $\mathbf{v}$. Classically, this flow is described by

$$\phi(\mathbf{x}) \ = \ \sqrt{n_s} \times \exp(im\mathbf{v} \cdot \mathbf{x}) \tag{76}$$

while quantum mechanically, we have a coherent pile up of atoms into the $\mathbf{k} = m\mathbf{v}$ mode, while other atoms form pairs with momenta $\mathbf{k}_{1,2} = m\mathbf{v} \pm \mathbf{k}_{\text{rel}}$. Altogether, we have a quantum state very much like the state of the superfluid at rest, except all atom's momenta are shifted by $m\mathbf{v}$. In other words, the state of the flowing superfluid obtains from the ground state of the superfluid at rest via *Galilean boost* of velocity $\mathbf{v}$.

The excitation spectrum of the moving superfluid also obtains via Galilean boost of the Hamiltonian,

$$\hat{H}' \ = \ \hat{H} \ + \ \mathbf{v} \cdot \hat{\mathbf{P}}_{\text{net}} \ + \ \tfrac{1}{2}\mathbf{v}^2 M_{\text{net}} \,. \tag{77}$$

For simplicity, let's ignore the interactions between the quasiparticles and focus on their free Hamiltonian. In quasiparticle terms,

$$\hat{H}_{\text{free}} \ = \ \text{const} \ + \sum_{\mathbf{k}} \hbar\omega(\mathbf{k})\, \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} \,, \tag{78}$$

$$\hat{P}_{\text{net}} \ = \sum_{\mathbf{k}} \hbar\mathbf{k}\, \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} \,, \tag{79}$$

hence for a moving superfluid

$$\hat{H}'(\mathbf{v}) \;=\; \text{const}' \;+\; \sum_{\mathbf{k}_{\rm rel}} \hbar\bigl(\omega(\mathbf{k}_{\rm rel}) + \mathbf{v}\cdot\mathbf{k}_{\rm rel}\bigr)\hat{b}^\dagger_{\mathbf{k}_{\rm rel}}\hat{b}_{\mathbf{k}_{\rm rel}} \tag{80}$$

where $\hbar\mathbf{k}_{\rm rel}$ denotes the quasiparticle momenta *in the frame of the moving superfluid*; in the lab frame, the quasiparticle momentum is $\mathbf{p} = \hbar\mathbf{k}_{\rm rel} + m\mathbf{v}$.

Before we apply eq. (80) to the liquid Helium II, consider the ideal gas. For the ideal gas at rest, the Hamiltonian also has form (78) where the 'quasiparticles' are the atoms and $\omega(k) = \hbar k^2/2m$. Consequently, the uniformly flowing gas has

$$\hat{H}'(\mathbf{v}) \;=\; \text{const} \;+\; \sum_{\mathbf{k}_{\rm rel}} \left(\frac{\hbar^2{\mathbf{k}_{\rm rel}}^2}{2m} + \mathbf{v}\cdot\hbar\mathbf{k}_{\rm rel}\right)\hat{b}^\dagger_{\mathbf{k}_{\rm rel}}\hat{b}_{\mathbf{k}_{\rm rel}} \tag{81}$$

where the frequencies

$$\omega'(\mathbf{k}_{\rm rel}) \;=\; \frac{\hbar{\mathbf{k}_{\rm rel}}^2}{2m} \;+\; \mathbf{v}\cdot\mathbf{k}_{\rm rel} \tag{82}$$

are positive for some modes $\mathbf{k}_{\rm rel}$ and negative for other modes. In particular, for $\mathbf{k}_{\rm rel}$ in opposite direction from the gas flow $\mathbf{v}$ and of magnitude $\hbar k_{\rm rel} < 2mv$, the frequency $\omega'(\mathbf{k}_{\rm rel})$ is negative.

Now, while a harmonic oscillator with a positive frequency ω has a unique ground state, the oscillator with a negative frequency has energy spectrum unlimited from below. Which means that any perturbation — however small it might be — would cause transitions building up the number of quanta while lowering the energy. For the ideal gas, this means spontaneous build up of atoms with $\omega'(\mathbf{k}_{\rm rel}) < 0$ — *i.e.*, with lab-frame velocities

$$\left|\mathbf{v}_{\rm qp}(\mathbf{k}_{\rm rel}) \;=\; \mathbf{v} \;+\; \frac{\hbar\mathbf{k}_{\rm rel}}{m}\right| \;<\; |\mathbf{v}|, \tag{83}$$

by taking them out of the coherent motion with the gas. In other words, any interactions with the outside world (for example, the walls of the pipe the gas flows through) would spontaneously knock the atoms out of the coherent flow of the gas and slow them down. Such slowed-down atoms would dissipate the net energy and the net momentum of the flowing gas; it is this dissipation that we experimentally observe as *resistance to the flow.*

Now consider the superfluid Helium with $\omega(\mathbf{k})$ as on the diagram (74). Unlike the ideal gas with $\omega \propto \mathbf{k}^2$, the superfluid has $\omega \propto |\mathbf{k}|$ at low momenta, and for any quasiparticle momenta $\omega(\mathbf{k}) > v_c \times |\mathbf{k}|$ for some positive v_c. Consequently, as long as the Helium flows at speed v less than the critical speed v_c, we have

$$\omega'(\mathbf{k}_{\text{rel}}) = \omega(\mathbf{k}_{\text{rel}}) + \mathbf{v}\cdot\mathbf{k}_{\text{rel}} > 0 \quad \text{for all } \mathbf{k}_{\text{rel}}\,. \tag{84}$$

Indeed,

$$\omega(\mathbf{k}_{\text{rel}}) + \mathbf{v}\cdot\mathbf{k}_{\text{rel}} > \omega(k_{\text{rel}}) - vk_{\text{rel}} > \omega(k_{\text{rel}}) - v_c k_{\text{rel}} \geq 0. \tag{85}$$

Consequently, there are no negative-energy quasiparticles — like the slowed-down atoms — so there are no microscopic transitions lowering the superfluid's net energy. This means no energy dissipation, and that's why the superfluid flows without resistance, hence the name *superfluidity*.

APPENDIX: Proofs of the Lemmas

Lemma 1: the bosonic commutation relations (45) for the quasiparticle creation and annihilation operators. Starting from the bosonic commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0, \quad [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0, \quad [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'} \tag{86}$$

for the operators creating and annihilating the atoms, and treating eqs.

$$\begin{aligned} \hat{b}_{\mathbf{k}} &= \cosh(t_{\mathbf{k}}) \times \hat{a}_{\mathbf{k}} + \sinh(t_{\mathbf{k}}) \times \hat{a}^\dagger_{-\mathbf{k}}\,, \\ \hat{b}^\dagger_{\mathbf{k}} &= \cosh(t_{\mathbf{k}}) \times \hat{a}^\dagger_{\mathbf{k}} + \sinh(t_{\mathbf{k}}) \times \hat{a}_{-\mathbf{k}}\,. \end{aligned} \tag{44}$$

as the definitions of the $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ operators, we immediately calculate

$$\begin{aligned} [\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}] &= \cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}'}) \times \left([\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{-\mathbf{k}'}] = \delta_{\mathbf{k},-\mathbf{k}'}\right) \\ &\quad + \sinh(t_{\mathbf{k}})\cosh(t_{\mathbf{k}'}) \times \left([\hat{a}^\dagger_{-\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = -\delta_{-\mathbf{k},\mathbf{k}'}\right) \\ &= \delta_{\mathbf{k}',-\mathbf{k}} \times \Big(\cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}'}) - \sinh(t_{\mathbf{k}})\cosh(t_{\mathbf{k}'}) = \sinh(t_{\mathbf{k}'} - t_{\mathbf{k}})\Big) \\ &= 0 \quad \text{because } t'_{\mathbf{k}} = t_{\mathbf{k}} \text{ when } \mathbf{k}' = -\mathbf{k}. \end{aligned} \tag{87}$$

In the same way, $[\hat{b}^\dagger_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] = 0$.

Finally,

$$
\begin{aligned}
[\hat{b}_{\mathbf{k}}, \hat{b}^{\dagger}_{\mathbf{k}'}] \;&=\; \cosh(t_{\mathbf{k}})\cosh(t_{\mathbf{k}'}) \times \big([\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}\big) \\
&\qquad +\; \sinh(t_{-\mathbf{k}})\sinh(t_{-\mathbf{k}'}) \times \Big([\hat{a}^{\dagger}_{-\mathbf{k}}, \hat{a}_{-\mathbf{k}'}] \;=\; -\delta_{-\mathbf{k},-\mathbf{k}'} \;=\; -\delta_{\mathbf{k},\mathbf{k}'}\Big) \\
&=\; \delta_{\mathbf{k},\mathbf{k}'} \times \Big(\cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{-\mathbf{k}}) \;= \cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{\mathbf{k}}) \;=\; 1\Big) \\
&=\; \delta_{\mathbf{k},\mathbf{k}'}\,.
\end{aligned}
\tag{88}
$$

Quod erat demonstrandum.

Lemma 2: bringing the Hamiltonian

$$
\hat{H} \;=\; \sum_{\mathbf{k}} \Big(A_{\mathbf{k}} \hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{k}} \;+\; \tfrac{1}{2} B_{\mathbf{k}} \big(\hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} \;+\; \hat{a}^{\dagger}_{\mathbf{k}} \hat{a}^{\dagger}_{-\mathbf{k}}\big)\Big), \tag{43}
$$

to the form

$$
\hat{H} \;=\; \sum_{\mathbf{k}} \hbar\omega(\mathbf{k})\, \hat{b}^{\dagger}_{\mathbf{k}} \hat{b}_{\mathbf{k}} \;+\; \text{constant} \tag{47}
$$

$$
\text{for} \quad \hbar\omega(\mathbf{k}) \;=\; \sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}}\,. \tag{48}
$$

Let's start by expressing the product $\hat{b}^{\dagger}_{\mathbf{k}} \hat{b}_{\mathbf{k}}$ in terms of the $\hat{a}^{\dagger}$ and $\hat{a}$ operators. Applying both definitions (44), we immediately obtain

$$
\begin{aligned}
\hat{b}^{\dagger}_{\mathbf{k}} \hat{b}_{\mathbf{k}} \;&=\; \cosh^2(t_{\mathbf{k}})\, \hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{k}} \;+\; \cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}})\, (\hat{a}^{\dagger}_{\mathbf{k}} \hat{a}^{\dagger}_{-\mathbf{k}} \;+\; \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}}) \\
&\qquad +\; \sinh^2(t_{\mathbf{k}})\, (\hat{a}_{-\mathbf{k}} \hat{a}^{\dagger}_{-\mathbf{k}} \;=\; \hat{a}^{\dagger}_{-\mathbf{k}} \hat{a}_{-\mathbf{k}} + 1).
\end{aligned}
\tag{89}
$$

Likewise,

$$
\begin{aligned}
\hat{b}^{\dagger}_{-\mathbf{k}} \hat{b}_{-\mathbf{k}} \;&=\; \cosh^2(t_{-\mathbf{k}})\, \hat{a}^{\dagger}_{-\mathbf{k}} \hat{a}_{-\mathbf{k}} \;+\; \cosh(t_{-\mathbf{k}})\sinh(t_{\mathbf{k}})\, (\hat{a}^{\dagger}_{-\mathbf{k}} \hat{a}^{\dagger}_{\mathbf{k}} \;+\; \hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}}) \\
&\qquad +\; \sinh^2(t_{-\mathbf{k}})\, (\hat{a}_{\mathbf{k}} \hat{a}^{\dagger}_{\mathbf{k}} \;=\; \hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{k}} + 1).
\end{aligned}
\tag{90}
$$

Assuming $t_{-\mathbf{k}} = t_{\mathbf{k}}$, we may combine

$$
\begin{aligned}
\hat{b}^{\dagger}_{\mathbf{k}} \hat{b}_{\mathbf{k}} \;+\; \hat{b}^{\dagger}_{-\mathbf{k}} \hat{b}_{-\mathbf{k}} \;&=\; \Big(\cosh^2(t_{\mathbf{k}}) + \sinh^2(t_{\mathbf{k}}) = \cosh(2t_{\mathbf{k}})\Big) \times (\hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{-\mathbf{k}} \hat{a}_{-\mathbf{k}}) \\
&\qquad +\; \Big(2\cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}}) = \sinh(2t_{\mathbf{k}})\Big) \times (\hat{a}^{\dagger}_{\mathbf{k}} \hat{a}^{\dagger}_{-\mathbf{k}} \;+\; \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}}) \;+\; \text{const.}
\end{aligned}
\tag{91}
$$

Now let's plug this formula into a Hamiltonian of the form (47) for some $\omega_{\mathbf{k}}$ and require that the result matches the original Hamiltonian (43). Assuming $\omega_{-\mathbf{k}} \equiv \omega_{\mathbf{k}}$, we obtain

$$\begin{aligned} \hat{H} &= \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} = \frac{1}{2} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} (\hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} + \hat{b}^\dagger_{-\mathbf{k}} \hat{b}_{-\mathbf{k}}) \\ &= \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \cosh(2t_{\mathbf{k}}) \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \tfrac{1}{2} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \sinh(2t_{\mathbf{k}}) \left(\hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}} + \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}} \right) + \text{const.} \end{aligned} \tag{92}$$

This formula must match (up to a constant) the original Hamiltonian (43), so we need to choose the parameters $\omega_{\mathbf{k}} = \omega_{-\mathbf{k}}$ and $t_{\mathbf{k}} = t_{-\mathbf{k}}$ such that

$$\hbar\omega_{\mathbf{k}} \cosh(2t_{\mathbf{k}}) = A_{\mathbf{k}} \quad \text{and} \quad \hbar\omega_{\mathbf{k}} \sinh(2t_{\mathbf{k}}) = B_{\mathbf{k}} . \tag{93}$$

These equations are easy to solve, and the solution exists as long as $A_{\mathbf{k}} = A_{-\mathbf{k}}$, $B_{\mathbf{k}} = B_{-\mathbf{k}}$, and $A_{\mathbf{k}} > |B_{\mathbf{k}}|$, namely

$$t_{\mathbf{k}} = \frac{1}{2} \operatorname{artanh} \frac{B_{\mathbf{k}}}{A_{\mathbf{k}}} \quad \text{and} \quad \hbar\omega_{\mathbf{k}} = \sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}} . \tag{94}$$

Quod erat demonstrandum.

Lemma 3: By the Baker–Hausdorff Lemma,

$$e^{+\hat{F}} \times \hat{a}_{\mathbf{k}} \times e^{-\hat{F}} = \hat{a}_{\mathbf{k}} + [\hat{F}, \hat{a}_{\mathbf{k}}] + \tfrac{1}{2} [\hat{F}, [\hat{F}, \hat{a}_{\mathbf{k}}]] + \tfrac{1}{6} [\hat{F}, [\hat{F}, [\hat{F}, \hat{a}_{\mathbf{k}}]]] + \cdots \tag{95}$$

and likewise for the $\hat{a}^\dagger_{\mathbf{k}}$. Specifically, for

$$\hat{F} = \frac{1}{2} \sum_{\mathbf{q}} t_{\mathbf{q}} \big(\hat{a}_{\mathbf{q}} \hat{a}_{-\mathbf{p}} - \hat{a}^\dagger_{\mathbf{q}} \hat{a}^\dagger_{-\mathbf{p}} \big), \tag{52}$$

we have

$$[\hat{F}, \hat{a}_{\mathbf{k}}] = -t_{\mathbf{k}} [\hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}}, \hat{a}_{\mathbf{k}}] = +t_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}} \tag{96}$$

and

$$[\hat{F}, \hat{a}^\dagger_{-\mathbf{k}}] = +t_{\mathbf{k}} [\hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}}, \hat{a}^\dagger_{-\mathbf{k}}] = +t_{\mathbf{k}} \hat{a}_{\mathbf{k}} . \tag{97}$$

Consequently, multiple commutators yield

$$[\hat{F},[\hat{F},\ldots[\hat{F},\hat{a}_{\mathbf{k}}]\cdots]]_{n\,\text{times}} \;=\; (t_{\mathbf{k}})^n \times \begin{cases} \hat{a}_{\mathbf{k}} & \text{for even } n, \\ \hat{a}^\dagger_{-\mathbf{k}} & \text{for odd } n. \end{cases}, \tag{98}$$

and therefore

$$\begin{aligned} e^{+\hat{F}} \times \hat{a}_{\mathbf{k}} \times e^{-\hat{F}} \;&=\; \sum_{\text{even}\,n} \frac{(t_{\mathbf{k}})^n}{n!} \times \hat{a}_{\mathbf{k}} \;+\; \sum_{\text{odd}\,n} \frac{(t_{\mathbf{k}})^n}{n!} \times \hat{a}^\dagger_{-\mathbf{k}} \\ &=\; \cosh(t_{\mathbf{k}}) \times \hat{a}_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}}) \times \hat{a}^\dagger_{-\mathbf{k}} \\ &=\; \hat{b}_{\mathbf{k}}\,. \end{aligned} \tag{99}$$

In exactly the same way

$$[\hat{F},[\hat{F},\ldots[\hat{F},\hat{a}^\dagger_{\mathbf{k}}]\cdots]]_{n\,\text{times}} \;=\; (t_{\mathbf{k}})^n \times \begin{cases} \hat{a}^\dagger_{\mathbf{k}} & \text{for even } n, \\ \hat{a}_{-\mathbf{k}} & \text{for odd } n. \end{cases}, \tag{100}$$

and therefore

$$\begin{aligned} e^{+\hat{F}} \times \hat{a}^\dagger_{\mathbf{k}} \times e^{-\hat{F}} \;&=\; \sum_{\text{even}\,n} \frac{(t_{\mathbf{k}})^n}{n!} \times \hat{a}^\dagger_{\mathbf{k}} \;+\; \sum_{\text{odd}\,n} \frac{(t_{\mathbf{k}})^n}{n!} \times \hat{a}_{-\mathbf{k}} \\ &=\; \cosh(t_{\mathbf{k}}) \times \hat{a}^\dagger_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}}) \times \hat{a}_{-\mathbf{k}} \\ &=\; \hat{b}^\dagger_{\mathbf{k}}\,. \end{aligned} \tag{101}$$

This establishes the unitary transform (51) of the creation and annihilation operators.

The unitary transforms (51) automatically preserve the commutation relations between the operators, so instead of going through the algebra of proving the Lemma 1, we may simply use

$$[\hat{b}_{\mathbf{k}},\hat{b}^\dagger_{\mathbf{k}'}] \;=\; e^{\hat{F}}[\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}]e^{-\hat{F}} \;=\; e^{\hat{F}}\delta_{\mathbf{k},\mathbf{k}'}e^{-\hat{F}} \;=\; \delta_{\mathbf{k},\mathbf{k}'} \tag{102}$$

and likewise for the $[\hat{b}_{\mathbf{k}},\hat{b}_{\mathbf{k}'}]$ and $[\hat{b}^\dagger_{\mathbf{k}},\hat{b}^\dagger_{\mathbf{k}'}]$.

In the same manner, the fact that the $|\text{coherent}\rangle$ state is annihilated by all the $\hat{a}_{\mathbf{k}}$ operators with $\mathbf{k}\neq 0$ automatically leads to the $|\text{ground}\rangle = e^{\hat{F}}\,|\text{coherent}\rangle$ state being annihilated by all the $\hat{b}_{\mathbf{k}}$ operators,

$$\hat{b}_{\mathbf{k}}\,|\text{ground}\rangle \;=\; e^{\hat{F}}\hat{a}_{\mathbf{k}}e^{-\hat{F}}\times e^{\hat{F}}\,|\text{coherent}\rangle \;=\; e^{\hat{F}}\hat{a}_{\mathbf{k}}\,|\text{coherent}\rangle \;=\; e^{\hat{F}}\times 0 \;=\; 0. \tag{103}$$

In other words, the $|\text{ground}\rangle = e^{\hat{F}}\,|\text{coherent}\rangle$ state is the quasi-particle vacuum state — it has no quasiparticles at all. And since all the quasiparticle energies $\hbar\omega_{\mathbf{k}}$ are positive, this state is the ground state of the (free part of the) excitation Hamiltonian (47).
Quod erat demonstrandum.

Lemma 4: The operator $\hat{F}$ — and hence its exponential $e^{\hat{F}}$ — creates and annihilates the atoms in $\pm\mathbf{k}$ pairs independently from all the other pairs. Consequently, the BEC ground state $|\text{ground}\rangle = e^{\hat{F}}\,|\text{coherent}\rangle$ can be written as a direct product of independent states of $\pm\mathbf{k}$ modes (and the coherent state for the $\mathbf{k}=0$ mode),

$$|\text{ground}\rangle \;=\; |\Psi_{\text{coherent}}(\mathbf{k}=0)\rangle\otimes\bigoplus_{\pm\mathbf{k}\,\text{pairs}}|\Psi(\pm\mathbf{k})\rangle\,, \tag{104}$$

where each

$$|\Psi\rangle\,(\pm\mathbf{k}) \;=\; \exp\bigl(t_{\mathbf{k}}(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}-\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}})\bigr)\,|0_{\mathbf{k}},0_{-\mathbf{k}}\rangle \;=\; \sum_{n=1}^{\infty}C_n(t_{\mathbf{k}})\,|n_{\mathbf{k}},n_{-\mathbf{k}}\rangle \tag{105}$$

for some coefficients $C_n(t_{\mathbf{k}})$. To calculate these coefficients, we use

$$\frac{d}{dt}e^{t\hat{F}}\,|0,0\rangle \;=\; \hat{F}e^{t\hat{F}}\,|0,0\rangle \tag{106}$$

hence

$$\begin{aligned}\frac{d}{dt}\sum_n C_n(t)\,|n,n\rangle \;&=\; (\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}-\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}})\sum_n C_n(t)\,|n,n\rangle\\ &=\sum_n C_n(t)\Bigl(n\,|n-1,n-1\rangle \;-\; (n+1)\,|n+1,n+1\rangle\Bigr)\\ &=\sum_n |n,n\rangle\times\Bigl((n+1)C_{n+1}(t)\;-\;nC_{n-1}(t)\Bigr),\end{aligned} \tag{107}$$

which leads us to differential equations

$$\frac{d}{dt}C_n(t) \;=\; (n+1)C_{n+1}(t) \;-\; nC_{n-1}(t) \tag{108}$$

subject to initial conditions $C_n(0)=\delta_{n,0}$. Instead of going through the long song and dance of solving the equations (108), let me simply give you the solutions

$$C_n(t) \;=\; \frac{(-\tanh(t))^n}{\cosh(t)} \tag{109}$$

and verify that they indeed solve the equations (108):

$$\begin{aligned}\frac{d}{dt}\,\frac{(-\tanh(t))^n}{\cosh(t)} &= n\frac{(-\tanh t)^{n-1}}{\cosh t}\times\frac{-1}{\cosh^2 t} \;-\; (-\tanh t)^n\times\frac{\sinh t}{\cosh^2 t}\\ &= \frac{1}{\cosh t}\Big(n(-\tanh t)^{n-1}\times(-1+\tanh^2 t) \;+\; (-\tanh t)^{n+1}\Big)\\ &= -n\,\frac{(-\tanh t)^{n-1}}{\cosh t} \;+\; (n+1)\,\frac{(-\tanh t)^{n+1}}{\cosh t}.\end{aligned} \tag{110}$$

Quod erat demonstrandum.

Now, given the quantum state (105) of the $\pm\mathbf{k}$ modes of the atoms, we may calculate the expectation value of the atom number in these two modes as

$$N_{\pm\mathbf{k}} \;=\; \sum_{n=0}^{\infty}(2n)\times C_n^2(t_{\mathbf{k}}). \tag{111}$$

Specifically, for the C_n coefficients as in eq. (109),

$$N_{\pm\mathbf{k}} \;=\; \frac{2}{\cosh^2 t_{\mathbf{k}}}\times\sum_n n(-\tanh t_{\mathbf{k}})^{2n} \;=\; \frac{2}{\cosh^2 t_{\mathbf{k}}}\times\frac{\tanh^2(t_{\mathbf{k}})}{(1-\tanh^2(t_{\mathbf{k}}))^2} \;=\; 2\sinh^2(t_{\mathbf{k}}). \tag{112}$$

Finally, combining all such $\pm\mathbf{k}$ pairs of modes, we get the net (average) number of atoms in all the $\mathbf{k}\neq 0$ modes as

$$N_{\mathbf{k}\neq 0} \;=\; \frac{1}{2}\sum_{\mathbf{k}} N_{\pm\mathbf{k}} \;=\; \sum_{\mathbf{k}}\sinh^2(t_{\mathbf{k}}). \tag{54}$$

Quod erat demonstrandum.

Lemma 5: the net momentum operator is

$$\hat{\mathbf{P}}_{\rm net} \;=\; \sum_{\mathbf{k}} \hbar\mathbf{k}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,. \tag{113}$$

Using eqs. (89) and (90) from the proof of Lemma 2 and $t_{-\mathbf{k}} = t_{\mathbf{k}}$, we immediately see that

$$\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;-\; \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} \;=\; \Bigl(\cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{\mathbf{k}}) = 1\Bigr) \times (\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;-\; \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}). \tag{114}$$

Consequently, for the momentum operator (113) we have

$$\begin{aligned}
\hat{\mathbf{P}} \;&=\; \sum_{\mathbf{k}} \hbar\mathbf{k} \times \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;=\; \sum_{\mathbf{k}} \hbar(-\mathbf{k}) \times \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}} \\
&=\; \frac{1}{2}\sum_{\mathbf{k}} \hbar\mathbf{k} \times (\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;-\; \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}) \\
&=\; \frac{1}{2}\sum_{\mathbf{k}} \hbar\mathbf{k} \times (\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;-\; \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}}) \\
&=\; \sum_{\mathbf{k}} \hbar\mathbf{k} \times \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,.
\end{aligned} \tag{115}$$

Quod erat demonstrandum.

Alternatively, we may use the unitary operator transform of Lemma 3 and the fact that the $\hat{F}$ operator commutes with the net momentum $\hat{\mathbf{P}}$. Indeed, for every $\mathbf{k}$ mode

$$\hat{\mathbf{P}}\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} \;=\; \hat{a}_{\mathbf{k}}(\hat{\mathbf{P}} - \hbar\mathbf{k})\hat{a}_{-\mathbf{k}} \;=\; \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}\hat{\mathbf{P}} \tag{116}$$

and likewise

$$\hat{\mathbf{P}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \;=\; \hat{a}^\dagger_{\mathbf{k}}(\hat{\mathbf{P}} + \hbar\mathbf{k})\hat{a}^\dagger_{-\mathbf{k}} \;=\; \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\hat{\mathbf{P}}, \tag{117}$$

hence $\hat{\mathbf{P}}\hat{F} = \hat{F}\hat{\mathbf{P}}$. Consequently, the BEC ground state $|\text{ground}\rangle = e^{\hat{F}}\,|\text{coherent}\rangle$ has the same momentum as the $|\text{coherent}\rangle$ state, namely zero, and the quasiparticles created by the $\hat{b}^\dagger_{\mathbf{k}}$ and annihilated by the $\hat{b}_{\mathbf{k}}$ carry the same definite momenta $\hbar\mathbf{k}$ as the atoms created by the $\hat{a}^\dagger_{\mathbf{k}}$ and annihilated by the $\hat{a}_{\mathbf{k}}$.

Lemma 6: the finite-range potential $V_2(\mathbf{x}-\mathbf{y})$ for the helium atoms. Consider the net potential operator

$$\hat{V} \;=\; \frac{1}{2}\int\!d^3\mathbf{x}\int\!d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y})\hat{\psi}(\mathbf{x}). \tag{118}$$

In terms of the shifted fields $\delta\hat{\psi}(\mathbf{x})=\hat{\psi}(\mathbf{x})-\sqrt{\bar{n}_s}$ and $\delta\hat{\psi}^\dagger(\mathbf{x})=\hat{\psi}^\dagger(\mathbf{x})-\sqrt{\bar{n}_s}$, we have

$$\begin{aligned}
\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y})\hat{\psi}(\mathbf{x}) \;=\;& \bar{n}_s^2 \;+\; \bar{n}_s^{3/2}\Big(\delta\hat{\psi}^\dagger(\mathbf{x}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{y}) \;+\; \delta\hat{\psi}(\mathbf{x}) \;+\; \delta\hat{\psi}(\mathbf{y})\Big)\\
&+\; \bar{n}_s\Big(\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{y})\delta\hat{\psi}(\mathbf{y})\Big)\\
&+\; \bar{n}_s\Big(\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{y})\delta\hat{\psi}(\mathbf{x})\Big)\\
&+\; \bar{n}_s\Big(\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) \;+\; \delta\hat{\psi}(\mathbf{y})\delta\hat{\psi}(\mathbf{x})\Big)\\
&+\; \text{cubic} \;+\; \text{quartic}.
\end{aligned} \tag{119}$$

The terms on the first two lines here depend only on the $\mathbf{x}$ or only on the $\mathbf{y}$, so when we plug them into the potential operator (118), we may immediately integrate over the other space position to obtain

$$[\text{@any fixed }\mathbf{y}]\;\int\!d^3\mathbf{x}\,V_2(\mathbf{x}-\mathbf{y}) \;=\; [\text{@any fixed }\mathbf{x}]\;\int\!d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y}) \;=\; W(0). \tag{120}$$

Consequently, integrating over the expansion (119) in the context of the potential (118) and making use of the $\mathbf{x}\leftrightarrow\mathbf{y}$ symmetry, we obtain

$$\begin{aligned}
\hat{V} \;=\;& \bar{n}_s\times W(0)\times\int\!d^3\mathbf{x}\Big(\tfrac{1}{2}\bar{n}_s \;+\; \sqrt{\bar{n}_s}\big(\delta\hat{\psi}^\dagger(\mathbf{x})+\delta\hat{\psi}(\mathbf{x})\big) \;+\; \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x})\Big)\\
&+\; \frac{\bar{n}_s}{2}\times\int\!d^3\mathbf{x}\int\!d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times\Big(2\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) \;+\; \delta\hat{\psi}(\mathbf{y})\delta\hat{\psi}(\mathbf{x})\Big)\\
&+\; \text{cubic} \;+\; \text{quartic}.
\end{aligned} \tag{121}$$

Now consider the other non-derivative term in the Helium's Hamiltonian

$$\hat{H}_{\text{net}} \;=\; \hat{K} \;+\; \hat{V} \;-\; \mu\hat{N}, \tag{122}$$

namely the chemical potential term,

$$\begin{aligned} -\mu\hat{N} &= -\mu\int d^3\mathbf{x}\,\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}) \\ &= -\mu\int d^3\mathbf{x}\Big(\bar{n}_s + \sqrt{\bar{n}_s}\big(\delta\hat{\psi}(\mathbf{x})+\delta\hat{\psi}^\dagger(\mathbf{x})\big) + \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x})\Big) \end{aligned} \tag{123}$$

If we generalize the $\mu = \lambda\bar{n}_s$ formula of the Landau–Ginzburg theory to the

$$\mu = W(0)\times\bar{n}_s\,, \tag{124}$$

then the chemical potential term (123) cancels the top line of the two-body potential (121) (except for the constant part), hence

$$\begin{aligned} \hat{V} - \mu\hat{N} = \frac{\bar{n}_2}{2}\int d^3\mathbf{x}\int d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times\Big(2\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) + \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) + \delta\hat{\psi}(\mathbf{y})\delta\hat{\psi}(\mathbf{x})\Big) \\ + \text{ constant } + \text{cubic } + \text{ quartic.} \end{aligned} \tag{125}$$

Thus altogether,

$$\hat{H} = \text{constant} + \hat{H}_{\text{free}} + \hat{H}_{\text{interactions}} \tag{126}$$

where

$$\begin{aligned} \hat{H}_{\text{free}} = {}& \frac{\hbar^2}{2m}\int d^3\mathbf{x}\,\nabla\delta\hat{\psi}^\dagger(\mathbf{x})\cdot\nabla\delta\hat{\psi}(\mathbf{x}) \\ &+ \frac{\bar{n}_2}{2}\int d^3\mathbf{x}\int d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times\Big(2\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) + \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) + \delta\hat{\psi}(\mathbf{y})\delta\hat{\psi}(\mathbf{x})\Big). \end{aligned} \tag{127}$$

This completes the proof of the first part of the Lemma 6 — the top two lines of the eq. (70).

To prove the second part of the Lemma (the bottom line of eq. (70)) we simply Fourier transform from the shifted creation and annihilation fields to the creation and annihilation operators for atoms with specific momenta $\mathbf{k}\neq 0$ (and ignore the shifted operators for the $\mathbf{k}=0$ mode)

$$\delta\hat{\psi}^\dagger(\mathbf{x}) = L^{-3/2}\sum_{\mathbf{k}\neq 0} e^{+i\mathbf{k}\mathbf{x}}\hat{a}^\dagger_{\mathbf{k}}\,, \qquad \delta\hat{\psi}(\mathbf{x}) = L^{-3/2}\sum_{\mathbf{k}\neq 0} e^{-i\mathbf{k}\mathbf{x}}\hat{a}_{\mathbf{k}}\,. \tag{128}$$

Consequently,

$$
\begin{aligned}
\int d^3\mathbf{x}\int d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;=&\\
&= \int d^3\mathbf{x}\int d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times L^{-3}\sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{k}\mathbf{x}-i\mathbf{k}'\mathbf{y}}\times\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}'}\\
&= \sum_{\mathbf{k},\mathbf{k}'}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}'}\times L^{-3}\int d^3\mathbf{x}\int d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times e^{i\mathbf{k}\mathbf{x}-i\mathbf{k}'\mathbf{y}}
\end{aligned}
\tag{129}
$$

where

$$
\begin{aligned}
L^{-3}\int d^3\mathbf{x}\int d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times e^{i\mathbf{k}\mathbf{x}-i\mathbf{k}'\mathbf{y}} \;=&\\
&= \; L^{-3}\int d^3\mathbf{y}\int d^3(\mathbf{z}=\mathbf{x}-\mathbf{y})\,V_2(\mathbf{z})\times e^{i\mathbf{k}(\mathbf{y}+\mathbf{z})-i\mathbf{k}'\mathbf{y}}\\
&= \int d^3\mathbf{z}\,V_2(\mathbf{z})e^{i\mathbf{k}\mathbf{z}}\times L^{-3}\int\limits_{\text{box}} d^3\mathbf{y}\,e^{i\mathbf{k}\mathbf{y}-i\mathbf{k}'\mathbf{y}}\\
&= \; W(\mathbf{k})\times\delta_{\mathbf{k},\mathbf{k}'}\,,
\end{aligned}
\tag{130}
$$

hence

$$
\int d^3\mathbf{x}\int d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;=\; \sum_{\mathbf{k}} W(\mathbf{k})\times\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,. \tag{131}
$$

In the same way we obtain

$$
\begin{aligned}
\int d^3\mathbf{x}\int d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) \;=&\\
&= \int d^3\mathbf{x}\int d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times L^{-3}\sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{k}\mathbf{x}-i\mathbf{k}'\mathbf{y}}\times\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\\
&= \sum_{\mathbf{k}} W(\mathbf{k})\times\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}
\end{aligned}
\tag{132}
$$

and likewise

$$
\int d^3\mathbf{x}\int d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times\delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;=\; \sum_{\mathbf{k}} W(\mathbf{k})\times\hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}\,. \tag{133}
$$

Combining all these formulae with the gradient term in the Hamiltonian (127),

$$\hat{K} \ = \ \frac{\hbar^2}{2m}\int d^3\mathbf{x}\, \nabla\psi^\dagger \cdot \nabla\psi \ = \ \frac{\hbar^2}{2m}\int d^3\mathbf{x}\, \nabla\delta\psi^\dagger \cdot \nabla\delta\psi \ = \sum_{\mathbf{k}} \frac{\hbar^2\mathbf{k}^2}{2m}\, \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,, \tag{134}$$

we finally assemble all quadratic terms to

$$\hat{H}_{\mathrm{free}} \ = \sum_{\mathbf{k}} \left(\left(\frac{\hbar^2\mathbf{k}^2}{2m} + W(\mathbf{k})\bar{n}_s \right) \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \ + \ \tfrac{1}{2}W(\mathbf{k})\bar{n}_s \big(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} + \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}} \big) \right) . \tag{135}$$

Quod erat demonstrandum.