SUPERCONDUCTIVITY

Superconductors are macroscopic systems that behave in some essentially quantum ways; many useful devices such as very sensitive magnetometers (SQUIDs) are based on such quantum features. The microscopic theory of superconductivity is quite complicated and took many years to develop; however, the macroscopic theory of superconductivity is much easier. The goal of these notes (and the exercises contained in them) is to give you a basic understanding of some of the phenomena involved.

For simplicity, let me focus on the superconducting metals rather than the high–T_c superconducting ceramics. In such metals, the electron-phonon interaction cause attractive forces between electrons with energies close to the Fermi surface, and at low temperatures a small fraction of these electrons form Cooper pairs — bound states of two electrons with near-opposite momenta and opposite spins. On the whole, a Cooper pair is a slowly moving spinless boson of electric charge $-2e$; it is the presence of these charged bosons that gives rise to superconductivity. Or rather, it's the Bose–Einstein condensate of the Cooper pairs which gives rise to the superconductivity. Indeed, the Cooper pairs hardly exist outside this condensate: the excitations of the superconductor's ground state break the pairs into individual electrons rather than kick a pair out of the condensate but keep it unbroken.

At the phenomenological level, we may describe the BEC of Cooper pairs by the Landau–Ginzburg complex classical field $\Psi(\mathbf{x},t)$, which has Hamiltonian function

$$H[\Psi,\Psi^*] \;=\; \int\! d^3\mathbf{x} \left(\frac{\hbar^2}{2M} \left|\vec{\mathcal{D}}\Psi\right|^2 \,+\, \frac{\lambda}{2}\,|\Psi|^4 \,-\, \mu\,|\Psi|^2 \right) \tag{1}$$

and obeys a Schrödinger-like but non-linear field equation

$$i\hbar D_t\Psi \;=\; -\,\frac{\hbar^2}{2M}\,\vec{\mathcal{D}}^2\Psi +\, \left(\lambda|\Psi|^2 - \mu\right)\Psi. \tag{2}$$

In these formulae, $M \neq 2m_e$ is the effective mass of a Cooper pair,

$$\vec{\mathcal{D}} \;=\; \nabla \,+\, \frac{2ei}{\hbar c}\mathbf{A}(\mathbf{x}) \quad \text{and} \quad \mathcal{D}_t \;=\; \frac{\partial}{\partial t} \,-\, \frac{2ei}{\hbar}\Phi(\mathbf{x}) \tag{3}$$

are the covariant derivatives for the electric charge $= -2e$, μ is the chemical potential, and λ parametrizes the short-distance repulsive forces between the Cooper pairs.

Heuristically, we may think of the BEC as having all Cooper pairs being in the same single-particle quantum state with a wave function $\psi(\mathbf{x}, t)$; in terms of this wave-function, the Landau–Ginzburg field is simply

$$\Psi(\mathbf{x}, t) \;=\; \sqrt{N_{\text{pairs}}} \times \psi(\mathbf{x}, t), \tag{4}$$

where the $\sqrt{N_{\text{pairs}}}$ factor makes $n_s = |\Psi|^2$ the local density of the Cooper pair condensate. The non-linear term in the field equation (2) stems from the Mean Field Theory approximation to the interactions between the pairs. That is, we neglect the rather weak interactions between individual pairs, but the collective effect of all the other pairs on any one pair gives rise to an effective potential

$$\mathcal{V}(\mathbf{x}, t) \;\approx\; \lambda|\Psi(\mathbf{x}, t)|^2 \;-\; \mu. \tag{5}$$

Combining this mean-field effective potential with the macroscopic electric and magnetic forces on a charged Cooper pair gives rise to the Schrödinger equation

$$i\hbar\mathcal{D}_t\psi \;=\; -\frac{\hbar^2}{2M}\vec{\mathcal{D}}^2\psi \;+\; \mathcal{V}\psi \tag{6}$$

for the wave function of a pair, and hence eq. (2) for the Landau–Ginzburg field.

The actual quantum origin of the semi-classical Landau–Ginzburg theory is a lot more complicated. In my extra lecture on 10/6, — *cf.* my notes on the subject — I explained how the LG theory of a superfluid emerges as classical limit of a quantum field theory of an arbitrary number of atoms, and the intermediate steps are quite complicated: The multi-boson Hilbert spaces, the creation and the annihilation operators, the quantum fields, the coherent state of the $\mathbf{k} = 0$ mode for the atoms, and the effect this coherent state has on the ground states of the other modes as well as their excited states. For the superconductivity, one goes through the similar hurdles, plus the the effect of the coherent state of on the unpaired electrons near the Fermi surface as well as on the other modes of the Cooper pairs. But at the end of the day, we end up with a similar semiclassical Landau–Ginzburg field theory, except for the covariant space derivatives (3) in the superconducting case.

Electric charges and currents

In the Landau–Ginzburg theory, the number density of Cooper pairs is $n_s = |\Psi|^2$ while their *kinematic* momentum density is

$$\vec{\mathcal{P}} = M n_s \mathbf{v} = \hbar\,\mathrm{Im}\big(\Psi^* \vec{\mathcal{D}} \Psi\big); \tag{7}$$

note the covariant gradient $\mathbf{D}$ in this formula. Consequently, the electric charge density and the electric current density due to superconducting Cooper pairs are

$$\rho_s = -2en_s = -2e\,|\Psi|^2\,, \tag{8}$$
$$\mathbf{J}_s = -2en_s\mathbf{v} = \frac{-2e}{M}\vec{\mathcal{P}} = -\frac{2e\hbar}{M}\,\mathrm{Im}\big(\Psi^* \vec{\mathcal{D}} \Psi\big). \tag{9}$$

Exercise (a):

Verify that these superconducting charge density and current density obey the continuity equation

$$\frac{\partial}{\partial t}\rho_s(\mathbf{x},t) + \nabla\cdot\mathbf{J}_s(\mathbf{x},t) = 0. \tag{10}$$

Hint: First prove that $\nabla\cdot\mathrm{Im}(\Psi^*\vec{\mathcal{D}}\Psi) = \mathrm{Im}(\Psi^*\vec{\mathcal{D}}^2\Psi)$, then use the field equation (2) for the LG field $\Psi(\mathbf{x},t)$.

But of course, the Cooper pair BEC is not the only charged ingredient in a superconductor, there is also a Fermi gas of normal (non-superconducting) electrons and the lattice of ion cores. Consequently, the net charge and current densities in a superconductor are

$$\begin{aligned} \rho_{\mathrm{net}} &= \rho_{\mathrm{ion}} + \rho_n + \rho_n\,, \\ \mathbf{J}_{\mathrm{net}} &= \mathbf{J}_n + \mathbf{J}_{\mathbf{s}}\,. \end{aligned} \tag{11}$$

Moreover, in all practical situations the net electric charge density in a superconducting metal is zero, $\rho_{\mathrm{net}} = 0$. On the other hand, the normal and the superconducting currents do not cancel each other. Instead, as long as superconductivity exists, the supercurrent $\mathbf{J}_s$ flows without resistance and shorts out the electric field $\mathbf{E} \to 0$, so by the Ohm's law $\mathbf{J}_n = \sigma\mathbf{E}$, the normal current does not flow. So the bottom line is: In a superconductor

$$\rho_{\mathrm{net}} = 0 \quad \text{but} \quad \mathbf{J}_{\mathrm{net}} = \mathbf{J}_s\,. \tag{12}$$

Exercise (b):

Let's describe the complex Landau–Ginzburg field in terms of its magnitude and phase as

$$\Psi(\mathbf{x},t) \;=\; \sqrt{n_s(\mathbf{x},t)}\,\exp(iS(\mathbf{x},t)/\hbar). \tag{13}$$

Show that in terms of these variables, the supercurrent becomes

$$\mathbf{J}_s(\mathbf{x},t) \;=\; \frac{-2en_s}{M}\left(\nabla S(\mathbf{x},t) \,+\, \frac{2e}{c}\mathbf{A}(\mathbf{x},t)\right). \tag{14}$$

The explicit presence of the vector potential in this formula gives rise to some rather spectacular effects. In particular, the magnetic field $\vec{B}$ cannot penetrate a bulk superconductor much beyond a certain depth. This is the *Meissner effect* and it's exhibited by all superconductors in weak magnetic fields; strong magnetic fields destroy the superconductivity.

Exercise (c): Assume uniform $n_s \neq 0$ for a bulk superconductor and a time-independent magnetic field $\mathbf{B}(\mathbf{x})$. Use Maxwell's equations together with eq. (14) for the supercurrent and show that the magnetic field in the superconductor obeys

$$\left(\vec{\nabla}^2 - \ell^{-2}\right)\mathbf{B}(\mathbf{x}) \;=\; 0 \tag{15}$$

and hence cannot penetrate the superconductor to a depth much beyond the so-called *London's penetration depth* ℓ. Compute ℓ in terms of Cooper pair density n and whatever constants you may need.

The Meissner effect leads to many other interesting phenomena, such as magnetic flux quantization. Indeed, consider a closed loop of superconducting wire: If the wire is thick enough to expel the magnetic field from its interior, the supercurrent would also be expelled from the wire's interior and flow through the wire's skin only. Hence, in the wire's interior

$\nabla S + \frac{2e}{c}\mathbf{A} = 0$; integrating this equation along the wire's center-line gives us

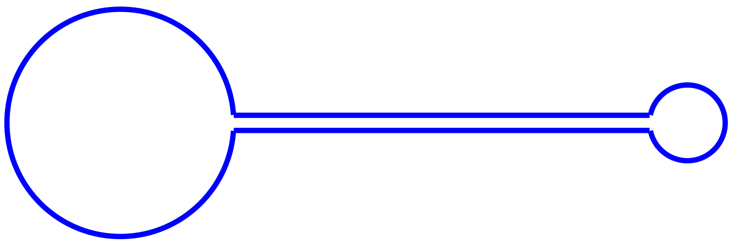
$$\oint_{\text{wire}} \mathbf{A}\cdot d\mathbf{x} \;=\; -\frac{c}{2e}\,\Delta S. \tag{16}$$

The left hand side of this equation is the magnetic flux F through the wire loop. The right hand side of eq. (16) involves the accumulated change of the phase $S/\hbar$ of the LG field; for a closed loop this total phase change must be an integer multiple of 2π. Therefore, eq. (16) tells us that *magnetic flux through a closed loop of a superconducting wire must be an integer multiple of*

$$F_0 \;=\; \frac{2\pi\hbar c}{2e} \tag{17}$$

This flux quantization condition is closely related to the Aharonov-Bohm effect.

Magnetic flux quantization is used in superconducting devices such as a magnetic amplifier, which is basically a loop of superconducting wire that looks like



(18)

Since the total flux through both loops is quantized, it cannot be changed adiabatically. Therefore, small adiabatic changes of the magnetic field going through the big loop result in much bigger changes of the field in the small loop. The amplification factor is given by (minus) the area ratio.

Josephson junctions and SQUIDS

A Josephson's junction is a week link in a superconducting wire. It can be a sharp point contact between two wires, or a very thin dielectric film separating two thick films of superconducting metal, or some other obstacle through which the Cooper pairs have to tunnel in order to get from one side of the junction to the other.

In the Landau-Ginsburg description, the junction appears as potential barrier: The effective potential $\mathcal{V}(\mathbf{x})$ now acquires an additional term $\Delta\mathcal{V}(\mathbf{x})$ that vanishes in the interior of the superconductor but become positive (and large, albeit finite) in the junction area, thus

$$\mathcal{V}(\mathbf{x},t) \;=\; \lambda|\Psi(\mathbf{x},t)|^2 \;-\; \mu \;+\; \Delta\mathcal{V}(\mathbf{x}) \;=\; \lambda(|\Psi(\mathbf{x},t)|^2 - n_0) \;+\; \Delta\mathcal{V}(\mathbf{x}). \tag{19}$$

where $n_0 = \mu/\lambda$ is the Cooper pair density in the bulk superconductor. Consequently, the stationary form of the Landau-Ginsburg equation (2) becomes

$$\frac{-\hbar^2}{2M}\vec{\mathcal{D}}^2\Psi(\mathbf{x}) \;+\; \lambda\big(|\Psi(\mathbf{x})|^2 - n_0\big)\Psi(\mathbf{x}) \;+\; \Delta\mathcal{V}(\mathbf{x})\Psi(\mathbf{x}) \;=\; 0 \tag{20}$$

Let's solve this equation — or rather get a general idea what the solution looks like — for the case of no magnetic field $\mathbf{B} = 0$. For simplicity, let's fix the gauge $\mathbf{A} = 0$, hence $\vec{\mathcal{D}}^2 = \nabla^2$, which makes (20) a real equation, but subject to complex boundary conditions:

- In the bulk of the wire#1, $\Psi(\mathbf{x}) \to \sqrt{n_0} \times \exp(i\phi_1)$ for a given phase ϕ_1.
- In the bulk of the wire#2, $\Psi(\mathbf{x}) \to \sqrt{n_0} \times \exp(i\phi_2)$ for a given phase ϕ_2.
- Away from both wires, $\Psi(\mathbf{x}) \to 0$.

Note that were it not for different phases $\phi_1 \neq \phi_2$, the solution of eq. (20) (for $\mathbf{A} = 0$) with these boundary conditions would be real up to some overall phase $e^{i\phi}$. However, as long as the barrier is sufficiently hard to tunnel through, the solution may be approximated as

$$\Psi(\mathbf{x}) \;=\; e^{i\phi_1} \times \Psi_1(\mathbf{x}) \;+\; e^{i\phi_2} \times \Psi_2(\mathbf{x}) \tag{21}$$

for some *real* functions $\Psi_1(\mathbf{x})$ and $\Psi_2(\mathbf{x})$.

Exercise (d):

Derive this formula by splitting the whole volume of the junction into 3 distinct regions:

(**A**) Interior and immediate vicinity of the first superconducting wire; in this region we may ignore the second wire.

(**B**) Interior and immediate vicinity of the second superconducting wire; in this region we may ignore the first wire.

(**C**) Everywhere else, including most of the space between the two wires. In this region, the Cooper pair density $n = |\Psi|^2$ is so small that we may neglect the non-linear term in eq. (20) and simplify it to

$$\frac{-\hbar^2}{2M}\nabla^2\Psi(\mathbf{x}) \;-\; \lambda n_0\Psi(\mathbf{x}) \;+\; \Delta\mathcal{V}(\mathbf{x})\Psi(\mathbf{x}) \;=\; 0. \tag{22}$$

Exercise (e):

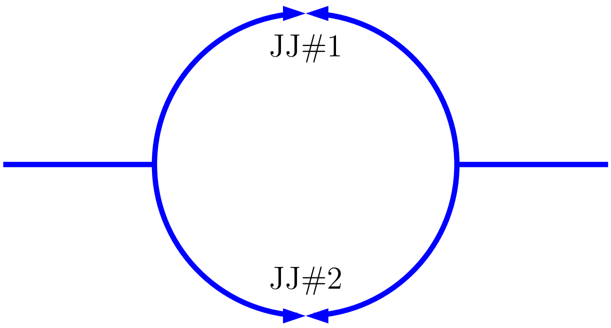
Use eq. (21) to show that the net supercurrent through the Josephson's junction is related to the phase difference $\phi_1 - \phi_2$ as

$$I \;=\; I_0\,\sin(\phi_1 - \phi_2). \tag{23}$$

where I_0 is a constant depending on the functions $\Psi_1(\mathbf{x})$ and $\Psi_2(\mathbf{x})$ and hence on the specifics of the junction's geometry.

Experimentally, I_0 is the maximal supercurrent that tunneling Cooper pairs can carry through the junction. Single electrons can carry a bigger electric current, but it would be a normal current, subject to resistance and thus needing a voltage drop. Moreover, according to the microscopic theory of superconductivity developed by Bardeen, Cooper and Schrieffer, a normal current cannot flow through a Josephson junction until the voltage drop exceeds some threshold value (typically, a few millivolts). Experimentally, this is indeed the case: As one increases the current through a Josephson's junction, the voltage stays exactly zero until the maximal supercurrent I_0 is reached, then suddenly jumps to a few millivolts; after that, it continues to grow with the current.

A **SQUID** is a Superconducting QUantum Interferometry Device. SQUIDs come in many shapes, but the simplest one consists of two Josephson's junctions in a single loop of superconducting wire:



(24)

In the absence of magnetic field, the maximal supercurrent that can flow through a symmetric SQUID is clearly $2I_0$; in the presence of magnetic field things are much more interesting.

Exercise (f): Show that in the presence of a magnetic field, the maximal supercurrent that can flow through the SQUID is

$$I_{\max}(B) \;=\; 2I_0 \left|\cos\frac{\pi F}{F_0}\right| \tag{25}$$

where F is the magnetic flux through the SQUID's loop and $F_0 = 2\pi\hbar c/2e$. Assume that the field is not so strong as to affect the junctions themselves (otherwise, I_0 would also change with the field) but only their interference.

Practically, SQUIDs are used as very sensitive magnetometers: According to eq. (25), tiny changes of the magnetic field through the SQUID's loop result in easily measurable changes in the maximal supercurrent $I_{\max}(B)$. And when even higher sensitivity is needed, one may combine a SQUID with a magnetic amplifier, or with a cascade arrangement of amplifiers; the engineering of magnetic couplings between SQUIDs and amplifier loops is tricky, but the physics is quite straightforward.