Wigner–Eckart Theorem

The Wigner–Eckart theorem concerns the matrix elements of scalar, vector, and tensor operators between states $|\alpha, j, m\rangle$ of definite angular momentum — or rather, definite J_z and definite $\mathbf{J}^2$. Most generally, the theorem gives us

$$\left\langle \alpha', j', m' \right| \hat{A}_{\text{indices}} \left| \alpha, j, m \right\rangle \;=\; \left\langle \alpha', j' \right|\right| \hat{A} \left|\left| \alpha, j \right\rangle \times \text{a\_known\_function\_of}(j, m, j', m', \text{indices}) \tag{1}$$

where 'indices' denotes 0, 1, or more vector indices of a scalar, vector, or tensor operator $\hat{A}_{\text{indices}}$, and $\langle \alpha', j' || \hat{A} || \alpha, j\rangle$ is the *reduced matrix element* which does not depend on m', m, or the indices of $\hat{A}$. As to the 'known function' in eq. (1), I shall spell it out in terms of the Clebbsch–Gordan coefficients later in these notes.

The Wigner–Eckart theorem drastically reduces the number of independent matrix elements one may need to calculate the hard way. For example, consider the transition in which the outer-most electron of an aluminum atom moves from the excited $L = 2$ state down to the ground $L = 1$ state while emitting a photon. The transition rate here depends on the matrix element of the electric dipole operator $\hat{\mathbf{d}}$, between the two states,

$$\left\langle \text{ground}, L' = 1, m'_L \right| \hat{d}_i \left| \text{excited}, L = 2, m_L \right\rangle \tag{2}$$

Alas, the initial state here can be any one of 5 degenerate states with $L = 2$ (but different m_L); likewise, the final state can be any one of 3 degenerate ground states with $L' = 1$; and the electric dipole vector has 3 components $\hat{d}_i$, so altogether there are $5 \times 3 \times 3 = 45$ matrix elements for the transition in question. For simplicity I ignore the spin here; otherwise we would need even more matrix elements. But even "only" 45 matrix elements are way too much work to calculate as explicit integrals. But thanks to the Wigner–Eckart theorem we do not need to work quite this hard. Instead, it's enough to calculate any one non-vanishing matrix element (2) the hard way, and the eq. (1) yields the reduced matrix element $\langle \text{ground}, L' = 1 || \hat{d} || \text{excited}, L = 2\rangle$ and hence all 45 matrix elements (2) for the transition in question.

Scalar Case

Let's start with the matrix elements of a scalar operator $\hat{A}$. The scalar operators commute with $\hat{\mathbf{J}}$ and $\hat{\mathbf{J}}^2$, so the state $\hat{A}\,|\alpha,j,m\rangle$ has exactly the same j and m as the original $|\alpha,j,m\rangle$ state. Consequently,

$$\langle\alpha',j',m'|\,\hat{A}\,|\alpha,j,m\rangle \;=\; 0 \quad \text{unless } j'=j \text{ and } m'=m. \tag{3}$$

Moreover, since the rotation operators $\hat{\mathcal{R}}(\mathbf{n},\phi)$ mix up states with different m's, the matrix element here does not depend on $m'=m$ (as long as $m'=m$), thus

$$\langle\alpha',j',m'|\,\hat{A}\,|\alpha,j,m\rangle \;=\; \delta_{j',j}\delta_{m',m}\times \text{function\_of}(\alpha',\alpha,j,\ \text{but not } m). \tag{4}$$

Altogether, there is a *selection rule* — the matrix element of a scalar operator vanishes unless $j'=j$, — and for any given α',α, and $j'=j$, all the $(2j+1)^2$ matrix elements corresponding to different m and m' are related via eq. (4).

Finally, the function of α',α, and $j'=j$ on the RHS of eq. (4) is called the *reduced matrix element* $\langle\alpha',j||\,\hat{A}\,||\alpha,j\rangle$ of the scalar operator $\hat{A}$. Or rather, in some conventions it's called the reduced matrix element, while in other conventions — like in the Sakurai's textbook — there is an extra factor of $\sqrt{2j+1}$, thus

$$\langle\alpha',j',m'|\,\hat{A}\,|\alpha,j,m\rangle \;=\; \delta_{j',j}\delta_{m',m}\times\frac{\langle\alpha',j||\,\hat{A}\,||\alpha,j\rangle}{\sqrt{2j+1}}\,. \tag{5}$$

Vector Case

Next, consider the matrix elements of the vector operators $\hat{\mathbf{V}}$. The Cartesian components $\hat{V}_{x,y,z}$ of a vector operator obey $[\hat{J}_i,\hat{V}_j]=i\hbar\epsilon_{i,j,k}\hat{V}_k$, but for our purposes it's more convenient to use the so-called spherical components

$$\hat{V}_{-1} \;=\; \frac{+\hat{V}_x-i\hat{V}_y}{\sqrt{2}}\,,\quad \hat{V}_0 \;=\; \hat{V}_z\,,\quad \hat{V}_{+1} \;=\; \frac{-\hat{V}_x-i\hat{V}_y}{\sqrt{2}}\,, \tag{6}$$

inspired by the $\ell=1$ spherical harmonics,

$$\sqrt{\frac{4\pi}{3}}\,rY_{1,-1}(\theta,\phi) \;=\; \frac{+x-iy}{\sqrt{2}}\,,\quad \sqrt{\frac{4\pi}{3}}\,rY_{1,0}(\theta,\phi) \;=\; z,\quad \sqrt{\frac{4\pi}{3}}\,rY_{1,+1}(\theta,\phi) \;=\; \frac{-x-iy}{\sqrt{2}}\,. \tag{7}$$

The commutators of the spherical components $\hat{V}_\mu$ ($\mu=-1,0,+1$) with the angular momentum operators $\hat{\mathbf{J}}$ exactly parallels the action of the $\hat{\mathbf{J}}$ operators on the $|j,m\rangle$ states with $j=1$:

$$\begin{aligned}
[\hat{J}_z,\hat{V}_\mu] &= \mu\hbar\hat{V}_\mu\,, & \hat{J}_z\,|j=1,m\rangle &= m\hbar\,|j=1,m\rangle\,,\\
[\hat{J}_\pm,\hat{V}_{\pm1}] &= 0, & \hat{J}_\pm\,|j=1,m=\pm1\rangle &= 0,\\
[\hat{J}_\pm,\hat{V}_0] &= \sqrt{2}\hbar\hat{V}_{\pm1}, & \hat{J}_\pm\,|j=1,m=0\rangle &= \sqrt{2}\hbar\,|j=1,m=\pm1\rangle\,,\\
[\hat{J}_\pm,\hat{V}_{\mp1}] &= \sqrt{2}\hbar\hat{V}_0\,, & \hat{J}_\pm\,|j=1,m=\mp1\rangle &= \sqrt{2}\hbar\,|j=1,m=0\rangle
\end{aligned} \tag{8}$$

(The proof is a part of your homework set#12). Consequently, for any unit vector $\mathbf{n}$

$$[(\mathbf{n}\cdot\hat{\mathbf{J}}),\hat{V}_\mu] \;=\; \sum_{\mu'}\hat{V}_{\mu'}\times\left\langle j=1,\mu'\right|(\mathbf{n}\cdot\hat{\mathbf{J}})\,|j=1,\mu\rangle \tag{9}$$

and hence for any finite rotation $R(\phi,\mathbf{n})$

$$\hat{\mathcal{R}}(\phi,\mathbf{n})\hat{V}_\mu\hat{\mathcal{R}}^\dagger(\phi,\mathbf{n}) \;=\; \sum_{\mu'}\hat{V}_{\mu'}\times\mathcal{D}^{(j=1)}_{\mu',\mu}(\phi,\mathbf{n}). \tag{10}$$

(Again, the proof is a part of the homework set#12).

Next, consider the vector operator $\hat{V}_\mu$ acting on states with definite j and m, $\hat{V}_\mu\,|\alpha,j,m\rangle$. When we rotate this state, we get

$$\begin{aligned}
\hat{\mathcal{R}}(\phi,\mathbf{n})\hat{V}_\mu\,|\alpha,j,m\rangle &= \hat{\mathcal{R}}(\phi,\mathbf{n})\hat{V}_\mu\hat{\mathcal{R}}^\dagger(\phi,\mathbf{n})\times\hat{\mathcal{R}}(\phi,\mathbf{n})\,|\alpha,j,m\rangle\\
&= \left(\sum_{\mu'}\hat{V}_{\mu'}\mathcal{D}^{(1)}_{\mu',\mu}(\phi,\mathbf{n})\right)\times\left(\sum_{m'}\left|\alpha,j,m'\right\rangle\mathcal{D}^{(j)}_{m',m}(\phi,\mathbf{n})\right)\\
&= \sum_{\mu',m'}\hat{V}_{\mu'}\left|\alpha,j,m'\right\rangle\times\mathcal{D}^{(1)}_{\mu',\mu}(\phi,\mathbf{n})\mathcal{D}^{(j)}_{m',m}(\phi,\mathbf{n})\,.
\end{aligned} \tag{11}$$

This transformation law under finite rotation looks exactly like transformation of two simultaneously rotated sets degrees of freedom, in multiplets $|j_1,m_1,j_2,m_2\rangle$ where for the case at hand $j_1=1$, $m_1=\mu$, $j_2=j$, and $m_2=m$. Consequently, from the net angular momentum point of view the state $\hat{V}_\mu\,|\alpha,j,m\rangle$ is a linear combination of states with $m_{\text{net}}=\mu+m$ while j_{net} runs from $|j-1|$ to $j+1$ by 1. Moreover, the coefficient of any particular j_{net}

component of this state is is the usual Clebbsch–Gordan coefficient $C(j_{\rm net},m_{\rm net}|1,\mu,j,m)$ for adding angular momenta $j_1=1$ and $j_2=j$. Although once we get the non-rotational degrees of freedom represented by α's into account, we get a more general formula

$$\hat{V}_\mu\,|\alpha,j,m\rangle \;=\; \sum_{j'=|j-1|}^{j+1}\sum_{\alpha'}\big|\alpha',j',m'=\mu+m\big\rangle\times C(j',m'|1,\mu,j,m)\times \\ \times\,{\rm function}(\alpha',j',\alpha,j\ \text{but not}\ m',\mu,m). \tag{12}$$

Or in terms of the matrix elements of the vector operator $\hat{\mathbf{V}}$,

$$\big\langle\alpha',j',m'\big|\,\hat{V}_\mu\,|\alpha,j,m\rangle \;=\; \frac{C(j',m'|1,\mu,j,m)}{\sqrt{2j'+1}}\times\big\langle\alpha',j'\big|\big|\,\hat{\mathbf{V}}\,||\alpha,j\rangle\,, \tag{13}$$

where the reduced matrix element $\langle\alpha',j'||\,\hat{\mathbf{V}}\,||\alpha,j\rangle$ depends on the multiplets (α,j) and (α',j') and on the vector operator $\hat{\mathbf{V}}$ but does not depend on the multiplet members m or m' of the specific component μ of the vector operator.

Also, the Clebbsch–Gordan coefficient in eq. (13) enforces the selection rules: the matrix elements of vector operators vanish unless $|j-1|\leq j'\leq j+1$ — or equivalently $|j'-j|=(0\text{ or }1)$ and $j'+j\geq 1$, — and also $m'=\mu+m$.

Spherical Tensors

Similar to the scalar and vector operators, there are Wigner–Eckart theorems for the tensor operators $\hat{T}_{ij}$, $\hat{T}_{ijk}$, *etc.*, with any numbers of indices. However, such tensors form reducible multiplets of the rotation group $SO(3)$, so they need to be reorganized into irreducible multiplets, and then each irreducible multiplet re-expressed in terms of spherical rather than Cartesian components.

For example, the irreducible parts of a general two-index tensor T_{ij} are:

1. The trace $S={\rm tr}(T)=T_{ii}$ (implicit $\sum_i$), which is a scalar.
2. The antisymmetric part $T^A_{ij}=\frac{1}{2}(T_{ij}-T_{ji})=-T^A_{ji}$, which is equivalent to a vector $V_i=\frac{1}{2}\epsilon_{ijk}T^A_{jk}=\frac{1}{2}\epsilon_{ijk}T_{jk}$, $T^A_{ij}=\epsilon_{ijk}V_k$.

3. The traceless symmetric part,

$$T^S_{ij} \;=\; \tfrac{1}{2}(T_{ij}+T_{ji}) \;-\; \tfrac{1}{3}\delta_{ij}\,\mathrm{tr}(T), \quad T^S_{ij} \;=\; +T^S_{ji}\,, \quad tr(T^S) \;=\; 0. \tag{14}$$

For the higher-rank tensors, their irreducible parts are also scalars, vectors, and traceless symmetric tensors of various ranks. For example, a general 3-index tensor T_{ijk} decomposes into a 3-index traceless totally-symmetric tensor, 2 different 2-index traceless symmetric tensors, 3 different vectors, and one scalar.

Next, consider the spherical harmonics, or rather $f_{\ell,m}(\mathbf{x}) = r^\ell Y_{\ell,m}(\theta,\phi)$. In terms of Cartesian coordinates x, y, z, each $f_{\ell,m}$ is a homogeneous polynomial of degree ℓ:

$$f_{\ell,m}(\mathbf{x}) \;=\; r^\ell Y_{\ell,m}(\theta,\phi) \;=\; T^{i_1,i_2,\ldots,i_\ell} x^{i_1} x^{i_2} \cdots x^{i_\ell}, \tag{15}$$

where the coefficients $T^{i_1,i_2,\ldots,i_\ell}$ form a totally symmetric ℓ–index tensor. Moreover, all such tensors are traceless since the $f_{\ell,m}(\mathbf{x})$ obey the Laplace equation:

$$0 \;=\; \nabla^2 f_{\ell,m}(\mathbf{x}) \;=\; \ell(\ell-1)T^{j,j,i_3,\ldots,i_\ell}\, x^{i_3}\cdots x^{i_\ell} \quad\Longrightarrow\quad T^{j,j,i_3,\ldots,i_\ell} \;=\; 0. \tag{16}$$

Also, the number of independent components of such an ℓ–index traceless symmetric tensor is $(2\ell+1)$, same as the number of independent spherical harmonics for a given ℓ. Indeed, in 3D a generic totally symmetric ℓ–index tensor has $\binom{\ell+2}{2} = \frac{1}{2}(\ell+1)(\ell+2)$ independent components, while the zero trace condition amounts to $\frac{1}{2}(\ell-2+1)(\ell-2+2)$ linear constraints, so a traceless symmetric tensor has only

$$\tfrac{1}{2}(\ell+1)(\ell+2) \;-\; \tfrac{1}{2}(\ell-2+1)(\ell-2+2) \;=\; 2\ell \;+\; 1 \tag{17}$$

independent components. Thus, eq. (15) provides a one-to-one map between the Cartesian components $T^{i_1,\ldots,i_\ell}$ and the spherical components $\mu = -\ell,\ldots,+\ell$. For example, for $\ell = 2$

$$\begin{aligned}
\sqrt{\frac{8\pi}{15}}\, r^2 Y_{2,\pm2}(\theta,\phi) \;&=\; \tfrac{1}{2}(\mp x - iy)^2\,, \\
\sqrt{\frac{8\pi}{15}}\, r^2 Y_{2,\pm1}(\theta,\phi) \;&=\; (\mp x - iy)z, \\
\sqrt{\frac{8\pi}{15}}\, r^2 Y_{2,0}(\theta,\phi) \;&=\; \sqrt{\tfrac{1}{6}}\,\big(2z^2 - x^2 - y^2\big),
\end{aligned} \tag{18}$$

so for any traceless symmetric 2-index tensor T_{ij} we define its spherical components as

$$\begin{aligned} T^{(2)}_{\pm 2} &= \tfrac{1}{2}(T_{xx} - T_{yy}) \ \pm\ iT_{xy}\,, \\ T^{(2)}_{\pm 1} &= \mp T_{xz} \ -\ iT_{yz}\,, \\ T^{(2)}_{0} &= \sqrt{\tfrac{1}{6}}\,\bigl(2T_{zz} - T_{xx} - T_{yy}\bigr). \end{aligned} \tag{19}$$

The bottom line of this exercise is that any ordinary tensor is equivalent to one or several *spherical tensors* $T^{(\ell)}_m$ which transform under rotations like the spherical harmonics $Y_{\ell,m}(\theta,\phi)$, namely

$$R : T^{(\ell)}_m \ \to\ \sum_{m'} \Bigl(\mathcal{D}^{(\ell)}_{m,m'}(R)\Bigr)^* \times T^{(\ell)}_{m'}\,, \tag{20}$$

similarly to

$$\begin{aligned} Y_{\ell,m}(R\mathbf{n}) &= \langle R\mathbf{n}|\ell,m\rangle \ = \ \langle \mathbf{n}|\,\hat{\mathcal{R}}^\dagger(R) = \hat{\mathcal{R}}(R^{-1})\,|\ell,m\rangle \\ &= \sum_{m'} \bigl\langle \mathbf{n}|\ell,m'\bigr\rangle \times \mathcal{D}^{(\ell)}_{m',m}(R^{-1}) \\ &= \sum_{m'} \bigl\langle \mathbf{n}|\ell,m'\bigr\rangle \times \Bigl(\mathcal{D}^{(\ell)}(R)\Bigr)^\dagger_{m',m} \\ &= \sum_{m'} \Bigl(\mathcal{D}^{(\ell)}_{m,m'}(R)\Bigr)^* Y_{\ell,m'}(\mathbf{n}). \end{aligned} \tag{21}$$

Tensor Operators

A spherical tensor operator $\hat{T}^{(k)}$ has $2k+1$ components $\hat{T}^{(k)}_\mu$ which transform into each other under rotations according to

$$\hat{\mathcal{R}}^\dagger(R)\hat{T}^{(k)}\hat{\mathcal{R}}(R) \ = \ \sum_{\mu'} \Bigl(\mathcal{D}^{(k)}_{\mu,\mu'}(R)\Bigr)^* \times \hat{T}^{(k)}_{\mu'}\,, \tag{22}$$

or in reverse direction

$$\hat{\mathcal{R}}(R)\hat{T}^{(k)}\hat{\mathcal{R}}^\dagger(R) \ = \ \sum_{\mu'} \hat{T}^{(k)}_{\mu'} \times \mathcal{D}^{(k)}_{\mu',\mu}(R), \tag{23}$$

exactly as the quantum states in a $(j=k)$ multiplet,

$$\hat{\mathcal{R}}(R)\,|j=k,m=\mu\rangle \;=\; \sum_{\mu'} \big|j=k,m=\mu'\big\rangle \times \mathcal{D}^{(k)}_{\mu',\mu}(R). \tag{24}$$

By the Baker–Hausdorff lemma, the transformation laws (23) are equivalent to the commutation relations with the angular momenta,

$$\begin{aligned} \big[\hat{J}_z,\hat{T}^{(k)}_\mu\big] \;&=\; \mu\hbar\,\hat{T}^{(k)}_\mu\,, \\ \big[\hat{J}_\pm,\hat{T}^{(k)}_\mu\big] \;&=\; \sqrt{(k\mp\mu)(k+1\pm\mu)}\hbar\,\hat{T}^{(k)}_{\mu\pm1}\,, \end{aligned} \tag{25}$$

similarly to the action of the $\hat{\mathbf{J}}$ operators on $|j,m\rangle$ states in a $(j=k)$ multiplet,

$$\begin{aligned} \hat{J}_z\,|j=k,m=\mu\rangle \;&=\; \mu\hbar\,|j=k,m=\mu\rangle\,, \\ \hat{J}_\pm\,|j=k,m=\mu\rangle \;&=\; \sqrt{(k\mp\mu)(k+1\pm\mu)}\hbar\,|j=k,m=\mu\pm1\rangle\,. \end{aligned} \tag{26}$$

When the spherical tensor operators $\hat{T}^{(k)}_\mu$ act on the $|\alpha,j,m\rangle$ states, the resulting states transform under rotations R according to

$$\begin{aligned} \hat{\mathcal{R}}(R)\hat{T}^{(k)}_\mu\,|\alpha,j,m\rangle \;&=\hat{\mathcal{R}}(R)\hat{T}^{(k)}_\mu\hat{\mathcal{R}}^\dagger \times \hat{\mathcal{R}}\,|\alpha,j,m\rangle \\ &=\left(\sum_{\mu'}\hat{T}^{(k)}_{\mu'}\,\mathcal{D}^{(k)}_{\mu',\mu}(R)\right)\times\Big(\big|\alpha,j,m'\big\rangle\,\mathcal{D}^{(j)}_{m',m}(R)\Big) \\ &=\sum_{\mu',m'}\hat{T}^{(k)}_{\mu'}\,\big|\alpha,j,m'\big\rangle\times\mathcal{D}^{(k)}_{\mu',\mu}(R)\,\mathcal{D}^{(j)}_{m',m}(R), \end{aligned} \tag{27}$$

which works exactly like a $(j_1=k)\times(j_2=j)$ multiplet of two separate (but simultaneously rotated) sets of degrees of freedom. Similar to what we had in eq. (11) for the vector operators, this reducible multiplet splits into irreducible multiplets of definite $j_{\rm net}$ ranging from $|j-k|$ to $j+k$, and the decomposition of states with definite μ and m into states with definite $j_{\rm net}$ and $m_{\rm net}=\mu+m$ follows the Clebbsch–Gordan coefficients $C(j_{\rm net},m_{\rm net}|j_1=k,m_1=\mu,j_2=j,m_2=m)$. Allowing for an overall (μ,m)-independent coefficient — as well as changing $\alpha \to \alpha'$ — due to a non-trivial action of the tensor operator $\hat{T}^{(k)}$, we end up with

$$\hat{T}^{(k)}_\mu |\alpha, j, m\rangle \;=\; \sum_{j'=|j-k|}^{j+k} \sum_{\alpha'} |\alpha', j', m' = \mu + m\rangle \times C(j', m'|k, \mu, j, m) \times \\ \times \text{function}(\alpha', j', \alpha, j \text{ but not } m', \mu, m). \tag{28}$$

Or in terms of matrix elements of the spherical tensor operator,

$$\langle \alpha', j', m' | \hat{T}^{(k)}_\mu |\alpha, j, m\rangle \;=\; \frac{C(j', m'|k, \mu, j, m)}{\sqrt{2j'+1}} \times \langle \alpha', j' || \hat{T}^{(k)} ||\alpha, j\rangle , \tag{29}$$

where the reduced matrix element $\langle \alpha', j' || \hat{T}^{(k)} ||\alpha, j\rangle$ depends on the multiplets (α, j) and (α', j') and on the vector operator $\hat{\mathbf{V}}$ but does not depend on the multiplet members m or m' of the specific component μ of the tensor operator.

Implicit in the Clebbsch–Gordan coefficient in eq. (29) is the selection rule for the matrix elements of the spherical tensor operator: The matrix element vanishes unless $m' = \mu + m$ while j, j', and k obey the triangle rule,

$$j \,+\, j' \,\geq k, \quad j \,+\, k \,\geq j', \quad j' \,+\, k \,\geq j. \tag{30}$$

Equation (29) is the general form of the Wigner–Eckart theorem, the scalar operators and the vector operators we have discussed earlier in these notes are simply special cases of $k = 0$ for the scalars and $k = 1$ for the vectors.

Projection Theorem

The Wigner–Eckart theorem for the vector operators has a very useful corollary for their matrix elements between states with the same $j' = j$:

$$\langle \alpha', j, m' | \hat{V}_i |\alpha, j, m\rangle \;=\; \frac{\langle \alpha', j, m| (\hat{\mathbf{J}} \cdot \hat{\mathbf{V}}) |\alpha, j, m\rangle}{\hbar^2 j(j+1)} \times \langle j, m' | \hat{J}_i |j, m\rangle . \tag{31}$$

This formula is known as the *projection theorem* because for $\alpha' = \alpha$ and $|\psi\rangle = \sum_m c_m |\alpha, j, m\rangle$

eq. (31) becomes

$$\langle\psi|\,\hat{\mathbf{V}}\,|\psi\rangle \;=\; \frac{\langle\psi|\,(\hat{\mathbf{J}}\cdot\hat{\mathbf{V}})\,|\psi\rangle}{\langle\psi|\,\hat{\mathbf{J}}^2\,|\psi\rangle}\,\langle\psi|\,\hat{\mathbf{J}}\,|\psi\rangle\,, \tag{32}$$

i.e., the expectation value of the vector $\mathbf{V}$ is the projection of $\mathbf{V}$ onto the direction of the angular momentum.

Proof: By the Wigner–Eckart theorem

$$\left\langle\alpha',j,m'\right|\hat{V}^i\left|\alpha,j,m\right\rangle \;=\; \left\langle\alpha',j\right|\!\left|\,\hat{\mathbf{V}}\,\right|\!\left|\alpha,j\right\rangle \times C^i_{m',m}(j) \tag{33}$$

where the $C^i_{m',m}(j)$ factor combines the Clebbsch $C(j,m'|1,\mu,j,m)$, the $1/\sqrt{2j+1}$ factor, and the translation from the spherical basis V_μ for the vectors to the Cartesian basis $V_{x,y,z}$. Eq. (33) apples to matrix elements of any kind of a vector operator, including the angular momentum $\hat{\mathbf{J}}$ itself, thus

$$\left\langle j,m'\right|\hat{J}^i\left|j,m\right\rangle \;=\; \langle j||\,\hat{\mathbf{J}}\,||j\rangle \times C^i_{m',m}(j) \tag{34}$$

for exactly the same matrices $C^i_{m',m}$ as in eq. (33).

Next, for the dot product $\hat{\mathbf{J}}\cdot\hat{\mathbf{V}}$ we have

$$\begin{aligned}
\left\langle\alpha',j,m\right|(\hat{\mathbf{J}}\cdot\hat{\mathbf{V}})\left|\alpha,j,m\right\rangle \;&=\; \left\langle\alpha',j,m\right|\hat{J}^i\hat{V}^i\left|\alpha,j,m\right\rangle \\
&\qquad \langle\!\langle \text{ inserting a complete basis of states between } \hat{J}^i \text{ and } \hat{V}^i \,\rangle\!\rangle \\
&=\; \sum_{\alpha'',j',m'} \left\langle\alpha',j,m\right|\hat{J}^i\left|\alpha'',j',m'\right\rangle \times \left\langle\alpha'',j',m'\right|\hat{V}^i\left|\alpha,j,m\right\rangle \\
&\qquad \langle\!\langle \text{ where the first factor vanishes uness } \alpha''=\alpha' \text{ and } j'=j \,\rangle\!\rangle \\
&=\; \sum_{m'\text{ only}} \left\langle j,m\right|\hat{J}^i\left|j,m'\right\rangle \times \left\langle\alpha',j,m'\right|\hat{V}^i\left|\alpha,j,m\right\rangle \\
&=\; \sum_{m'} \langle j||\,\hat{\mathbf{J}}\,||j\rangle\; C^i_{m,m'}(j) \times \left\langle\alpha',j\right|\!\left|\,\hat{\mathbf{V}}\,\right|\!\left|\alpha,j\right\rangle \times C^i_{m',m}(j) \\
&=\; \left\langle\alpha',j\right|\!\left|\,\hat{\mathbf{V}}\,\right|\!\left|\alpha,j\right\rangle \times \langle j||\,\hat{\mathbf{J}}\,||j\rangle \times \sum_{m'} C^i_{m,m'}(j)\,C^i_{m',m}(j),
\end{aligned} \tag{35}$$

and in exactly similar fashion

$$\langle j,m|\,\hat{\mathbf{J}}^2\,|j,m\rangle \;=\; \langle j||\,\hat{\mathbf{J}}\,||j\rangle \times \langle j||\,\hat{\mathbf{J}}\,||j\rangle \times \sum_{m'} C^i_{m,m'}(j)\,C^i_{m',m}(j)\,. \tag{36}$$

Consequently, taking the ratio of the last two formulae, we arrive at

$$\frac{\langle\alpha',j,m|\,(\hat{\mathbf{J}}\cdot\hat{\mathbf{V}})\,|\alpha,j,m\rangle}{\langle j,m|\,\hat{\mathbf{J}}^2\,|j,m\rangle} \;=\; \frac{\langle\alpha',j||\,\hat{\mathbf{V}}\,||\alpha,j\rangle}{\langle j||\,\hat{\mathbf{J}}\,||j\rangle}\,. \tag{37}$$

Finally, taking the ratio of eqs. (33) and (34), we get

$$\frac{\langle\alpha',j,m'|\,\hat{V}_i\,|\alpha,j,m\rangle}{\langle j,m'|\,\hat{J}^i\,|j,m\rangle} \;=\; \frac{\langle\alpha',j||\,\hat{\mathbf{V}}\,||\alpha,j\rangle}{\langle j||\,\hat{\mathbf{J}}\,||j\rangle} \;=\; \frac{\langle\alpha',j,m|\,(\hat{\mathbf{J}}\cdot\hat{\mathbf{V}})\,|\alpha,j,m\rangle}{\langle j,m|\,\hat{\mathbf{J}}^2\,|j,m\rangle}\,, \tag{38}$$

and therefore

$$\left\langle\alpha',j,m'\right|\hat{V}_i\,|\alpha,j,m\rangle \;=\; \frac{\langle\alpha',j,m|\,(\hat{\mathbf{J}}\cdot\hat{\mathbf{V}})\,|\alpha,j,m\rangle}{\langle j,m|\,\hat{\mathbf{J}}^2\,|j,m\rangle = \hbar^2 j(j+1)} \times \left\langle j,m'\right|\hat{J}^i\,|j,m\rangle\,. \tag{39}$$

Quod erat demonstrandum.

Example: The magnetic moment of an electron is approximately

$$\hat{\vec{\mu}} \;=\; \frac{-e}{2mc}(\hat{\mathbf{L}}+2\hat{\mathbf{S}}), \tag{40}$$

so the magnetic moment operator for a whole atom is

$$\hat{\vec{\mu}}_{\rm net} \;=\; \frac{-e}{2mc}\big(\hat{\mathbf{L}}_{\rm net}+2\hat{\mathbf{S}}_{\rm net}\big). \tag{41}$$

Now consider the expectation value of this operator in a state of definite net orbital angular momentum L, definite net spin S, and definite net angular momentum J. The m_J quantum number can have any value, or we can have a linear combination of states with different m_J. By the projection theorem, in such a state

$$\left\langle \hat{\vec{\mu}} \right\rangle = \frac{\left\langle \hat{\vec{\mu}} \cdot \hat{\mathbf{J}} \right\rangle}{\left\langle \hat{\mathbf{J}}^2 \right\rangle} \left\langle \hat{\mathbf{J}} \right\rangle = \frac{-e}{2mc} \frac{\left\langle (\hat{\mathbf{L}} + 2\hat{\mathbf{S}}) \cdot \hat{\mathbf{J}} \right\rangle}{\left\langle \hat{\mathbf{J}}^2 \right\rangle} \left\langle \hat{\mathbf{J}} \right\rangle, \tag{42}$$

or in short,

$$\left\langle \hat{\vec{\mu}} \right\rangle = -\frac{ge}{2mc} \left\langle \hat{\mathbf{J}} \right\rangle \tag{43}$$

where

$$g = \frac{\left\langle (\hat{\mathbf{L}} + 2\hat{\mathbf{S}}) \cdot \hat{\mathbf{J}} \right\rangle}{\left\langle \hat{\mathbf{J}}^2 \right\rangle} \tag{44}$$

is the *gyromagnetic factor* of the atom.

To evaluate eq. (44), we notice that

$$(\hat{\mathbf{L}} + 2\hat{\mathbf{S}}) \cdot \hat{\mathbf{J}} = (\hat{\mathbf{J}} + \hat{\mathbf{S}}) \cdot \hat{\mathbf{J}} = \hat{\mathbf{J}}^2 + \hat{\mathbf{S}} \cdot \hat{\mathbf{J}} \tag{45}$$

and

$$2\hat{\mathbf{S}} \cdot \hat{\mathbf{J}} = \hat{\mathbf{J}}^2 + \hat{\mathbf{S}}^2 - (\hat{\mathbf{J}} - \hat{\mathbf{S}} = \hat{\mathbf{L}})^2, \tag{46}$$

thus

$$(\hat{\mathbf{L}} + 2\hat{\mathbf{S}}) \cdot \hat{\mathbf{J}} = \tfrac{3}{2}\hat{\mathbf{J}}^2 + \tfrac{1}{2}\hat{\mathbf{S}}^2 - \tfrac{1}{2}\hat{\mathbf{L}}^2. \tag{47}$$

Consequently, in a state with definite values of L, S, and J,

$$\left\langle (\hat{\mathbf{L}} + 2\hat{\mathbf{S}}) \cdot \hat{\mathbf{J}} \right\rangle = \tfrac{3}{2}\hbar^2 j(j+1) + \tfrac{1}{2}\hbar^2 S(S+1) - \tfrac{1}{2}\hbar^2 L(L+1) \tag{48}$$

and therefore

$$g = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}. \tag{49}$$

For example, an oxygen atom in its ground state has $L = 1$, $S = 1$, $J = 2$, and hence $g = \frac{3}{2}$.