The same infinite-dimensional Hilbert space can have both discrete and continuous bases. For example, the Hilbert space of a quantum particle moving in one space dimension has a continuous *position* basis $\{|x\rangle\}$ and an equally continuous *momentum* basis $\{|p\rangle\}$. However, it also may have discrete bases, and the purpose of this homework is to explicitly construct a discrete basis $\{|n\rangle\}$ $(n = 0, 1, \ldots)$ for this Hilbert space.

The most common way to construct a basis of a Hilbert space involves eigenstates of some hermitian operator. In this homework we shall use the Hamiltonian operator of a one-dimensional harmonic oscillator:

$$\hat{H} \;=\; \frac{1}{2m}\hat{P}^2 \;+\; \frac{m\omega^2}{2}\hat{X}^2 \tag{1}$$

where $\hat{P}$ and $\hat{X}$ are respectively the momentum and the position operators.

1. Let's start by solving the eigenvalue equation $\hat{H}\,|n\rangle = E_n\,|n\rangle$ and writing down the position-basis wave-functions $\psi_n(x)$ of the eigenstates $|n\rangle$. Our goal in this problem is to show that

$$E_n \;=\; \hbar\omega(n + \tfrac{1}{2}) \quad \text{for } n = 0, 1, 2, \ldots \tag{2}$$

while

$$\langle x|n\rangle \;=\; \psi_n(x) \;=\; C_n H_n(\alpha x)\exp(-\tfrac{1}{2}\alpha^2 x^2) \tag{3}$$

where

$$\alpha \;=\; \sqrt{\frac{m\omega}{\hbar}}\,, \tag{4}$$

C_n is some normalization factor keeping $\langle n|n\rangle = 1$, and H_n is the n^{th} Hermite polynomial, to be explained below.

(a) Spell out the eigenvalue equation $\hat{H}\,|n\rangle = E_n\,|n\rangle$ in the coordinate basis, *i.e.* in terms of the wave-function $\psi_n(x)$.
Then verify that the the ground state — with $\psi_0(x) = C_0\exp(-\frac{1}{2}\alpha^2 x^2)$ since $H_0 \equiv 1$ — indeed obeys the eigenvalue equation for $E_0 = \frac{1}{2}\hbar\omega$.

The Hermite polynomials $H_n(\xi)$ are defined as

$$H_n(\xi) \stackrel{\text{def}}{=} (-1)^n e^{+\xi^2} \times \frac{d^n}{d\xi^n} e^{-\xi^2}. \tag{5}$$

Each $H_n(\xi)$ is a polynomial of degree n, and it can be recursively constructed using

$$H_n(\xi) = 1, \qquad H_{n+1}(\xi) = 2\xi \times H_n(\xi) - \frac{d}{d\xi} H_n(\xi). \tag{6}$$

(b) Verify this recursion relation. Also, let

$$f^{(n)}(\xi) = (-1)^n e^{-\xi^2} \times H_n(x) = \frac{d^n}{d\xi^n} e^{-\xi^2} \tag{7}$$

and prove another recursion relation

$$f^{(n+2)}(\xi) + 2\xi f^{(n+1)}(\xi) + 2(n+1) f^{(n)}(\xi) = 0 \tag{8}$$

by induction in n.

(c) Verify that the wave-functions (3) are indeed eigenfunctions of the Hamiltonian (1) for the eigenvalues $E_n = \hbar\omega(n + \frac{1}{2})$.
Hint: write the wave-functions $\psi_n(x)$ in terms of $f^{(n)}(\xi = \alpha x)$, rewrite the eigenvalue equation as a differential equation for the $f^{(n)}(\xi)$, and use the recursion relation (8).

2. Eigenstates of any hermitian operator that corresponds to different eigenvalues are guaranteed to be orthogonal to each other (this is a theorem).

(a) Verify that the quantum states $|n\rangle$ described by the wave functions (3) are indeed orthogonal to each other:

$$\langle n|m\rangle \equiv \int dx\, \Psi_n^*(x)\, \Psi_m(x) = 0 \qquad \text{for any } n \neq m. \tag{9}$$

Hint: Use eq. (5) and the fact that H_n is a polynomial of degree n, so for $m > n$ the m^{th} derivative of the H_n must vanish.

(b) Show that the states $|n\rangle$ are normalized, *i.e.* $\langle n|n\rangle = 1$, provided we set

$$C_n^2 \;=\; \frac{1}{2^n n!} \times \frac{\alpha}{\sqrt{\pi}}\,. \tag{10}$$

Altogether, the quantum states $|n\rangle$, $n = 0, 1, \ldots$ form an *orthonormal* set:

$$\langle n|m\rangle \;\equiv\; \int dx\, \Psi_n^*(x)\, \Psi_m(x) \;=\; \delta_{n,m}\ , \qquad n, m = 0, 1, 2, \ldots\,. \tag{11}$$

3. As discussed in class, an infinite orthonormal set of vectors in a Hilbert space $\mathcal{H}$ does not necessary make a complete basis. The purpose of this problem is to verify that the basis $\{|n\rangle\}$ constructed in the first problem is indeed complete, that is, that *any* vector of $\mathcal{H}$ is a linear combination of the $|n\rangle$.

   (a) Prove the Lemma:

$$\Psi_n(x) \;=\; C_n \times \frac{(-i)^n \sqrt{\pi}}{\alpha^{n+1}} \times \exp\bigl(+\tfrac{1}{2}\alpha^2 x^2\bigr) \times \int\limits_{-\infty}^{+\infty} \frac{dk}{2\pi}\, k^n \times \exp\left(ixk \,-\, \frac{k^2}{4\alpha^2}\right). \tag{12}$$

   (b) Use the Lemma (12) to show that

$$\sum_{n=0}^{\infty} \Psi_n^*(x')\Psi_n(x'') \;=\; \delta(x' - x''). \tag{13}$$

   Hints: Use eq. (12) for both $\Psi_n^*(x')$ and $\Psi_n(x'')$ and sum the series before taking the integrals. The sum should have an exponential form, so combine all the exponents together. The net exponent should have form $\mathcal{E}(k', k'') = \mathcal{E}_1(k' - k'') + \mathcal{E}_2(k'')$, so changing the integration variable k' to $q = k' - k''$ should factorize the double integral into a product of $\int dq$ and $\int dk''$.

(c) Finally, show that the formula (13) implies that for *any* wave-function $\Phi(x)$,

$$\sum_n \langle n|\Phi\rangle \; \Psi_n(x) \;\; = \;\; \Phi(x) \tag{14}$$

and hence for *any* vector $|\Phi\rangle \in \mathcal{H}$,

$$\sum_n |n\rangle \, \langle n|\Phi\rangle \;\; = \;\; |\Phi\rangle \, . \tag{15}$$

In other words, eq. (13) implies that the set $\{|n\rangle\}$ (for $n = 0, 1, \ldots$) is a *complete basis* of the Hilbert space.