

The same infinite-dimensional Hilbert space can have both discrete and continuous bases. For example, the Hilbert space of a quantum particle moving in one space dimension has a continuous *position* basis $\{|x\rangle\}$ and an equally continuous *momentum* basis $\{|p\rangle\}$. However, it also may have discrete bases, and the purpose of this homework is to explicitly construct a discrete basis $\{|n\rangle\}$ ($n = 0, 1, \dots$) for this Hilbert space.

The most common way to construct a basis of a Hilbert space involves eigenstates of some hermitian operator. In this homework we shall use the Hamiltonian operator of a one-dimensional harmonic oscillator:

$$\hat{H} = \frac{1}{2m} \hat{P}^2 + \frac{m\omega^2}{2} \hat{X}^2 \quad (1)$$

where \hat{P} and \hat{X} are respectively the momentum and the position operators.

1. Let's start by solving the eigenvalue equation $\hat{H}|n\rangle = E_n|n\rangle$ and writing down the position-basis wave-functions $\psi_n(x)$ of the eigenstates $|n\rangle$. Our goal in this problem is to show that

$$E_n = \hbar\omega(n + \frac{1}{2}) \quad \text{for } n = 0, 1, 2, \dots \quad (2)$$

while

$$\langle x|n\rangle = \psi_n(x) = C_n H_n(\alpha x) \exp(-\frac{1}{2}\alpha^2 x^2) \quad (3)$$

where

$$\alpha = \sqrt{\frac{m\omega}{\hbar}}, \quad (4)$$

C_n is some normalization factor keeping $\langle n|n\rangle = 1$, and H_n is the n^{th} Hermite polynomial, to be explained below.

- (a) Spell out the eigenvalue equation $\hat{H}|n\rangle = E_n|n\rangle$ in the coordinate basis, *i.e.* in terms of the wave-function $\psi_n(x)$.

Then verify that the the ground state — with $\psi_0(x) = C_0 \exp(-\frac{1}{2}\alpha^2 x^2)$ since $H_0 \equiv 1$ — indeed obeys the eigenvalue equation for $E_0 = \frac{1}{2}\hbar\omega$.

The Hermite polynomials $H_n(\xi)$ are defined as

$$H_n(\xi) \stackrel{\text{def}}{=} (-1)^n e^{+\xi^2} \times \frac{d^n}{d\xi^n} e^{-\xi^2}. \quad (5)$$

Each $H_n(\xi)$ is a polynomial of degree n , and it can be recursively constructed using

$$H_n(\xi) = 1, \quad H_{n+1}(\xi) = 2\xi \times H_n(\xi) - \frac{d}{d\xi} H_n(\xi). \quad (6)$$

(b) Verify this recursion relation. Also, let

$$f^{(n)}(\xi) = (-1)^n e^{-\xi^2} \times H_n(x) = \frac{d^n}{d\xi^n} e^{-\xi^2} \quad (7)$$

and prove another recursion relation

$$f^{(n+2)}(\xi) + 2\xi f^{(n+1)}(\xi) + 2(n+1)f^{(n)}(\xi) = 0 \quad (8)$$

by induction in n .

(c) Verify that the wave-functions (3) are indeed eigenfunctions of the Hamiltonian (1) for the eigenvalues $E_n = \hbar\omega(n + \frac{1}{2})$.

Hint: write the wave-functions $\psi_n(x)$ in terms of $f^{(n)}(\xi = \alpha x)$, rewrite the eigenvalue equation as a differential equation for the $f^{(n)}(\xi)$, and use the recursion relation (8).

2. Eigenstates of any hermitian operator that corresponds to different eigenvalues are guaranteed to be orthogonal to each other (this is a theorem).

(a) Verify that the quantum states $|n\rangle$ described by the wave functions (3) are indeed orthogonal to each other:

$$\langle n|m\rangle \equiv \int dx \Psi_n^*(x) \Psi_m(x) = 0 \quad \text{for any } n \neq m. \quad (9)$$

Hint: Use eq. (5) and the fact that H_n is a polynomial of degree n , so for $m > n$ the m^{th} derivative of the H_n must vanish.

(b) Show that the states $|n\rangle$ are normalized, *i.e.* $\langle n|n\rangle = 1$, provided we set

$$C_n^2 = \frac{1}{2^n n!} \times \frac{\alpha}{\sqrt{\pi}}. \quad (10)$$

Altogether, the quantum states $|n\rangle$, $n = 0, 1, \dots$ form an *orthonormal* set:

$$\langle n|m\rangle \equiv \int dx \Psi_n^*(x) \Psi_m(x) = \delta_{n,m}, \quad n, m = 0, 1, 2, \dots \quad (11)$$

3. As discussed in class, an infinite orthonormal set of vectors in a Hilbert space \mathcal{H} does not necessarily make a complete basis. The purpose of this problem is to verify that the basis $\{|n\rangle\}$ constructed in the first problem is indeed complete, that is, that *any* vector of \mathcal{H} is a linear combination of the $|n\rangle$.

(a) Prove the Lemma:

$$\Psi_n(x) = C_n \times \frac{(-i)^n \sqrt{\pi}}{\alpha^{n+1}} \times \exp(+\frac{1}{2}\alpha^2 x^2) \times \int_{-\infty}^{+\infty} \frac{dk}{2\pi} k^n \times \exp\left(ikx - \frac{k^2}{4\alpha^2}\right). \quad (12)$$

(b) Use the Lemma (12) to show that

$$\sum_{n=0}^{\infty} \Psi_n^*(x') \Psi_n(x'') = \delta(x' - x''). \quad (13)$$

Hints: Use eq. (12) for both $\Psi_n^*(x')$ and $\Psi_n(x'')$ and sum the series before taking the integrals. The sum should have an exponential form, so combine all the exponents together. The net exponent should have form $\mathcal{E}(k', k'') = \mathcal{E}_1(k' - k'') + \mathcal{E}_2(k'')$, so changing the integration variable k' to $q = k' - k''$ should factorize the double integral into a product of $\int dq$ and $\int dk''$.

(c) Finally, show that the formula (13) implies that for *any* wave-function $\Phi(x)$,

$$\sum_n \langle n|\Phi\rangle \Psi_n(x) = \Phi(x) \quad (14)$$

and hence for *any* vector $|\Phi\rangle \in \mathcal{H}$,

$$\sum_n |n\rangle \langle n|\Phi\rangle = |\Phi\rangle. \quad (15)$$

In other words, eq. (13) implies that the set $\{|n\rangle\}$ (for $n = 0, 1, \dots$) is a *complete basis* of the Hilbert space.