1. Consider the spin states of an electron (or a spin $\frac{1}{2}$ atom, such as silver). The spin is a vector (in 3–d sense) $\hat{\mathbf{S}}$ whose components are operators $(\hat{S}_x, \hat{S}_y, \hat{S}_z)$. In the standard basis of the Hilberst space of spin states, the matrices of these components are $S_i = (\hbar/2)\sigma_i$ (for $i = x, y, z$) where the σ_i are the *Pauli matrices*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

(a) Verify that the Pauli matrices and hence the $\hat{\sigma}_i$ operators obey

$$\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k \tag{2}$$

where I is the unit matrix, ϵ_{ijk} is the 3d Levi–Civita antisymmetric tensor, and the repeated index k is summed over.

⋆ Warning: If your index-handling skills are rusty, please hone them up in a hurry. Future homeworks will have 3–d space indices in truckloads.

(b) Suppose the 3 components v_i of some 3–vector $\mathbf{v}$ commute with each other and also with all the Pauli matrices. Show that in this case $(\mathbf{v} \cdot \vec{\sigma})^2 = \mathbf{v}^2 I$.

(c) Calculate $\exp(\mathbf{v} \cdot \vec{\sigma})$ under the assumptions of part (b).

Now consider an electron in a uniform magnetic field $\mathbf{B}$. For simplicity, let's ignore the electron's motion and focus on its spin degrees of freedom. The electron has magnetic moment $\mathbf{m} = -(e/M_e c)\mathbf{S}$ (Gauss units) — or in terms of the Bohr magneton $m_B = e\hbar/2M_e c$, $\mathbf{m} = -m_B\vec{\sigma}$, — so in the uniform magnetic field it has spin Hamiltonian

$$\hat{H} = -\mathbf{B} \cdot \hat{\mathbf{m}} = +m_B \mathbf{B} \cdot \hat{\vec{\sigma}}. \tag{3}$$

(d) Show that the time evolution operator for the electron's spin is

$$\hat{U}(t, t_0) = \cos\frac{\omega(t - t_0)}{2}\hat{I} - i\sin\frac{\omega(t - t_0)}{2}(\mathbf{n} \cdot \hat{\vec{\sigma}}) \tag{4}$$

where $\omega = (eB/M_e c)$ is the classical precession frequency of the magnetic moment and $\mathbf{n}$ is the unit vector in the direction of the magnetic field.

(e) Given an arbitrary initial quantum state $|\Psi(t_0)\rangle$, compute the expectation value of the magnetic moment for all future times and show that it precesses with the classical frequency ω around the magnetic field's direction. For simplicity, assume that direction to be the z axis.

2. Consider three hermitian operators $\hat{H}$, $\hat{A}_1$, and $\hat{A}_2$ such that

$$[\hat{H}, \hat{A}_1] \;=\; 0, \quad [\hat{H}, \hat{A}_2] \;=\; 0, \quad \text{but} \quad [\hat{A}_1, \hat{A}_2] \;\neq\; 0. \tag{5}$$

(a) Show that the eigenvalue spectrum of $\hat{H}$ must be degenerate.

(b) Can *some* of the eigenvalues of $\hat{H}$ be non-degenerate? What is a necessary condition (in terms of $i[\hat{A}_1, \hat{A}_2]$) for this to happen?

(c) Consider a three-dimensional Hilbert space. Suppose in some orthonormal basis $\hat{H}$ has diagonal matrix

$$H \;=\; \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \quad \text{with} \quad h_1 = h_2 \neq h_3\,. \tag{6}$$

Give an example of operators $\hat{A}_1$ and $\hat{A}_2$ — or rather, of their matrices in the same basis — which both commute with the $\hat{H}$ but do not commute with each other.

3. The commutator bracket $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ has several important algebraic properties. Two of them — the antisymmetry $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$ and the bi-linearity with respect to both $\hat{A}$ and $\hat{B}$, — are quite obvious from the definition of the commutator. Your task in this problem is to prove two not-so-obvious properties, namely the Leibniz rules

$$\begin{aligned} [\hat{A}, \hat{B}\hat{C}] \;&=\; [\hat{A}, \hat{B}]\hat{C} \;+\; \hat{B}[\hat{A}, \hat{C}], \\ [\hat{A}\hat{B}, \hat{C}] \;&=\; \hat{A}[\hat{B}, \hat{C}] \;+\; [\hat{A}, \hat{C}]\hat{B}, \end{aligned} \tag{7}$$

and the Jacobi identity

$$[\hat{A}, [\hat{B}, \hat{C}]] \;+\; [\hat{B}, [\hat{C}, \hat{A}]] \;+\; [\hat{C}, [\hat{A}, \hat{B}]] \;=\; 0. \tag{8}$$

4. Finally, a harder exercise in the commutator algebra. Suppose two operators $\hat{X}$ and $\hat{P}$ are Hermitian and the commutator $[\hat{X},\hat{P}] = i\hbar$, but make no assumptions about the physical nature of these operators. Using nothing but the commutator $[\hat{X},\hat{P}]$ and the Hermiticity of $\hat{X}$ and $\hat{P}$, prove the following:

   (a) For any analytic function $f(P)$,

$$[\hat{X}, f(\hat{P})] \;=\; i\hbar f'(\hat{P}) \quad \text{where} \quad f'(p) \;=\; \frac{df}{dp}\,. \tag{9}$$

   Hint: decompose $f(\hat{P})$ into a power series in $\hat{P}$.

   (b) Likewise, for any analytic function $g(X)$,

$$[g(\hat{X}), \hat{P}] \;=\; i\hbar g'(\hat{X}) \quad \text{where} \quad g'(x) \;=\; \frac{dg}{dx}\,. \tag{10}$$

   (c) Show that $\hat{P} \times e^{ik\hat{X}} = e^{ik\hat{X}} \times (\hat{P} + k\hbar)$ and $\hat{X} \times e^{i\lambda\hat{P}} = e^{i\lambda\hat{P}} \times (\hat{X} - \lambda\hbar)$.

   (d) For any analytic functions $f(\hat{P})$ and $g(\hat{X})$,

$$e^{-ik\hat{X}} f(\hat{P}) e^{ik\hat{X}} \;=\; f(\hat{P} + k\hbar) \quad \text{and} \quad e^{-i\lambda\hat{P}} g(\hat{X}) e^{i\lambda\hat{P}} \;=\; g(\hat{X} - \lambda\hbar). \tag{11}$$

   (e) Show that for any two eigenstates $|x_1\rangle$ and $|x_2\rangle$ of $\hat{X}$,

$$\langle x_1|\, e^{-i\lambda\hat{P}}\, |x_2\rangle \;=\; 0 \quad \text{unless} \quad x_1 - x_2 = \lambda\hbar. \tag{12}$$

   Hint: compute $\langle x_1|\,[\hat{X}, e^{-i\lambda\hat{P}}]\,|x_2\rangle$.

Now suppose $\hat{X}$ is a position operator in 1 dimension. The result of part (e) shows that $\exp(-ia\hat{P}/\hbar)$ is the translation operator which moves a state localized at $X = x$ into the state localized at $X = x + a$.

   (f) Suppose the phases of the position eigenstates $|x\rangle$ are chosen such that

$$\langle x_1|\exp(-ia\hat{P}/\hbar)\,|x_2\rangle \;=\; \delta(x_1 - x_2 - a). \tag{13}$$

   Show that in such a position basis, the $\hat{P}$ operator acts on the wavefunctions as the usual momentum operator,

$$\langle x|\,\hat{P}\,|\Psi\rangle \;=\; -i\hbar\frac{d}{dx}\,\langle x|\Psi\rangle\,, \quad i.e., \quad \hat{P}\psi(x) \;-i\hbar\frac{d}{dx}\psi(x). \tag{14}$$