

1. Consider the spin states of an electron (or a spin $\frac{1}{2}$ atom, such as silver). The spin is a vector (in 3-d sense) $\hat{\mathbf{S}}$ whose components are operators ($\hat{S}_x, \hat{S}_y, \hat{S}_z$). In the standard basis of the Hilbert space of spin states, the matrices of these components are $S_i = (\hbar/2)\sigma_i$ (for $i = x, y, z$) where the σ_i are the *Pauli matrices*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1)$$

- (a) Verify that the Pauli matrices and hence the $\hat{\sigma}_i$ operators obey

$$\sigma_i \sigma_j = \delta_{ij} I + i \epsilon_{ijk} \sigma_k \quad (2)$$

where I is the unit matrix, ϵ_{ijk} is the 3d [Levi-Civita antisymmetric tensor](#), and the repeated index k is summed over.

★ Warning: If your index-handling skills are rusty, please hone them up in a hurry. Future homeworks will have 3-d space indices in truckloads.

- (b) Suppose the 3 components v_i of some 3-vector \mathbf{v} commute with each other and also with all the Pauli matrices. Show that in this case $(\mathbf{v} \cdot \vec{\sigma})^2 = \mathbf{v}^2 I$.

- (c) Calculate $\exp(\mathbf{v} \cdot \vec{\sigma})$ under the assumptions of part (b).

Now consider an electron in a uniform magnetic field \mathbf{B} . For simplicity, let's ignore the electron's motion and focus on its spin degrees of freedom. The electron has magnetic moment $\mathbf{m} = -(e/M_e c)\mathbf{S}$ (Gauss units) — or in terms of the Bohr magneton $m_B = e\hbar/2M_e c$, $\mathbf{m} = -m_B \vec{\sigma}$, — so in the uniform magnetic field it has spin Hamiltonian

$$\hat{H} = -\mathbf{B} \cdot \hat{\mathbf{m}} = +m_B \mathbf{B} \cdot \vec{\sigma}. \quad (3)$$

- (d) Show that the time evolution operator for the electron's spin is

$$\hat{U}(t, t_0) = \cos \frac{\omega(t-t_0)}{2} \hat{I} - i \sin \frac{\omega(t-t_0)}{2} (\mathbf{n} \cdot \vec{\sigma}) \quad (4)$$

where $\omega = (eB/M_e c)$ is the classical precession frequency of the magnetic moment and \mathbf{n} is the unit vector in the direction of the magnetic field.

- (e) Given an arbitrary initial quantum state $|\Psi(t_0)\rangle$, compute the expectation value of the magnetic moment for all future times and show that it precesses with the classical frequency ω around the magnetic field's direction. For simplicity, assume that direction to be the z axis.

2. Consider three hermitian operators \hat{H} , \hat{A}_1 , and \hat{A}_2 such that

$$[\hat{H}, \hat{A}_1] = 0, \quad [\hat{H}, \hat{A}_2] = 0, \quad \text{but} \quad [\hat{A}_1, \hat{A}_2] \neq 0. \quad (5)$$

- (a) Show that the eigenvalue spectrum of \hat{H} must be degenerate.
- (b) Can *some* of the eigenvalues of \hat{H} be non-degenerate? What is a necessary condition (in terms of $i[\hat{A}_1, \hat{A}_2]$) for this to happen?
- (c) Consider a three-dimensional Hilbert space. Suppose in some orthonormal basis \hat{H} has diagonal matrix

$$H = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \quad \text{with} \quad h_1 = h_2 \neq h_3. \quad (6)$$

Give an example of operators \hat{A}_1 and \hat{A}_2 — or rather, of their matrices in the same basis — which both commute with the \hat{H} but do not commute with each other.

3. The commutator bracket $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ has several important algebraic properties. Two of them — the antisymmetry $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$ and the bi-linearity with respect to both \hat{A} and \hat{B} , — are quite obvious from the definition of the commutator. Your task in this problem is to prove two non-so-obvious properties, namely the Leibniz rules

$$\begin{aligned} [\hat{A}, \hat{B}\hat{C}] &= [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}], \\ [\hat{A}\hat{B}, \hat{C}] &= \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}, \end{aligned} \quad (7)$$

and the Jacobi identity

$$[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0. \quad (8)$$

4. Finally, a harder exercise in the commutator algebra. Suppose two operators \hat{X} and \hat{P} are Hermitian and the commutator $[\hat{X}, \hat{P}] = i\hbar$, but make no assumptions about the physical nature of these operators. Using nothing but the commutator $[\hat{X}, \hat{P}]$ and the Hermiticity of \hat{X} and \hat{P} , prove the following:

(a) For any analytic function $f(P)$,

$$[\hat{X}, f(\hat{P})] = i\hbar f'(\hat{P}) \quad \text{where} \quad f'(p) = \frac{df}{dp}. \quad (9)$$

Hint: decompose $f(\hat{P})$ into a power series in \hat{P} .

(b) Likewise, for any analytic function $g(X)$,

$$[g(\hat{X}), \hat{P}] = i\hbar g'(\hat{X}) \quad \text{where} \quad g'(x) = \frac{dg}{dx}. \quad (10)$$

(c) Show that $\hat{P} \times e^{ik\hat{X}} = e^{ik\hat{X}} \times (\hat{P} + k\hbar)$ and $\hat{X} \times e^{i\lambda\hat{P}} = e^{i\lambda\hat{P}} \times (\hat{X} - \lambda\hbar)$.

(d) For any analytic functions $f(\hat{P})$ and $g(\hat{X})$,

$$e^{-ik\hat{X}} f(\hat{P}) e^{ik\hat{X}} = f(\hat{P} + k\hbar) \quad \text{and} \quad e^{-i\lambda\hat{P}} g(\hat{X}) e^{i\lambda\hat{P}} = g(\hat{X} - \lambda\hbar). \quad (11)$$

(e) Show that for any two eigenstates $|x_1\rangle$ and $|x_2\rangle$ of \hat{X} ,

$$\langle x_1 | e^{-i\lambda\hat{P}} | x_2 \rangle = 0 \quad \text{unless} \quad x_1 - x_2 = \lambda\hbar. \quad (12)$$

Hint: compute $\langle x_1 | [\hat{X}, e^{-i\lambda\hat{P}}] | x_2 \rangle$.

Now suppose \hat{X} is a position operator in 1 dimension. The result of part (e) shows that $\exp(-ia\hat{P}/\hbar)$ is the translation operator which moves a state localized at $X = x$ into the state localized at $X = x + a$.

(f) Suppose the phases of the position eigenstates $|x\rangle$ are chosen such that

$$\langle x_1 | \exp(-ia\hat{P}/\hbar) | x_2 \rangle = \delta(x_1 - x_2 - a). \quad (13)$$

Show that in such a position basis, the \hat{P} operator acts on the wavefunctions as the usual momentum operator,

$$\langle x | \hat{P} | \Psi \rangle = -i\hbar \frac{d}{dx} \langle x | \Psi \rangle, \quad \text{i. e.,} \quad \hat{P}\psi(x) = -i\hbar \frac{d}{dx} \psi(x). \quad (14)$$