1. Consider a spinless charged particle in a uniform magnetic field **B**. For simplicity, assume the particle moves freely in the xy plane but cannot move at all in the z direction, while the magnetic field is directed along the z axis. As we saw in the previous homework (set#4, problem 4), the Hamiltonian operator for this particle is

$$\hat{H} = \frac{\hat{\pi}_x^2 + \hat{\pi}_y^2}{2M} \tag{1}$$

where

$$\begin{aligned} [\hat{x}, \hat{y}] &= 0, \\ [\hat{x}_i, \hat{\pi}_i] &= i\hbar\delta_{ij} \quad (\text{for } i, j = x, y), \\ [\hat{\pi}_x, \hat{\pi}_y] &= i\frac{QB\hbar}{c}. \end{aligned} \tag{2}$$

(a) Let

$$\hat{a} = \sqrt{\frac{c}{2\hbar|QB|}}\big(\hat{\pi}_x + i\,\mathrm{sign}(QB)\,\hat{\pi}_y\big) \tag{3}$$

and show that this non-Hermitian operator obeys $[\hat{a}, \hat{a}^\dagger] = 1$.

(b) Rewrite the Hamiltonian (1) in terms of the $\hat{a}$ and $\hat{a}^\dagger$ operators, then show that its spectrum consists of discrete *Landau levels*

$$E_n = \hbar\Omega(n + \tfrac{1}{2}), \quad n = 0, 1, 2, \ldots \tag{4}$$

where

$$\Omega = \frac{|QB|}{Mc} \tag{5}$$

is the classical *cyclotron frequency* of the particle moving in a Larmor circle in the magnetic field.

(c) Show that for a classical particle moving in a Larmor circle, the circle's center is located at

$$x_c = x + \frac{c}{QB}\pi_y\,, \quad y_c = y - \frac{c}{QB}\pi_x\,. \tag{6}$$

(d) Show that the quantum analogues $\hat{x}_c$ and $\hat{y}_c$ of the center's coordinate commute with both $\hat{\pi}_x$ and $\hat{\pi}_y$ and hence with the Hamiltonian $\hat{H}$. In other words, both $\hat{x}_c$ and $\hat{y}_c$ are conserved operators.

(e) Show that the $\hat{x}_c$ and $\hat{y}_c$ do not commute with each other; instead

$$[\hat{x}_c, \hat{y}_c] = \frac{-i\hbar c}{QB}\,. \tag{7}$$

(f) Use the commutator (7) to show that each Landau energy level is infinitely degenerate. Hint: build Harmonic-oscillator-like operators $\hat{b}$ and $\hat{b}^\dagger$ with $[\hat{b}, \hat{b}^\dagger] = 1$ from $\hat{x}_c$ and $\hat{y}_c$, then show that an entire infinite tower of oscillator-like states must exist at every Landau level.

2. Now let's learn about the *coherent states* of a harmonic oscillator.

(a) First, a lemma about functions of $\hat{a}$ or $\hat{a}^\dagger$ operators. Let $f(\xi)$ be any analytic function of a complex number ξ and $f'(x) = df/d\xi$ its derivative. Show that

$$[\hat{a}, f(\hat{a}^\dagger)] = f'(\hat{a}^\dagger) \quad \text{and} \quad [\hat{a}^\dagger, f(\hat{a})] = -f'(\hat{a}). \tag{8}$$

Next, for any complex number ξ we define the coherent state $|\xi\rangle$ as

$$|\xi\rangle \stackrel{\text{def}}{=} e^{-|\xi|^2/2} \exp\bigl(\xi\hat{a}^\dagger\bigr)\,|0\rangle \tag{9}$$

where $|0\rangle = |n = 0\rangle$ is the oscillator's ground state.

(b) Calculate $\langle n|\xi\rangle$ for all $n = 0, 1, 2.\ldots$, then show that the state (9) is normalized, *i.e.* $\langle\xi|\xi\rangle = 1$.

The operators $\hat{a}$ and $\hat{a}^\dagger$ cannot be diagonalized. However, $\hat{a}$ has an eigen-ket for any complex eigenvalue while $\hat{a}^\dagger$ has an eigen-bra for any complex eigenvalue. On the other hand, $\hat{a}^\dagger$ has no eigen-kets at all, while $\hat{a}$ has no eigen-bras.

(c) Show that the coherent state $|\xi\rangle$ is the eigen-ket of $\hat{a}$ for the eigenvalue ξ; likewise, $\langle\xi|$ is an eigen-bra of $\hat{a}^\dagger$ for the eigenvalue ξ^*:

$$\hat{a}\,|\xi\rangle \;=\; \xi\,|\xi\rangle\,, \quad \langle\xi|\,\hat{a}^\dagger \;=\; \xi^*\,\langle\xi|\,. \tag{10}$$

Hint: use part (a) to show that $\hat{a}\exp(\xi\hat{a}^\dagger) = \exp(\xi\hat{a}^\dagger)(\hat{a}+\xi)$, then apply both sides of this equation to $|0\rangle$.

(d) Show that $\hat{a}^\dagger$ has no eigen-kets for any complex eigenvalues while $\hat{a}$ has no eigen-bras. Hint: show that if $\hat{a}^\dagger\,|\psi\rangle = \lambda\,|\psi\rangle$ then $|\psi\rangle$ is un-normalizable because $|\langle n|\psi\rangle|^2$ increases with n.

Coming back to the coherent states, in any coherent state ξ, the expectation value of any *normal-ordered* product of raising and lowering operators — *i.e.*, a product $(\hat{a}^\dagger)^m(\hat{a})^n$ in which all raising operators are to the left of all the lowering operators — is simply

$$\langle\xi|\,(\hat{a}^\dagger)^m(\hat{a})^n\,|\xi\rangle \;=\; \xi^{*m}\xi^n. \tag{11}$$

(e) Prove this.

A coherent state $|\xi\rangle$ does not have a definite energy (except for $\xi = 0$). However, for the highly excited coherent state with $\langle E\rangle \gg \hbar\omega$, the *relative* energy uncertainty becomes small, $\Delta E \ll \langle E\rangle$.

(f) Calculate $\langle E\rangle$ and ΔE in a coherent state ξ.
Hint: prove and use $\hat{n}^2 = (\hat{a}^\dagger)^2(\hat{a})^2 + \hat{n}$, then use eq. (11).

Since the operator $\hat{a}$ is not Hermitian, its eigen-kets are not orthogonal to each other. Nevertheless, the overlap between 2 coherent states $|\xi\rangle$ and $|\eta\rangle$ becomes exponentially small for large $|\xi-\eta|$.

(g) Calculate the overlap and show that $|\langle\eta|\xi\rangle|^2 = \exp\bigl(-|\xi-\eta|^2\bigr)$.

(h) Finally, show that the set of all the coherent states $|\xi\rangle$ (for all complex ξ) forms an over-complete basis of the harmonic oscillator's Hilbert space,

$$\int \frac{d^2\xi}{\pi}\,|\xi\rangle\langle\xi| \;=\; \hat{1} \tag{12}$$

where $d^2\xi = d(\Re\xi)d(\Im\xi) = |\xi|\,d(|\xi|)d(\arg\xi)$.

Hint: calculate the matrix elements $\langle m|$ integral (12) $|n\rangle$.

3. Finally, consider the dynamics of coherent states. If the initial state of a harmonic oscillator is coherent, then it remain a coherent state at all future times, but for a time-dependent $\xi(t)$, namely

$$\xi(t) \;=\; \xi_0 \times e^{-i\omega t}. \tag{13}$$

(a) Show that the state

$$|\psi\rangle(t) \;=\; e^{-i\omega t/2}\,|\xi(t)\rangle \tag{14}$$

where $\xi(t)$ evolves according to eq. (13) obeys the Schrödinger equation

$$i\hbar\frac{d}{dt}\,|\psi\rangle(t) \;=\; \hat{H}\,|\psi\rangle(t). \tag{15}$$

(b) Calculate the expectation values $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ of the position and momentum in a coherent state ξ. Then show that when $\xi(t)$ evolves according to eq. (13), these expectation values obey the classical equations of motion.

(c) Calculate the uncertainties Δq and Δp in a coherent state and show that $\Delta q \times \Delta p = \frac{1}{2}\hbar$, the minimum allowed by the Heisenberg's uncertainty principle.

In an earlier homework set#1 we saw that that the Heisenberg bound is saturated by the Gaussian wave packets (with real coefficients of $-x^2$ in the exponent). The coherent state also saturate this bound because they are Gaussian wave packets of this kind.

(d) Solve the equation $(\hat{a}-\xi)|\psi\rangle = 0$ in the coordinate basis, and show that the solution is indeed the Gaussian wave packet

$$\psi(q) \;=\; C\times\exp\left(-\frac{m\omega}{2\hbar}\times(q-\bar{q})^2\;+\;\frac{i\bar{p}}{\hbar}\times q\right) \tag{16}$$

where $\bar{q} = \langle\xi|\,\hat{q}\,|\xi\rangle$, $\bar{p} = \langle\xi|\,\hat{p}\,|\xi\rangle$, and C is a constant overall factor.

Note: the magnitude of the constant C obtains from the normalization condition $\langle\psi|\psi\rangle = 1$, but determining the phase of C takes extra information, for example requiring $|\psi\rangle = |\xi\rangle$ having exactly the same overall phase as in eq. (9). The correct answer is

$$C \;=\; \sqrt[4]{\frac{m\omega}{\pi\hbar}}\times e^{-i\bar{p}\bar{q}/2\hbar}, \tag{17}$$

but deriving this formula is **not** a part of this homework assignment.

The bottom line is, the best way to see the near-classical oscillations in quantum mechanics is to look at the coherent states $|\xi\rangle$ with $\xi(t) = \xi_0 e^{i\omega t}$. These states provide for minimal uncertainties Δq and δp while the expectation values $\langle q\rangle(t)$ and $\langle p\rangle(t)$ oscillate in a classical manner. Also, while the coherent states do not have definite energies, the *relative* energy uncertainty becomes small for the highly excited states (*cf.* problem 2(e)).

By comparison, the stationary states $|n\rangle$ do not show any classical-like motion. Indeed, not only there is no motion at all in a stationary state, but also

$$\langle n|\,\hat{q}\,|n\rangle \;=\; \langle n|\,\hat{p}\,|n\rangle \;=\; 0 \tag{18}$$

while the uncertainties grow with n:

$$(\Delta q)^2 \;=\; \frac{\hbar}{2m\omega}\times(2n+1), \quad (\Delta p)^2 \;=\; \frac{\hbar\omega m}{2}\times(2n+1) \quad\Longrightarrow\quad \Delta q\times\Delta p \;=\; \hbar\times(\tfrac{1}{2}+n). \tag{19}$$

(e) Verify all these formulae.