- This homework is about one-dimensional square wells and barriers. For most students, this should be a refresher of undergraduate quantum mechanics. Also, here are some reminders in case I do not go over them in class on 9/28:

  (α) The wave function $\psi(x)$ of any stationary state is continuous everywhere, even at the discontinuities of the potential $V(x)$.

  (β) As long as the potential $V(x)$ is finite, the wave function's first derivative $\psi'(x)$ should also be continuous everywhere. However, the second derivative $\psi''(x)$ becomes discontinuous at the discontinuities of the potential.

  And here is the algorithm for calculating the reflection and transmission coefficients of a potential barrier:

  (a) Find a general solution of the Schrödinger equation, write its asymptotic behavior for $x \to \pm\infty$ in the form

  $$\begin{aligned} \text{for } x \to -\infty: \quad \psi(x) &\approx A_1 e^{+ik_1 x} + B_1 e^{-ik_1 x}, \\ \text{for } x \to +\infty: \quad \psi(x) &\approx A_2 e^{+ik_2 x} + B_2 e^{-ik_2 x}, \end{aligned} \tag{1}$$

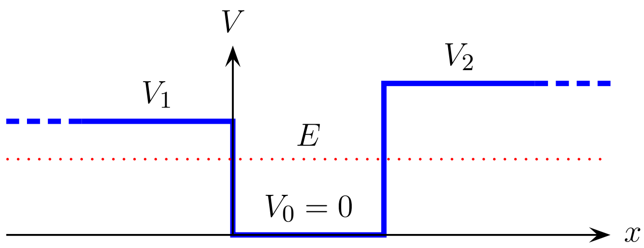
  for some coefficients A_1, B_1, A_2, B_2, and obtain two linear relations between these 4 coefficients.

  (b) Add another linear equation $B_2 = 0$ (for the incident particles coming only from the left), then these 3 linear equations for the B_1/A_1 and A_2/A_1 ratios.

  (c) The reflection probability R and the transmission probability T obtain from these ratios as

  $$R = \left|\frac{B_1}{A_1}\right|^2, \qquad T = \frac{k_2}{k_1} \times \left|\frac{A_2}{A_1}\right|^2. \tag{2}$$

1. Consider the bound states in an asymmetric square well,



$$V(x) \;=\; \begin{cases} V_1 > 0 & \text{for } x < 0, \\ V_0 = 0 & \text{for } 0 < x < L, \\ V_2 > 0 & \text{for } x > L. \end{cases} \tag{3}$$

A symmetric potential well in one dimension always has at least one bound state, but this is not always true for the asymmetric wells with $V_1 \neq V_2$.

(a) Solve the Schrödinger equation in each region of x, apply the 'boundary' conditions at $x \to \pm\infty$, and spell out the continuity conditions at $x = 0$ and $x = L$. Assume a bound state, thus $E < V_1 \leq V_2$.

(b) Solve the continuity conditions and show that the bound states energies — and the corresponding k, κ_1, and κ_2 — must obey

$$kL \;=\; \arctan\frac{\kappa_1}{k} \;+\; \arctan\frac{\kappa_2}{k} \;+\; n\times\pi \;=\; \arccos\sqrt{\frac{E}{V_1}} \;+\; \arccos\sqrt{\frac{E}{V_2}} \;+\; n\times\pi \tag{4}$$

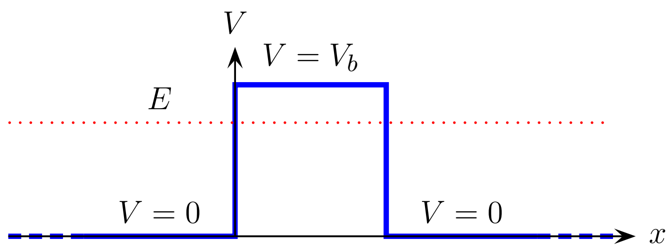
for an integer n

The best way to solve eq. (4) is graphic: plot both sides of the equation as functions of k — or as functions of $\sqrt{E/V_1} = k \times \text{const}$ — and look for the intersections. Note that the RHS has a separate branch for each n, so there could be several intersections and hence several bound states with different energies.

(c) Show that for a symmetric well with $V_2 = V_1$ and any $L > 0$, the branch with $n = 0$ always has an intersection. Thus, a symmetric well always has at least one bound state.

(d) On the other hand, an asymmetric well with $V_1 \neq V_2$ and small enough L has no bound state at all. Show this, and also calculate the smallest width L_{min} of the asymmetric well that does have a bound state.

2. Next, consider a symmetric square barrier,



$$V(x) = \begin{cases} V_b & \text{for } 0 < x < L, \\ 0 & \text{for } x < 0 \text{ or } x > L. \end{cases} \tag{5}$$

Calculate the reflection and transmission coefficients R and T for this barrier as functions of the unbound state energy $E > 0$ for three cases:

(a) $E < V_b$, tunneling under the barrier.

(b) $E > V_b$, flying over the barrier.

(c) The barrier is actually a well with $V_b < 0$.
Hint: show that in terms of k' (inside the barrier/well) and k (outside), the wave function and the continuity equations are exactly in part (b), then re-use the result of part (b) instead of re-calculating it from scratch.

($\star$) Also, show that the reflection and transmission coefficients as functions of energy in all 3 cases are related by analytic continuation.