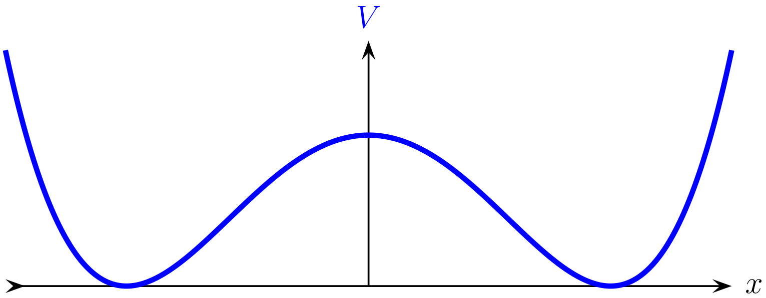
1. Consider a symmetric double-well potential



(1)

In the limit of deep and wide wells, each well has its own spectrum of bound states up to the top of the barrier. The (modernized) Bohr–Sommerfeld rule

$$S_{\text{bounce}}(E_n) \;=\; 2\pi\hbar(n+\tfrac{1}{2}), \tag{2}$$

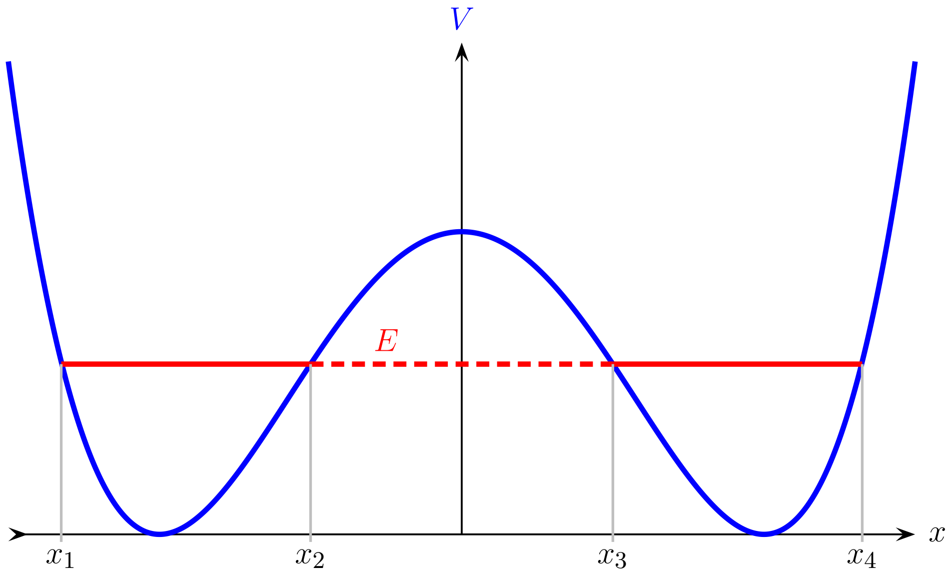
gives us the approximate energies of these bound states, and thanks to the symmetry between the two wells we have $E_n^{\text{left well}} = E_n^{\text{right well}}$, thus double degeneracy of the bound states of the whole double-well system.

However, the quantum tunneling through the barrier between the wells lifts this degeneracy: Each E_n level splits into a pair of non-degenerate levels $E_n \mp \delta E_n$, with the corresponding eigenstates being approximately

$$\begin{aligned} &\frac{|\text{right well}:n\rangle + |\text{left well}:n\rangle}{\sqrt{2}} \quad \text{for} \quad E \;=\; E_n - \delta E_n \\ \text{and} \quad &\frac{|\text{right well}:n\rangle - |\text{left well}:n\rangle}{\sqrt{2}} \quad \text{for} \quad E \;=\; E_n + \delta E_n\,. \end{aligned} \tag{3}$$

In this problem we shall see how this work and calculate the energy split δE_n.

Let's pick some energy level $E = E_n$ below the top of the barrier, and let x_1, x_2, x_3, x_4 denote the classical turning points in the double well for this energy level:



(4)

(a) Start with the WKB-approximate solutions which in the classically forbidden zone (between x_2 and x_3) look like

$$\Psi_\pm(x) \;=\; \frac{C}{\sqrt{\kappa(x)}}\left(\exp\left(+\int_0^x \kappa(x')\,dx'\right) \;\pm\; \exp\left(-\int_0^x \kappa(x')\,dx'\right)\right) \tag{5}$$

for some overall constant C. Extend these solutions to the classically allowed zones between x_1 and x_2 and between x_3 and x_4 and show that

$$\text{for}\quad x_3 < x < x_4: \quad \Psi_\pm(x) \;=\; \frac{D}{\sqrt{k(x)}} \times \sin\left(\frac{\pi}{4} \pm \delta\phi + \int_{x_3}^x k(x')\,dx'\right), \tag{6}$$

$$\text{for}\quad x_1 < x < x_2: \quad \Psi_\pm(x) \;=\; \pm\frac{D}{\sqrt{k(x)}} \times \sin\left(\frac{\pi}{4} \pm \delta\phi + \int_x^{x_2} k(x')\,dx'\right), \tag{7}$$

where $D \neq C$ is some overall constant, and the phase shift $\delta\phi$ follows from the bounce action $S_{\text{bounce}}^{\text{barrier}} = 2\hbar w$ in the reversed potential inside the barrier:

$$w = \int_{x_2}^{x_3} \kappa(x)\,dx, \qquad \delta\phi = \arctan\left(\tfrac{1}{2}e^{-w}\right) \approx \tfrac{1}{2}e^{-w} \ll 1. \tag{8}$$

(b) Show that $\delta\phi \neq 0$ modifies the Bohr–Sommerfeld rule for the bound state energy of an individual well from eq. (2) to

$$\text{for } E = E_n \mp \delta E_n : \quad \frac{1}{2\hbar} S_{\text{bounce}}^{\text{well}} = \int_{x_1}^{x_2} k(x)\,dx = \int_{x_3}^{x_4} k(x)\,dx = \pi(n + \tfrac{1}{2}) \mp \delta\phi\,. \tag{9}$$

Classically, the bounce action — *i.e.*, the net action over a complete period of the bound motion — depends on energy as

$$\frac{dS(E)}{dE} = T(E) = \text{time period of the bound motion.} \tag{10}$$

(c) Combine this rule with the modified Bohr–Sommerfeld rule (9) to show that

$$\delta E_n \approx \frac{2\hbar}{T(E_n)} \times \delta\phi(E_n). \tag{11}$$

2. Read my notes on superconductivity and solve exercises (a) through (f) embedded in those notes.