

1. This exercise is about the $SO(3)$ group of rotations in three space dimensions.

- (a) A vector \mathbf{v} rotated through an infinitesimal angle $d\alpha$ around axis \mathbf{n} becomes $\mathbf{v}' = \mathbf{v} + (d\alpha)\mathbf{n} \times \mathbf{v}$. Show that a rotation through a *finite* angle α results in

$$\mathbf{v}' = \cos \alpha \mathbf{v} + \sin \alpha \mathbf{n} \times \mathbf{v} + (1 - \cos \alpha)\mathbf{n}(\mathbf{n} \cdot \mathbf{v}). \quad (1)$$

Hint: $\mathbf{n} \times \mathbf{n} \times \mathbf{v} = -\mathbf{v} + \mathbf{n}(\mathbf{n} \cdot \mathbf{v})$ and hence $\mathbf{n} \times \mathbf{n} \times \mathbf{n} \times \mathbf{v} = -\mathbf{n} \times \mathbf{v}$.

- (b) Re-express formula (1) as $v'_i = R_{ij}(\alpha, \mathbf{n})v_j$ and write down the explicit form of the rotation matrix $R_{ij}(\mathbf{n}, \alpha)$ and show that it is an $SO(3)$ matrix — real, orthogonal 3×3 matrix with determinant $\det(R) = +1$. (Orthogonality is the real-number analogue of unitarity: An orthogonal matrix satisfies $RR^\top = R^\top R = 1$.)

- (c) Optional exercise: Show that any $SO(3)$ matrix R is a rotation matrix $R(\alpha, \mathbf{n})$ for some angle α and some axis \mathbf{n} .

Hint: Show that an $SO(3)$ matrix has eigenvalues $(e^{+i\alpha}, e^{-i\alpha}, +1)$, then identify α as the rotation angle and the eigenvector for the $+1$ eigenvalues as the axis of rotations.

In the $SO(3)$ matrix language, the multiplication law for successive rotations is simply the matrix product $R_3 = R_2 R_1$, or in index terms $R_{ik}^{(3)} = R_{ij}^{(2)} R_{jk}^{(1)}$. Thus, if we first rotate through angle α_1 around axis \mathbf{n}_1 and then rotate through angle α_2 around axis \mathbf{n}_2 , then the net effect is the $SO(3)$ matrix

$$R(\alpha_2, \mathbf{n}_2)R(\alpha_1, \mathbf{n}_1) = R_3 = R(\alpha_3, \mathbf{n}_3) \quad \text{for some } \alpha_3 \text{ and } \mathbf{n}_3, \quad (2)$$

where the second equality follows from part (c). Alas, calculating the net rotation's angle α_3 and axis \mathbf{n}_3 directly from this formula is painfully tedious.

Instead, there is a simpler Cayley–Klein method; originally, it involved [quaternions](#), but later was rephrased in terms of the $SU(2)$ matrices, — *i.e.*, complex unitary 2×2 matrices of unit determinant. Here is how it works: For any rotation $R(\alpha, \mathbf{n})$, let's define an $SU(2)$ matrix

$$Q(\alpha, \mathbf{n}) = \exp\left(-i\frac{\alpha}{2} \mathbf{n} \cdot \vec{\sigma}\right) = \cos \frac{\alpha}{2} - i \sin \frac{\alpha}{2} \mathbf{n} \cdot \vec{\sigma}. \quad (3)$$

where $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices, *cf.* [homework set#3](#) (problem#1).

(d) Show that

$$Q^\dagger(\alpha, \mathbf{n}) \sigma_i Q(\alpha, \mathbf{n}) = R_{ij}(\alpha, \mathbf{n}) \sigma_j. \quad (4)$$

(e) Now suppose the angles $\alpha_{1,2,3}$ and the unit vectors $\mathbf{n}_{1,2,3}$ satisfy the Cayley–Klein equation

$$Q(\mathbf{n}_3, \alpha_3) = Q(\mathbf{n}_2, \alpha_2) Q(\mathbf{n}_1, \alpha_1). \quad (5)$$

Use eq. (4) to show that the corresponding 3×3 rotation matrices satisfy eq. (2).

(f) Finally, solve the Cayley-Klein equation (5) for the (\mathbf{n}_3, α_3) in terms of the (\mathbf{n}_2, α_2) and the (\mathbf{n}_1, α_1) .

2. In the Heisenberg picture of the rotational symmetries, a rotation through angle α around axis \mathbf{n} transforms an operator \hat{A} into

$$\hat{A}' = \hat{\mathcal{R}}^\dagger(\mathbf{n}, \alpha) \hat{A} \hat{\mathcal{R}}(\mathbf{n}, \alpha). \quad (6)$$

Consequently, a scalar operator \hat{S} must be invariant under all rotations,

$$\hat{\mathcal{R}}^\dagger(\mathbf{n}, \alpha) \hat{S} \hat{\mathcal{R}}(\mathbf{n}, \alpha) = \hat{S}, \quad (7)$$

while the 3 components $(\hat{V}_x, \hat{V}_y, \hat{V}_z)$ of a vector operator $\hat{\mathbf{V}}$ transform into each other as

$$\hat{\mathcal{R}}^\dagger(\mathbf{n}, \alpha) \hat{V}_i \hat{\mathcal{R}}(\mathbf{n}, \alpha) = R_{ij}(\mathbf{n}, \alpha) \hat{V}_j. \quad (8)$$

If fact, eqs. (7) is a definition of a scalar operator while eq. (8) is a definition of a vector operator.

These definitions of scalar and vector operators can be restated in terms of commutation relations with the angular momentum operators $\hat{J}_{x,y,z}$:

$$\hat{S} \text{ is a scalar iff } [\hat{S}, \hat{J}_i] = 0, \quad (9)$$

$$\hat{\mathbf{V}} \text{ is a vector iff } [\hat{V}_i, \hat{J}_j] = i\hbar \epsilon_{ijk} \hat{V}_k. \quad (10)$$

(a) Show that eqs. (7) and (9) for a scalar operator are equivalent to each other: if \hat{S} remains invariant under all rotations as in eq. (7) then it must commute with the $\hat{J}_{x,y,z}$, and conversely if \hat{S} commutes with all 3 $\hat{J}_{x,y,z}$ then it's invariant under all rotations.

For the vector operators, eqs. (8) and (10) are also equivalent to each other, but this takes a bit more work to prove. In particular, we need the Baker–Hausdorff lemma: For any two operators \hat{B} and \hat{C} ,

$$e^{\hat{B}}\hat{C}e^{-\hat{B}} = \hat{C} + [\hat{B}, \hat{C}] + \frac{1}{2}[\hat{B}, [\hat{B}, \hat{C}]] + \cdots + \frac{1}{n!}[\hat{B}, [\hat{B}, \dots, [\hat{B}, \hat{C}] \dots]]_n + \cdots. \quad (11)$$

- (b) Prove this lemma. Hint: Let $\hat{C}_\lambda = e^{\lambda\hat{B}}\hat{C}e^{-\lambda\hat{B}}$, show that $\frac{d}{d\lambda}\hat{C}_\lambda = [\hat{B}, \hat{C}_\lambda]$, iterate for the higher derivatives, and then expand into a series in powers of λ .
- (c) Show that if the 3 components \hat{V}_i of a vector operator $\hat{\mathbf{V}}$ transform into each other under space rotations according to eq. (8), then their commutation relations with the angular momentum components should be as in eq. (10). Hint: consider infinitesimally small α and work to the first order in α .
- (d) Now, suppose three component operators $\hat{V}_{x,y,z}$ satisfy the commutation relations (10). Use the Baker-Hausdorff lemma to verify eq. (8) for any finite rotation.
Hint: $\hat{\mathcal{R}}(\mathbf{n}, \alpha) = \exp(-i\frac{\alpha}{\hbar}\mathbf{n} \cdot \hat{\mathbf{J}})$.

Note that for spin-half system with no other degrees of freedom, $\hat{\mathbf{J}}$ is simply the spin $\hat{\mathbf{S}} = \frac{\hbar}{2}\vec{\sigma}$ and hence $\hat{\mathcal{R}}(\mathbf{n}, \alpha) = U(\mathbf{n}, \alpha)$ (*cf.* eq. (3)). Thus, eq. (4) is simply a special case of the general formula (8).

- (e) Next, consider the tensor operators. By definition, the component operators $\hat{T}_{i_1, i_2, \dots, i_n}$ form an n -index tensor operator if and only if for any rotation,

$$\hat{\mathcal{R}}^\dagger \hat{T}_{i_1, i_2, \dots, i_n} \hat{\mathcal{R}} = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_n j_n} \hat{T}_{j_1, j_2, \dots, j_n}. \quad (12)$$

Show that $\hat{T}_{i_1, i_2, \dots, i_n}$ form a tensor if and only if

$$\left[\hat{T}_{i_1, i_2, \dots, i_n}, \hat{J}_j \right] = i\hbar \epsilon_{i_1 j k} \hat{T}_{k, i_2, \dots, i_n} + \cdots + i\hbar \epsilon_{i_n j k} \hat{T}_{i_1, \dots, i_{n-1}, k}. \quad (13)$$

- (f) Finally, use $R^\top R = 1$ and $\det(R) = 1$ to show that for any two-index tensor operator \hat{A}_{ij} , $\hat{B} = \text{tr}(\hat{A}) = \delta_{ij} \hat{A}_{ij}$ is a scalar and $\hat{C}_i = \epsilon_{ijk} \hat{A}_{jk}$ is a vector. Then use these facts to show that for any two vector operators $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$ — regardless of whether they commute with each other or not, — the dot product $\hat{\mathbf{F}} \cdot \hat{\mathbf{G}}$ is a scalar operator and the cross product $\hat{\mathbf{F}} \times \hat{\mathbf{G}}$ is a vector operator.