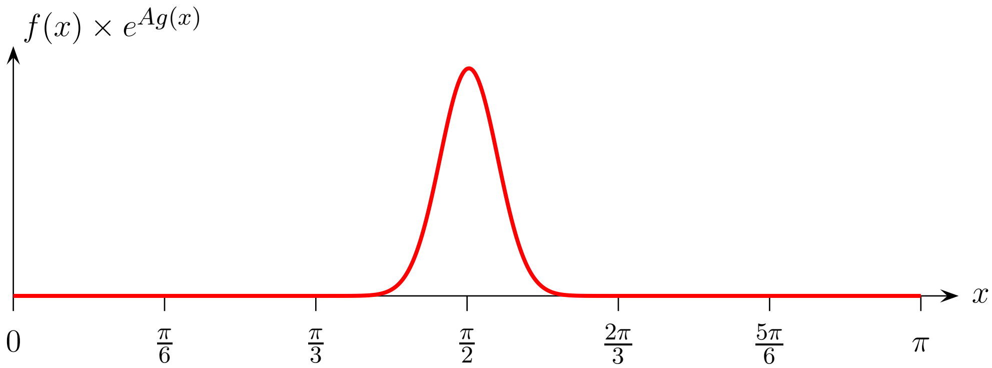
1. Saddle Point Method of Asymptotic Expansion

1.1 The Real Case.

Consider an integral of the form

$$I(A) = \int_{x_1}^{x_2} dx\, f(x)\, e^{Ag(x)} \qquad (1.1)$$

where f and g are some real functions of x and $A > 0$ is a parameter. For large values of A the integrand has narrow sharp peaks like this



(in this particular example $f(x) = x$, $g(x) = \sin x$ and $A = 100$), and the integral is completely dominated by the biggest peak. Each peak is located at a maximum of $g(x)$, and its width is $O(1/\sqrt{A})$. So let x_0 be the location of the biggest maximum of g between x_1 and x_2, and let's change the integration variable from x to y according to

$$x = x_0 + \frac{y}{\sqrt{A}}. \qquad (1.2)$$

Expanding $Ag(x)$ in powers of y, we have

$$Ag(x) = Ag(x_0) + \tfrac{1}{2}y^2 g''(x_0) + \frac{y^3 g'''(x_0)}{6\sqrt{A}} + \cdots \qquad (1.3)$$

(the first-derivative term is missing here because x_0 is a maximum of g). Treating this expansion as expansion in powers of $1/\sqrt{A}$ rather than y, we expand the exponential $e^{Ag(x)}$

as

$$e^{Ag(x)} \;=\; e^{Ag(x_0)} \cdot e^{y^2 g''(x_0)/2} \cdot \left(1 \;+\; \frac{y^3 g'''(x_0)}{6\sqrt{A}} \;+\; \frac{3y^4 g'''' + y^6 (g''')^2}{72A} \;+\; \cdots\right). \tag{1.4}$$

Similarly, assuming $f(x_0) \neq 0$, we have

$$f(x) \;=\; f(x_0) \cdot \left(1 \;+\; \frac{y f'(x_0)}{f(x_0)\sqrt{A}} \;+\; \frac{y^2 f''(x_0)}{2f(x_0)A} \;+\; \cdots\right). \tag{1.5}$$

Substituting eqs. (1.4) and (1.5) into (1.1) gives us

$$I(A) \;=\; \frac{f(x_0)\, e^{Ag(x_0)}}{\sqrt{A}} \int_{y_1}^{y_2} dy\, e^{y^2 g''(x_0)/2} \cdot \left(1 \;+ \sum_{n=1}^{\infty} A^{-n/2} P_n(y)\right), \tag{1.6}$$

where P_n are some polynomial functions of y; it is easy to show that $P_n(y)$ are odd polynomials for odd n and even polynomials for even n.

We assume that $x_1 < x_0 < x_2$ — *i.e.*, the maximum of g occurs strictly between x_1 and x_2 and not at one of the end points. Then, in the large A limit, $y_1 \to -\infty$ and $y_2 \to +\infty$ as $O(\sqrt{A})$, and since the Gaussian factor $e^{y^2 g''(x_0)/2}$ decreases very rapidly for $y \to \pm\infty$ (note $g''(x_0) < 0$ because x_0 is a maximum of g), we may extend the integration range of the integral (1.6) to the entire real axis. This is in-exact for a finite A, but the relative error in $I(A)$ due to this extension decrease with A faster than any power of A. Therefore,

$$\begin{aligned} I(A) \;&\approx\; \frac{f(x_0)\, e^{Ag(x_0)}}{\sqrt{A}} \int_{-\infty}^{+\infty} dy\, e^{y^2 g''(x_0)/2} \cdot \left(1 \;+ \sum_{n=1}^{\infty} A^{-n/2} P_n(y)\right) \\ &=\; f(x_0)\, e^{Ag(x_0)} \sqrt{\frac{2\pi}{-Ag''(x_0)}} \times \left(1 \;+ \sum_{n=1}^{\infty} \frac{C_n}{A^{n/2}}\right), \end{aligned} \tag{1.7}$$

where

$$C_n \;=\; \frac{\int_{-\infty}^{+\infty} dy\, e^{g''(x_0) y^2/2} \times P_n(y)}{\int_{-\infty}^{+\infty} dy\, e^{g''(x_0) y^2/2}}. \tag{1.8}$$

Moreover, since all the odd-numbered $P_{2n-1}(y)$ polynomial are odd WRT $y \to -y$, all the odd C_{2n-1} vanish, so only the even C_{2n} contribute to the series (1.7), thus

$$I(A) \;=\; f(x_0)\, e^{Ag(x_0)} \sqrt{\frac{2\pi}{-Ag''(x_0)}} \times \left(1 \;+\; \sum_{n=1}^{\infty} \frac{C_{2n}}{A^n}\right). \tag{1.9}$$

Working out explicit formulae for the C_{2n} in terms of the derivatives of the f and g functions at x_0 is a straightforward (but rather boring) exercise; for example

$$C_2 \;=\; \left(-\frac{f''}{2fg''} \;+\; \frac{f'g'''}{2f(g'')^2} \;+\; \frac{g^{(4)}}{8(g'')^2} \;-\; \frac{5(g''')^2}{24(g'')^3}\right) @x_0\,. \tag{1.10}$$

Fortunately, we won't need such formulae in these notes, so let me skip their derivation.

The series in eq. (1.9) usually has zero radius of convergence, so it cannot be actually summed up for any finite value of A. Instead, it's an *asymptotic series* whose partial sums have the right asymptotic behavior in the large A limit,

$$1+\sum_{n=1}^{\infty}\frac{C_{2n}}{A^n} \;=\; 1+O(1/A) \;=\; 1+C_2/A+O(1/A^2) \;=\; 1+C_2/A+C_4/A^2+O(1/A^3) \;=\; \cdots \tag{1.11}$$

in the strict mathematical sense of $O(1/A^n)$ — it's no larger than const$/A^n$ but only for a large enough A. Thus the precise meaning of eq. (1.9) is

$$\begin{aligned} I(A) \;&=\; f(x_0)\, e^{Ag(x_0)} \sqrt{\frac{2\pi}{-Ag''(x_0)}} \cdot \big(1+O(1/A)\big) \\ &=\; f(x_0)\, e^{Ag(x_0)} \sqrt{\frac{2\pi}{-Ag''(x_0)}} \cdot \big(1+C_2/A+O(1/A^2)\big) \\ &=\; \cdots \end{aligned} \tag{1.12}$$

in the large A limit.

1.2 The Complex Case.

Now consider the case of complex $f(x)$ and $g(x)$. Again, in the large A limit the integrand is sharply peaked near the maximum of $\mathrm{Re}\, g(x)$, so it seems like we could proceed similarly to the real case. There is however one crucial difference — the maximum of $\mathrm{Re}\, g(x)$ is not necessarily the stationary point of the phase $\mathrm{Im}\, g(x)$, so we have to add a purely imaginary term $\sqrt{A}\, y\, g'(x_0)$ to the expansion (1.3) for the $Ag(x)$. Consequently, the integral (1.6) becomes

$$I(A) \;=\; \frac{f(x_0)\, e^{Ag(x_0)}}{\sqrt{A}} \int dy\, e^{y^2 g''(x_0)/2} e^{y\sqrt{A} g'(x_0)} \cdot \left(1 \;+ \sum_{n=1}^{\infty} A^{-n/2} P_n(y)\right), \tag{1.13}$$

and the rapidly (in the large A limit) oscillating phase factor $e^{y\sqrt{A}g'(x_0)}$ severely suppresses the asymptotic behavior of the integral. Specifically, the leading term in the expansion now gives us

$$I(A) \;\xrightarrow[A\to\infty]{?}\; f(x_0)\, e^{Ag(x_0)} \sqrt{\frac{2\pi}{-Ag''(x_0)}} \cdot \exp\left(-A{g'}^2(x_0)/g''(x_0)\right), \tag{1.14}$$

and the last factor here is very small because the real part of ${g'}^2(x_0)/g''(x_0)$ is always positive. Consequently, a maximum of $\mathrm{Re}\, g(x)$ does not contribute at full strength unless it also happen to be a stationary point of the phase $\mathrm{Im}\, g(x)$. The suppression is so strong that the region around a maximum of $\mathrm{Re}\, g$ that is not a stationary point of the phase may no longer dominate the large A asymptotic behavior of the integral. This calls for a different approach in the complex case.

Indeed, proper complexification of the integral (1.1) goes beyond making f and g complex functions of a real variable x. Instead, we should take f and g to be *complex analytic functions of a complex variable*, and write a contour integral

$$I(A) \;= \int\limits_{\Gamma} dz\, f(z) \times e^{Ag(z)} \tag{1.15}$$

over some contour Γ in the complex plane. A fundamental theorem of complex analysis states that contour integrals of analytic functions are invariant under any continuous deformation of the contour that does not affects its end points (if any) and does not drag contour over any singularities of the integrand. Thus for the problem at hand, we deform the contour until the maximum of $\mathrm{Re}\,g$ along the contour is also a stationary point of the phase $\mathrm{Im}\,g$. Often, such deformation turns a real interval from x_1 to x_2 into a non-real contour in the complex plane. This may seem like making the problem even more complex (in both senses of the word) than it is, but in fact this leads to an easily obtainable large A asymptotics.

The points in the complex plane where a maximum of $\mathrm{Re}\,g$ (along some contour) coincides with a stationary point of the phase $\mathrm{Im}\,g$ are the zeros of the complex derivative $g'(z)$. Near such a point $\mathrm{Re}\,g(z)$ looks like a saddle or the top of a mountain pass — it has a maximum along some directions in the complex plane and a minimum along other directions — hence the two names for the asymptotic method described here: *the saddle point method* or *the mountain pass method*. The mountain pass analogy is particularly apt, for a properly routed contour not only goes through a zero of g', but crosses that zero in the manner of a highway crossing a mountain pass, by starting in a valley (of low $\mathrm{Re}\,g$), going up till it reaches the top of the pass, then going down into another valley. Although a mountain goat might think of a pass as a low point on a trail from one hill to another, thinking like a goat does not work for computing integrals.

Once you have the right contour Γ, obtaining the large A asymptotics of the integral (1.15) becomes analogous to the real case. First, we change the integration variable from z to a new complex variable y related to z via

$$z \;=\; z_0 \;+\; \frac{\eta y}{\sqrt{A}}\,, \tag{1.16}$$

where z_0 is a zero of $g'(z)$ and η is a unimodular complex number, $|\eta| = 1$. Second, we expand the integrand of (1.15) into powers of $1/\sqrt{A}$ in the same manner as we did in the real case; this gives us

$$I(A) \;=\; \frac{\eta f(z_0)\, e^{Ag(z_0)}}{\sqrt{A}} \int\limits_{\Gamma'} dy\, e^{y^2\eta^2 g''(z_0)/2} \cdot \left(1 \;+ \sum_{n=1}^{\infty} A^{-n/2} P_n(\eta y)\right), \tag{1.17}$$

where Γ' is the integration contour in the y plane. The Γ' always crosses the $y = 0$ point, but the direction of that crossing depends on η; to simplify our arguments, let us choose the η that would make the direction of the Γ' contour — *i.e.*, the direction of dy along the Γ' —

real and positive at $y=0$. Consequently, for the original contour Γ (in the z plane) which crosses the saddle point z_0 as a mountain highway rather than as a goat trail, the coefficient of $y^2/2$ in the exponent in eq. (1.17) has a negative real part,

$$\mathrm{Re}(\eta^2 \times g''(z_0)) \ < \ 0. \tag{1.18}$$

Also, the Γ' contour in the y plane is tangent to the real axis at $y=0$, so in the $A\to\infty$ limit it becomes the real axis itself, plus some appendages at infinity. Similar to the real case, contributions of these appendages to the integral (1.17) has relative magnitude smaller than any negative power of A, so we may safely neglect it. In other words, we may replace Γ' with the real axis, hence

$$\begin{aligned} I(A) \ &\approx \ \frac{\eta f(z_0)\, e^{Ag(z_0)}}{\sqrt{A}} \int\limits_{-\infty}^{+\infty} dy\, e^{y^2\eta^2 g''(z_0)/2} \times \left(1 \ + \sum_{n=1}^{\infty} A^{-n/2} P_n(\eta y)\right) \\ &\qquad \langle\!\langle \text{ where the integral converges thanks to the inequality (1.18)} \rangle\!\rangle \\ &= \ \sqrt{\frac{2\pi}{A}} \exp\bigl(Ag(z_0)\bigr) \times \frac{\eta f(z_0)}{\sqrt{-\eta^2 g''(z_0)}} \times \left(1 \ + \sum_{n=1}^{\infty} \frac{C_{2n}}{A^n}\right), \end{aligned} \tag{1.19}$$

in complete analogy to the real formula (1.12). Note that the η parameter in eq. (1.19) essentially cancels itself out, except that it helps determine the sign of the complex square root $\sqrt{-\eta^2 g''(z_0)}$ — the real part of this root should be positive.

Although formulae (1.12) and (1.19) differ very little, there is one important difference between large A asymptotics of real and complex integrals $I(A)$. The asymptotics of a real integral (1.1) is always dominated by the global maximum of $g(x)$ *within the integration range*, which can be either the biggest local maximum x_0 strictly within the range, or one of the end points of that range (in which case eq. (1.12) does not apply). Local maxima of $g(x)$ *outside* the integration range of (1.1) never play any role in the asymptotic expansion even if they are bigger than any maximum within the range.

For the complex integrals (1.15), determining which of the saddle points of $g(z)$ in the complex plane dominates the integral's asymptotics is not so straightforward. Given the freedom to deform the integration contour Γ, one cannot simply say that a saddle point z_0 is "within the integration range" while another saddle point is "outside the integration range", because Γ can always be deformed to cross any point we like. Usually, the general direction of the original contour and the phases of g'' at saddle points which control the directions in which those saddle points should be traversed give sufficient clues to determine which saddle point is dominant and how to deform the contour to go through it. However, such determination is somewhat of a black art best explained on specific examples; one such example — the asymptotic behavior of Airy functions — shall be discussed in the next section.

2. Airy Functions

2.1 Construction.

The Airy functions Ai and Bi are solutions of the linear differential equation

$$\Psi''(z) \ - z\Psi(z) \ = \ 0. \tag{2.1}$$

Physically, it is the scale-invariant form of the Schrödinger equation for a quantum particle subject to a constant force, *i.e.*, linear potential $V(x) = -Fx$,

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} \ - \ Fx \times \Psi(x) \ = \ E\Psi(x). \tag{2.2}$$

The relation between z and the particle's coordinate x is

$$z \ = \ -\,\mathrm{sign}(F)\sqrt[3]{\frac{2m|F|}{\hbar^2}} \times (x - x_0) \tag{2.3}$$

where $x_0 = -E/F$ is the classical turning point where $V(x_0) = E$.

For Bessel functions experts, the easiest way to solve the equation (2.1) is to substitute

$$z \ = \tfrac{2i}{3} y^{3/2}, \quad \Psi(z) \ = \ y^{1/3} J(y). \tag{2.4}$$

In terms of $J(y)$ equation (2.1) becomes

$$J''(y) \ + \ \frac{J'(y)}{y} \ + \ \left(1 - \frac{1}{9y^2}\right) J(y), \tag{2.5}$$

which is the Bessel equation of the order 1/3. Thus, $J(y)$ is a linear combination of the Bessel functions $J_{+1/3}(y)$ and $J_{-1/3}(y)$.

However, it is more instructive to solve the equation (2.1) in a different way. Let us perform a Laplace-like transform and look for a solution $\Psi(z)$ in the form of a contour integral

$$\Psi(z) \;=\; \int\limits_{\Gamma} dt\, e^{tz}\Phi(t); \tag{2.6}$$

here Γ is some z-independent contour in the complex t plane and $\Phi(t)$ is an analytic function of t that does not have any singularities on the contour Γ. With these assumptions, the second derivative $\Psi''(z)$ is related to $t^2\Phi(t)$ via

$$\Psi''(z) \;=\; \int\limits_{\Gamma} dt\, e^{tz} t^2 \Phi(t). \tag{2.7}$$

On the other hand, $z\Psi(z)$ is related to the first derivative of $\Phi(t)$ via

$$\begin{aligned} z\Psi(z) \;&=\; \int\limits_{\Gamma} dt\, \frac{\partial e^{tz}}{\partial t} \times \Phi(t) \\ &\quad\langle\!\langle \text{ integrating by parts } \rangle\!\rangle \\ &=\; \Big[e^{tz}\Phi(t) \Big]_{\text{start}(\Gamma)}^{\text{end}(\Gamma)} \;-\; \int\limits_{\Gamma} dt\, e^{tz}\Phi'(t). \end{aligned} \tag{2.8}$$

Therefore, in terms of $\Phi(t)$, the second order eq. (2.1) is equivalent to a first order equation

$$t^2\Phi(t) \;+\; \Phi'(t) \;=\; 0, \tag{2.9}$$

plus a boundary condition

$$e^{tz}\Phi(t) \;=\; 0 \quad \text{on the boundary of the contour } \;\Gamma. \tag{2.10}$$

The general solution of the equation (2.9) is

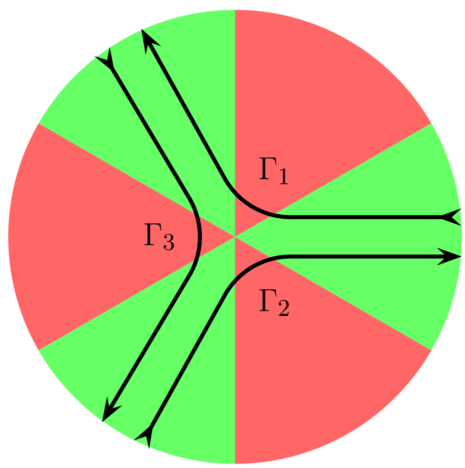
$$\Phi(t) \;=\; C \exp\left(-t^3/3\right), \tag{2.11}$$

where C is a constant; therefore, we have formally solved the Airy equation (2.1) in terms of the contour integral

$$\Psi(z) \;=\; C \int\limits_{\Gamma} dt\, \exp\left(tz - \tfrac{1}{3}t^3\right), \tag{2.12}$$

and all we need now is to specify the integration contour Γ.

Since the integrand of eq. (2.12) has no singularities for any finite t, only the end points of the contour would affect the integral; in particular, any closed Γ would lead to the trivial solution $\Psi(z) \equiv 0$. On the other hand, an open Γ with finite end points would violate the boundary condition (2.10). Hence, both end points of the contour Γ must be at the complex ∞, and the directions in which the two ends of the contour approach the ∞ would completely determine the integral (2.12) (the latter follows from the lack of finite singularities of the integrand). Those directions of approach are controlled by two considerations: First, one should approach the infinity along directions in which the integrand decreases rather than increases; for the problem at hand, this allows angles of approach between $-\pi/6$ and $+\pi/6$, between $+\pi/2$ and $+5\pi/6$, and between $-5\pi/6$ and $-\pi/2$, that is, within one of the three green sectors on the following diagram:



(2.13)

The second consideration is that all approaches within the same sector are equivalent; consequently, the two ends of the contour must belong to different sectors. These two considerations give us a choice of three contours — Γ_1, Γ_2 and Γ_3 on figure (2.13) — corresponding to three different solutions $\Psi_1(z)$, $\Psi_2(z)$ and $\Psi_3(z)$ of the Airy equation (2.1). Since the combined contour $\Gamma_1 + \Gamma_2 + \Gamma_3$ is shrinkable, it follows that $\Psi_1(z) + \Psi_2(z) + \Psi_3(z) \equiv 0$, so only two of the solutions are independent. The Airy functions Ai and Bi are the two independent solutions which happen to be real for real z, namely

$$\begin{aligned} Ai(z) &= \frac{i}{2\pi} \int\limits_{\Gamma_3} dt\, \exp\left(tz - \tfrac{1}{3}t^3\right), \\ Bi(z) &= \frac{1}{2\pi} \int\limits_{\Gamma_2 - \Gamma_1} dt\, \exp\left(tz - \tfrac{1}{3}t^3\right), \end{aligned} \tag{2.14}$$

where the integral over $\Gamma_2 - \Gamma_1$ is the integral over Γ_2, plus the integral over $-\Gamma_1$, the latter being Γ_1 traversed in the direction opposite to the arrow on figure (2.13).

2.2 Asymptotics

In quantum mechanics, the Airy functions help to match the WKB approximations to the wave functions on two sides of a classical turning point. To do that, we need to know the asymptotic behaviors of the Airy functions for large real z, both positive (for the classically forbidden side) and negative (for the classically allowed side). The easiest way to obtain this information is to use the saddle point method described in the previous section.

Although the integrals in eq. (2.14) do not have the exact form (1.15), we may bring them to that form by changing the integration variable from t to $\tau \equiv t/\sqrt{|z|}$, thus

$$\begin{aligned} Ai(z) &= \frac{i}{2\pi} \times |z|^{1/2} \int\limits_{\Gamma_3} d\tau \, \exp\left(|z|^{3/2}\left(\tfrac{z}{|z|}\tau - \tfrac{1}{3}\tau^3\right)\right), \\ Bi(z) &= \frac{1}{2\pi} \times |z|^{1/2} \int\limits_{\Gamma_2 - \Gamma_1} d\tau \, \exp\left(|z|^{3/2}\left(\tfrac{z}{|z|}\tau - \tfrac{1}{3}\tau^3\right)\right); \end{aligned} \tag{2.15}$$

note that the contours $\Gamma_{1,2,3}$ are essentially scale invariant, so they may be used without change in both t and τ planes. For the integrals (2.15), large $|z|^{3/2}$ plays the role of the large A parameter while $\frac{z}{|z|}\tau - \frac{1}{3}\tau^3$ acts as A-independent $g(\tau)$, exactly as in eq. (1.15). However, this $g(\tau)$ also depends on the phase of a complex z — or the sign of a real z, so we should have different asymptotics for $z \to +\infty$ than for $z \to -\infty$.

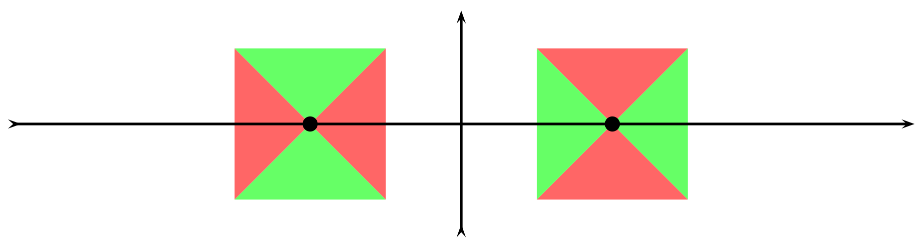
Let's start with the positive $z \to +\infty$ for which $g(\tau) = +\tau - \frac{1}{3}\tau^3$ has stationary points at $\tau_1 = +1$ and $\tau_2 = -1$. The positive saddle point τ_1 has a larger value of $\mathrm{Re}\,g(\tau_1)$ than the negative saddle point τ_2,

$$g(\tau_1 = +1) = +\tfrac{2}{3} \quad \textit{versus} \quad g(\tau_2 = -1) = -\tfrac{2}{3}, \tag{2.16}$$

so *naively* one expects both integrals (2.15) to be dominated by the τ_1 saddle point. However, the actual situation is more complicated due to the $\tau \to \tau^*$ symmetry of the integrals and the mountain-pass directions in which the two saddle points should be traversed. Indeed,

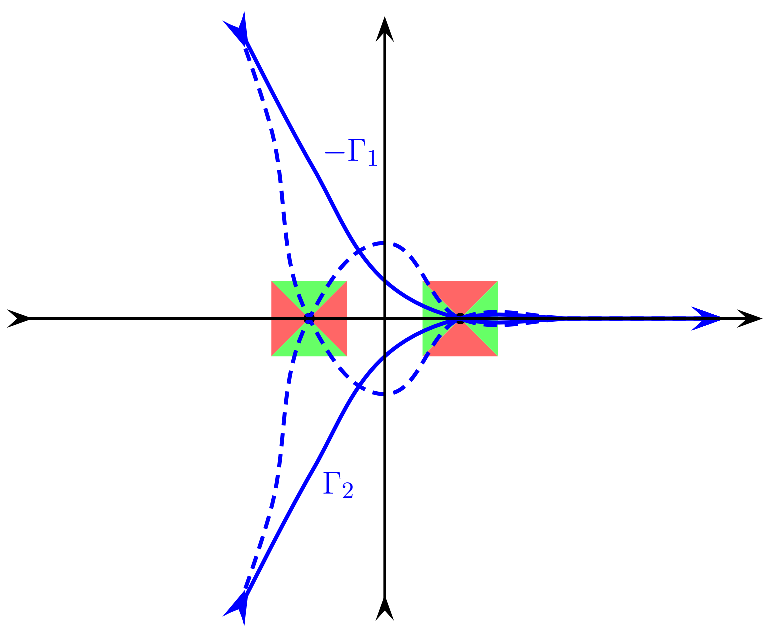
$$g''(\tau_1) = -2, \quad g''(\tau_2) = +2, \tag{2.17}$$

so the mountain-pass directions through τ_1 are within 45° of the horizontal while the mountain-pass directions through τ_2 are within 45° of the vertical; graphically,



(2.18)

For the Γ_2 and the $-\Gamma_1$ contours in the complex plane, it is easy to deform them so they run through the $\tau_1 = +1$ saddle point in the horizontal left-to-right direction, *cf.* solid blue lines on the diagram below:



(2.19)

We may also make each of the two contours run through both saddle points in mountain-pass directions — *cf.* the dashed blue lines — but they would run in the same left-to-right direction through the $\tau_1 = +1$ saddle point but in the opposite vertical direction through the $\tau_2 = -1$: the Γ_2 contour would run upward while the $-\Gamma_1$ contour would run downward.

Thanks to the $\tau \to \tau^*$ symmetry, this means that the contributions of the τ_2 saddle point to the integrals over the Γ_2 and the $-\Gamma_1$ contours would cancel each other from the $Bi(z)$ Airy function, so we may just as well re-route the two contours to skip the τ_2 point altogether and run only through the τ_1 saddle point. (The solid blue lines on the above diagram).

At the $\tau_1 = +1$ point, we have $g(\tau_1) = +\frac{2}{3}$, $f(\tau) \equiv 1$, $-g''(\tau_1) = +2$, $\eta = +1$ for both $+\Gamma_2$ and $-\Gamma_1$ contours, hence in the large $A = |z|^{3/2}$ limit

$$\sqrt{\frac{2\pi}{A}} \times \exp(Ag(\tau_1)) \times \frac{\eta f(\tau_1)}{\sqrt{-\eta^2 g''(\tau_1)}} \;=\; \frac{\sqrt{2\pi}}{|z|^{3/4}} \times \exp\bigl(+\tfrac{2}{3}|z|^{3/2}\bigr) \times \frac{1}{\sqrt{2}} \tag{2.20}$$

and therefore

$$\left.\begin{array}{l} \displaystyle\int\limits_{+\Gamma_2} d\tau\, \exp\bigl(|z|^{3/2} \times (\tau - \tfrac{1}{3}\tau^3)\bigr) \\ \displaystyle\int\limits_{-\Gamma_1} d\tau\, \exp\bigl(|z|^{3/2} \times (\tau - \tfrac{1}{3}\tau^3)\bigr) \end{array}\right\} \longrightarrow \frac{\sqrt{\pi}}{|z|^{3/4}} \times \exp\bigl(+\tfrac{2}{3}|z|^{3/2}\bigr) \times \bigl(1 + O(|z|^{-3/2}\bigr). \tag{2.21}$$

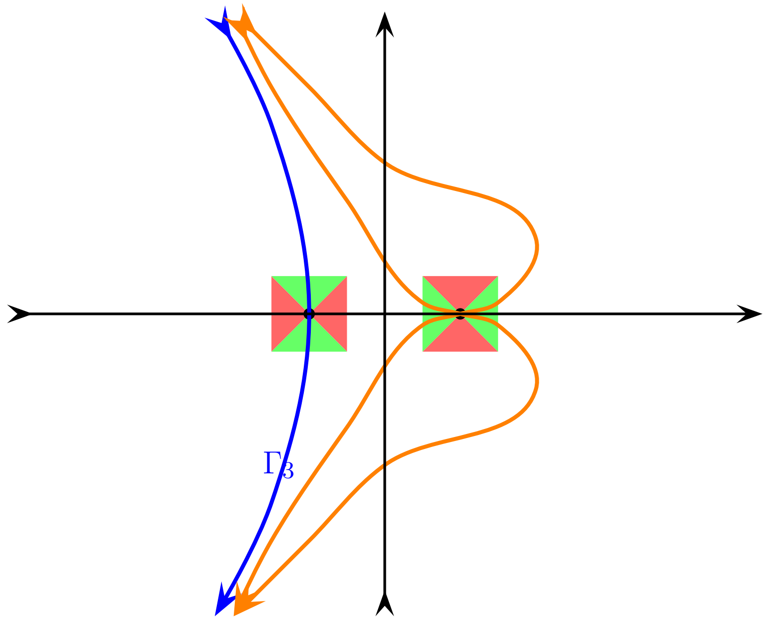
In terms of the *irregular Airy function*

$$Bi(z) \;=\; \frac{|z|^{1/2}}{2\pi} \times \sum \begin{pmatrix} \text{2 contour} \\ \text{integrals} \end{pmatrix}, \tag{2.22}$$

this gives us its asymptotic behavior for $z \to +\infty$ as

$$Bi(z) \;\xrightarrow[z\to+\infty]{}\; \frac{1}{\sqrt{\pi}|z|^{1/4}} \times \exp\bigl(+\tfrac{2}{3}|z|^{3/2}\bigr) \times \bigl(1 + O(|z|^{-3/2}\bigr). \tag{2.23}$$

For the regular Airy function $Ai(z)$, we have a very different challenge re-routing the Γ_3 contour to run through the saddle points $\tau_1 = +1$ and $\tau_2 = -1$. This time, it is quite natural to let Γ_3 run through the τ_2 in the mountain-pass-like downward direction — the blue line on the diagram below — but much less natural to let it run through the τ_1 in mountain-pass-like horizontal direction, as in the orange lines below:



(2.24)

Moreover, the two orange lines cross the τ_1 saddle point in opposite directions, so its contribution to the integrals over the two orange contours would have opposite sign. But by the $\tau \to \tau^*$ symmetry of the integrand, the two orange lines should be completely equivalent as integration contours, so we may just as well take the average (of the two integrals), from which the τ_1 contribution would completely cancel out! Thus, despite the τ_1 saddle point having a larger $\mathrm{Re}\, g(\tau)$ than the τ_2 saddle point, it's the τ_2 point which dominates the integral over the Γ_3 contour in the $A = |z|^{3/2} \to \infty$ limit.

Thus, for the regular Airy function $Ai(z)$ for $z \to +\infty$, we draw the Γ_3 contour as the blue line on the diagram (2.24) and focus on the contribution of the $\tau_2 = -1$ saddle point. At that point, $g(\tau_2) = -\frac{2}{3}$, $-g''(\tau_2) = -2$, $f(\tau) \equiv 1$, and $\eta = -i$ for the Γ_3 contour, thus

$$\sqrt{\frac{2\pi}{A}} \times \exp(Ag(\tau_2)) \times \frac{\eta f(\tau_2)}{\sqrt{-\eta^2 g''(\tau_2)}} \;=\; \frac{\sqrt{2\pi}}{|z|^{3/4}} \times \exp\bigl(-\tfrac{2}{3}|z|^{3/2}\bigr) \times \frac{-i}{\sqrt{2}}, \tag{2.25}$$

hence

$$\int\limits_{\Gamma_3} d\tau\, \exp\Bigl(|z|^{3/2}(\tau - \tfrac{1}{3}\tau^3)\Bigr) \;\approx\; \frac{-i\sqrt{\pi}}{|z|^{3/4}} \times \exp\bigl(-\tfrac{2}{3}z^{3/2}\bigr) \times \bigl(1 + O(z^{-3/2})\bigr) \tag{2.26}$$

and therefore

$$Ai(z) \;=\; \frac{i|z|^{1/2}}{2\pi} \times (2.26) \;=\; \frac{1}{2\pi^{1/2}\,|z|^{1/4}} \times \exp\bigl(-\tfrac{2}{3}|z|^{3/2}\bigr) \times \bigl(1 + O(z^{-3/2})\bigr). \qquad (2.27)$$

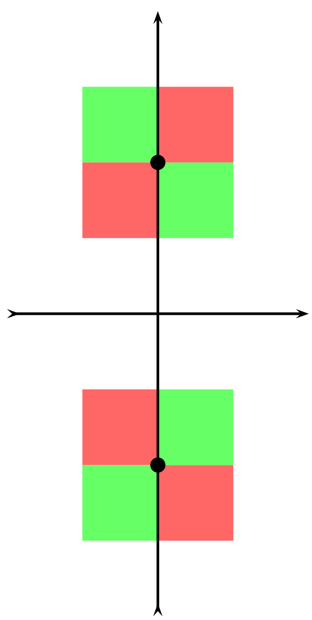
The asymptotic limits (2.27) and (2.23) of Airy functions are for the large *positive* values of z. For a *negative* z, we have

$$g(\tau) \;=\; -\tau \;-\; \frac{\tau^3}{3}\,, \qquad (2.28)$$

so the saddle points are at imaginary locations $\tau_{3,4} = \pm i$ rather than the real real $\tau_{1,2} = \pm 1$. At these saddle points, we have

$$\begin{aligned} g(\tau_3 = +i) \;&=\; -\tfrac{2}{3}i, \qquad & g''(\tau_3) \;&=\; -2i,\\ g(\tau_4 = -i) \;&=\; +\tfrac{2}{3}i, \qquad & g''(\tau_4) \;&=\; +2i. \end{aligned} \qquad (2.29)$$

In particular, both saddle points have the same value of $\mathrm{Re}g(\tau_{3,4}) = 0$, so we expect the two saddle points to be co-dominant for both Airy functions $Ai(z)$ and $Bi(z)$ in the $z \to -\infty$ limit. Also, the second derivatives indicate the mountain-pass-like directions for traversing the two saddle points:

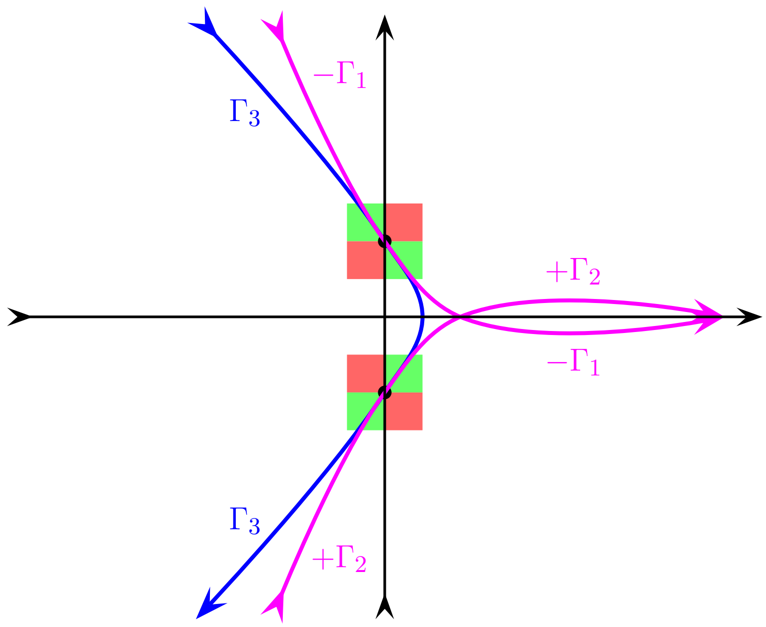


NW to SE or SE to NW
through $\tau_3 = +i$

(2.30)

NE to SW or SW to NE
through $\tau_4 = -i$

Given the general directions of the three integration contours $-\Gamma_1$, $+\Gamma_2$, and Γ_3, this means that that the contours should be re-routed as shown on the diagram below:



(2.31)

In particular, the $-\Gamma_1$ contour crosses only one saddle point τ_3 in the direction $\eta = e^{-\pi i/4}$, hence

$$\sqrt{\frac{2\pi}{A}} \times \exp(Ag(\tau_3)) \times \frac{\eta f(\tau_3)}{\sqrt{-\eta^2 g''(\tau_3)}} \;=\; \frac{\sqrt{2\pi}}{|z|^{3/4}} \times \exp\bigl(-\tfrac{2}{3}i|z|^{3/2}\bigr) \times \frac{e^{-\pi i/4}}{\sqrt{2}} \tag{2.32}$$

and therefore

$$\int\limits_{-\Gamma_1} d\tau\, \exp\bigl(|z|^{3/2}(-\tau - \tfrac{1}{3}\tau^3)\bigr) \;\approx\; \frac{\sqrt{\pi}}{|z|^{3/4}} \times \exp\bigl(-i\tfrac{\pi}{4} \,-\, \tfrac{2}{3}i|z|^{3/2}\bigr) \times \bigl(1 + O(|z|^{-3/2})\bigr). \tag{2.33}$$

Likewise, the $+\Gamma_2$ contour also crossed only one saddle point, but τ_4 rather than τ_3 and in the direction $\eta = e^{+\pi i/4}$, hence

$$\sqrt{\frac{2\pi}{A}} \times \exp(Ag(\tau_4)) \times \frac{\eta f(\tau_4)}{\sqrt{-\eta^2 g''(\tau_4)}} \;=\; \frac{\sqrt{2\pi}}{|z|^{3/4}} \times \exp\bigl(+\tfrac{2}{3}i|z|^{3/2}\bigr) \times \frac{e^{+\pi i/4}}{\sqrt{2}} \tag{2.34}$$

and therefore

$$\int\limits_{+\Gamma_2} d\tau\, \exp\bigl(|z|^{3/2}(-\tau - \tfrac{1}{3}\tau^3)\bigr) \;\approx\; \frac{\sqrt{\pi}}{|z|^{3/4}} \times \exp\bigl(+i\tfrac{\pi}{4} \,+\, \tfrac{2}{3}i|z|^{3/2}\bigr) \times \bigl(1 + O(|z|^{-3/2})\bigr). \tag{2.35}$$

Altogether, this gives us the $z\to-\infty$ limit of the irregular Airy function,

$$Bi(z)\;=\;\frac{|z|^{1/2}}{2\pi}\big((2.33)+(2.35)\big)\;\longrightarrow\;\frac{1}{\pi^{1/2}|z|^{1/4}}\times\cos\!\big(\tfrac{\pi}{4}+\tfrac{2}{3}|z|^{3/2}\big)\times\big(1+O(|z|^{-3/2})\big). \tag{2.36}$$

Finally, the Γ_3 contour crosses both saddle points $\tau_{3,4}$, and both of them integrate to the integral over this contour. At the $\tau_3=+i$ saddle point the contour's direction is $\eta=e^{-\pi i/4}$, hence

$$\sqrt{\frac{2\pi}{A}}\times\exp(Ag(\tau_3))\times\frac{\eta f(\tau_3)}{\sqrt{-\eta^2 g''(\tau_3)}}\;=\;\frac{\sqrt{2\pi}}{|z|^{3/4}}\times\exp\!\big(-\tfrac{2}{3}i|z|^{3/2}\big)\times\frac{e^{-\pi i/4}}{\sqrt{2}} \tag{2.37}$$

while at the $\tau_4=-i$ saddle point Γ_3 runs in the $\eta=e^{-3\pi i/4}=-e^{+\pi i/4}$ direction, thus

$$\sqrt{\frac{2\pi}{A}}\times\exp(Ag(\tau_4))\times\frac{\eta f(\tau_4)}{\sqrt{-\eta^2 g''(\tau_4)}}\;=\;\frac{\sqrt{2\pi}}{|z|^{3/4}}\times\exp\!\big(+\tfrac{2}{3}i|z|^{3/2}\big)\times\frac{-e^{+\pi i/4}}{\sqrt{2}}\,. \tag{2.38}$$

Altogether,

$$\begin{aligned}\int\limits_{\Gamma_3}d\tau\,\exp\!\big(|z|^{3/2}(-\tau-\tfrac{1}{3}\tau^3)\big)\;&=\;\frac{\sqrt{2\pi}}{|z|^{3/4}}\times\exp\!\big(-\tfrac{2}{3}i|z|^{3/2}\big)\times\frac{e^{-\pi i/4}}{\sqrt{2}}\times\big(1+O(|z|^{-3/2})\big)\\&\quad+\;\frac{\sqrt{2\pi}}{|z|^{3/4}}\times\exp\!\big(+\tfrac{2}{3}i|z|^{3/2}\big)\times\frac{-e^{+\pi i/4}}{\sqrt{2}}\times\big(1+O(|z|^{-3/2})\big)\\&=\;\frac{\sqrt{\pi}}{|z|^{3/2}}\times(-2i)\sin\!\big(\tfrac{\pi}{4}\;+\;\tfrac{2}{3}|z|^{3/2}\big)\times\big(1+O(|z|^{-3/2})\big),\end{aligned} \tag{2.39}$$

and therefore the regular Airy function Ai behaves as

$$Ai(z)\;=\;\frac{i|z|^{1/2}}{2\pi}\times(2.39)\;=\;\frac{1}{\pi^{1/2}\,|z|^{1/4}}\,\sin\!\big(\tfrac{\pi}{4}\;+\;\tfrac{2}{3}|z|^{3/2}\big)\times\big(1+O(|z|^{-3/2})\big). \tag{2.40}$$

2.3 Airy Functions and the WKB Approximation.

Finally, consider 1D motion of a quantum particle in some potential $V(x)$. In the WKB approximation, the particle's wave function in the classically allowed region (where $V(x) < E$) looks like

$$\Psi(x) = \sum_{\pm} \frac{C_{\pm}}{\sqrt{k(x)}} \times \exp\left(\pm i \int dx\, k(x)\right), \qquad k(x) = \frac{\sqrt{2m(E - V(x))}}{\hbar}. \tag{2.41}$$

The approximation is valid when the potential is smooth so that $k(x)$ changes slowly on the $1/k$ scale of distance. A similar approximation exists in the classically forbidden region where $V(x) > E$, namely

$$\Psi(x) = \sum_{\pm} \frac{C_{\pm}}{\sqrt{\kappa(x)}} \times \exp\left(\pm \int dx\, \kappa(x)\right), \qquad \kappa(x) = \frac{\sqrt{2m(V(x) - E)}}{\hbar}; \tag{2.42}$$

again, this approximation is valid as long as $\kappa(x)$ changes slowly on the $1/\kappa$ scale.

Near a classical turning point x_0 where $V(x_0) = E$, both approximations (2.41) and (2.42) break down. Instead, near x_0 we treat the force $F = -dV/dx$ as approximately constant, so the wave function is approximately an Airy function of

$$z = -\operatorname{sign}(F) \sqrt[3]{\frac{2m|F|}{\hbar^2}} \times (x - x_0). \tag{2.3}$$

For $z \to \pm\infty$, x if far enough from the turning point x_0 so the Airy function of z should match the WKB approximation for a linear potential $V = -F(x - x_0)$. Specifically, $z \to +\infty$ corresponds to the classically forbidden side of the turning point x_0, so the Airy-function solution should match eq. (2.42), while the $z \to -\infty$ corresponds to the classically allowed side so the Airy function should match the eq. (2.41).

Let's see how this works. For the sake of definiteness, let $F > 0$, so the classically allowed region is to the right of x_0 while the classically forbidden region is to the left of x_0. On the forbidden side $x < x_0$, $z > 0$ we have:

$$\kappa(x) = \frac{\sqrt{2mF(x_0 - x)}}{\hbar} = \sqrt[3]{\frac{2mF}{\hbar^2}} \times z^{1/2}, \tag{2.43}$$

$$\int_x^{x_0} \kappa(x')\,dk' \;=\; \frac{2}{3}(x_0-x)^{3/2}\frac{\sqrt{2mF}}{\hbar} \;=\; \frac{2}{3}\,z^{3/2}, \tag{2.44}$$

hence the two WKB wave functions (2.42) become

$$\begin{aligned}
\Psi_1(x) &= \frac{C_1}{2\sqrt{\kappa(x)}}\exp\left(-\int_x^{x_0}\kappa(x')\,dk'\right) \;\to\; \frac{C_1}{2}\sqrt[6]{\frac{\hbar^2}{2mF}}\times z^{-1/4}\exp\bigl(-\tfrac{2}{3}z^{3/2}\bigr),\\
&\qquad \langle\!\langle\,\text{note factors of 2 in the denominator}\,\rangle\!\rangle \\
\Psi_2(x) &= \frac{C_2}{\sqrt{\kappa(x)}}\exp\left(+\int_x^{x_0}\kappa(x')\,dk'\right) \;\to\; C_2\sqrt[6]{\frac{\hbar^2}{2mF}}\times z^{-1/4}\exp\bigl(+\tfrac{2}{3}z^{3/2}\bigr),
\end{aligned} \tag{2.45}$$

in perfect agreement with the $z\to+\infty$ limits of the two Airy functions (2.27) and (2.23). Specifically,

$$\text{for } z\to+\infty,\quad \Psi_1(x) \;=\; C_1\sqrt{\pi}\sqrt[6]{\frac{\hbar^2}{2mF}}\times Ai(z), \qquad \Psi_2(x) \;=\; C_2\sqrt{\pi}\sqrt[6]{\frac{\hbar^2}{2mF}}\times Bi(z). \tag{2.46}$$

Likewise, in the allowed side $x>x_0,\ z<0$ we have:

$$k(x) \;=\; \frac{\sqrt{2mF(x-x_0)}}{\hbar} \;=\; \sqrt[3]{\frac{2mF}{\hbar^2}}\times|z|^{1/2}, \tag{2.47}$$

$$\int_{x_0}^{x} k(x')\,dk' \;=\; \frac{2}{3}(x-x_0)^{3/2}\frac{\sqrt{2mF}}{\hbar} \;=\; \frac{2}{3}\,|z|^{3/2}, \tag{2.48}$$

hence the two WKB wave functions (2.41), — or rather, their real linear combinations

$$\begin{aligned}
\Psi_1(x) &= \frac{C_1}{\sqrt{k(x)}}\times\sin\left(\frac{\pi}{4}+\int_{x_0}^{x}k(x')\,dx'\right),\\
\Psi_2(x) &= \frac{C_2}{\sqrt{k(x)}}\times\cos\left(\frac{\pi}{4}+\int_{x_0}^{x}k(x')\,dx'\right),
\end{aligned} \tag{2.49}$$

become

$$\begin{aligned}
\Psi_1(x) &\to C_1\sqrt[6]{\frac{\hbar^2}{2mF}}\times|z|^{-1/4}\sin\bigl(\tfrac{\pi}{4}+\tfrac{2}{3}|z|^{3/2}\bigr),\\
\Psi_2(x) &\to C_2\sqrt[6]{\frac{\hbar^2}{2mF}}\times|z|^{-1/4}\cos\bigl(\tfrac{\pi}{4}+\tfrac{2}{3}|z|^{3/2}\bigr),
\end{aligned}\tag{2.50}$$

in perfect agreement with the $z\to-\infty$ limits of the Airy functions (2.27) and (2.23). Specifically,

$$\text{for } z\to-\infty,\quad \Psi_1(x)\;=\;C_1\sqrt{\pi}\sqrt[6]{\frac{\hbar^2}{2mF}}\times Ai(z),\qquad \Psi_2(x)\;=\;C_2\sqrt{\pi}\sqrt[6]{\frac{\hbar^2}{2mF}}\times Bi(z).\tag{2.51}$$

Moreover, comparing eqs. (2.46) and (2.51) to each other, we immediately learn how to match the WKB solutions on the allowed and forbidden sides of the classical turning point x_0:

$$\begin{aligned}
&\text{forbidden side } x<x_0 && \longleftrightarrow && \text{allowed side } x>x_0\\
&\Psi_1(x)\;=\;\frac{C_1}{2\sqrt{\kappa(x)}}\exp\left(-\int_x^{x_0}\kappa(x')\,dx'\right) && \longleftrightarrow && \Psi_1(x)\;=\;\frac{C_1}{\sqrt{k(x)}}\sin\left(\frac{\pi}{4}+\int_{x_0}^{x}k(x')\,dx'\right),\\
&\Psi_2(x)\;=\;\frac{C_2}{\sqrt{\kappa(x)}}\exp\left(+\int_x^{x_0}\kappa(x')\,dx'\right) && \longleftrightarrow && \Psi_2(x)\;=\;\frac{C_2}{\sqrt{k(x)}}\cos\left(\frac{\pi}{4}+\int_{x_0}^{x}k(x')\,dx'\right).
\end{aligned}\tag{2.52}$$