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Unitary $N \times N$ matrix

complex matrix, preserves norm of any complex vector

$\vec{x} = (x_1 \cdots x_n)$, $\vec{x}' = Ux$

$$x_i' = \sum_j u_{ij} x_j$$

$$\sum_i |x_i'|^2 = \sum |x_i|^2$$

$$\sum_i x_i'^* y_i' = \sum_i x_i^* y_i \qquad \vec{x}' = Ux \quad \vec{y}' = Uy$$

$$\sum_{ikl} x_k^* u_{ik}^* \cdot u_{il} y_l$$

$$x^\dagger u^\dagger u y = x^\dagger y$$

True $\forall x, y \Rightarrow \boxed{u^\dagger u = 1}$

For A unitary u, $u^\dagger = u^{-1}$

A linear operator $\hat{u}$ in some Hilbert space is unitary iff it is invertible and preserves Dirac product;

$$\hat{u}^\dagger \hat{u} = 1.$$

For a ~~...~~ finite matrix $u^\dagger u = 1$ guarantees u is invertible & $u^{-1} = u^\dagger$

~~not true for infinite matrices~~ or operators.

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Example: space of $(x_0, x_1, x_2, \cdots \infty)$

basis of $|n\rangle$ $n = 0, 1, 2, \cdots \infty$

Let $\hat{U}|n\rangle = |n+1\rangle$

$\langle m|\hat{U}|n\rangle = \delta_{m,n+1}$

$\langle k|\hat{U}^\dagger|l\rangle = \delta_{k+1,l}$

$\langle k|U^\dagger U|n\rangle = \sum_m \langle k|U^\dagger|m\rangle\langle m|U|n\rangle$

$= \sum_{m=0}^{\infty} \delta_{k+1,m}\,\delta_{m,n+1} = \delta_{k+1,n+1} = \delta_{k,n}$

$\Rightarrow \hat{U}^\dagger\hat{U} = 1$

<u>But</u> $\hat{U}$ is not invertible

because no $\hat{U}|\psi\rangle = |0\rangle$

Also $\hat{U}\hat{U}^\dagger \neq 1$

$\langle k|\hat{U}\hat{U}^\dagger|n\rangle = \sum_m \langle k|U|m\rangle\langle m|U^\dagger|n\rangle$

$= \sum_{m=0}^{\infty} \delta_{m+1,k}\,\delta_{m+1,n}$

$m+1 \to m'$

$= \sum_{m'=1}^{\infty} \delta_{k m'}\,\delta_{m' n} = \delta_{k,n} \times \begin{cases} 1 & \text{if } k > 0 \\ 0 & \text{if } k = 0 \end{cases}$

$\neq \delta_{kn}$

$\hat{U}\hat{U}^\dagger \neq 1$

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Unitarity criteria:

U is unitary iff

$\Rightarrow$ either 1) $U^\dagger U = 1$ and U is invertible

or 2) $U^\dagger U = 1$ and $UU^\dagger = 1$.

or both

Any unitary operator is diagonalizable

$UU^\dagger = 1 = U^\dagger U \Rightarrow [U^\dagger, U] = 0$

$\rightarrow U$ has def. values in some $\perp$ bases.

$$\hat{U} = \sum_i |i\rangle \underbrace{u_i}_{\langle i|U|i\rangle} \langle i|$$

Eigenvalues u_i are unimodular

$|u_i| = 1$

$b\text{ec} \; 1 = U^\dagger U = \sum_i |i\rangle \underbrace{u_i^* u_i}_{\langle i|1|i\rangle} \langle i| \Rightarrow u_i^* u_i = 1.$

$\forall$ hermitian operator $\hat{H} = \hat{H}^\dagger$

& any real α

$U = \exp(i\alpha\hat{H})$ is unitary.

$\Rightarrow U^\dagger = \left(\sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \hat{H}^n\right)^\dagger = \sum_n \frac{(-i\alpha)^n}{n!} \underbrace{(\hat{H}^\dagger)^n}_{\hat{H}^n}$

$= \exp(-i\alpha\hat{H}).$

but also $U^{-1} = \exp(-i\alpha\hat{H}) \Rightarrow \exists U^{-1}$ & $U^{-1} = U^\dagger$.

bec $\frac{1}{\exp(i\alpha x)} = \exp(-i\alpha x)$

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Symmetries

A symmetry of a system:
a rule by which a system is transformed into an equivalent state.

Symmetries always form A group.
group product: consecutive action

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[rr, bend right, "yx"'] & B \arrow[r, "y"] & C \end{tikzcd}$$

associative product ~~$(xy)z = x(yz)$~~

$$z(yx) = (zy)x = zyx$$

$$\begin{tikzcd} B \arrow[r, "y"] \arrow[dr, "zy"] & C \arrow[d, "z"] \\ A \arrow[u, "x"] \arrow[ur, "yx"] \arrow[r] \arrow[r, bend right, "z(yx) = (zy)x"'] & D \end{tikzcd}$$

Group $\ni$ Trivial symmetry (no change @all)

$$A \xrightarrow{1} A$$

All symmetries are invertible

$$\forall x \; \exists \, x^{-1} \ni : x^{-1}x = xx^{-1} = 1$$

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Symmetries in QM.

Any symmetry must preserve superposition principle → it must act as a linear operator

$|\psi\rangle \to \hat{S}|\psi\rangle$

A symmetry must preserve probability overlaps $|\langle\psi'|\phi'\rangle|^2 = |\langle\psi|\phi\rangle|^2$

→ either $\langle\psi'|\phi'\rangle = \langle\psi|\phi\rangle$ (1)

or $\langle\psi'|\phi'\rangle = \langle\psi|\phi\rangle^*$ (2).

Most symmetries : case 1)

→ $\hat{S}$ must be unitary

~~$\langle\psi\psi|$~~ $\langle\psi'| = \langle\psi|\hat{S}^\dagger$, $|\phi'\rangle = \hat{S}|\phi\rangle$

$\Rightarrow \langle\psi'|\phi'\rangle = \langle\psi|S^\dagger S|\phi\rangle$

$= \text{must} = \langle\psi|\phi\rangle \quad \forall \psi, \phi$

$\Rightarrow \hat{S}^\dagger S = 1$

Also, $\hat{S}$ is invertible } $\hat{S}$ is unitary.

(2) Time reversal (motion reversal)

: anti-linear and anti-unitary.

i.e. $\tilde{S}(\alpha|\phi\rangle + \beta|\chi\rangle) = \alpha^* \hat{S}|\phi\rangle + \beta^* \hat{S}|\chi\rangle$

All other symmetries:

linear & unitary.

Translation symmetries

① $\vec{x} \to \vec{x} + \vec{a}$

Parametrised by 3-vectors $\vec{a}$

$$\left.\begin{array}{l} T_{\vec{a}} T_{\vec{b}} = T_{\vec{b}} T_{\vec{a}} = T_{\vec{a}+\vec{b}} \\ (T_{\vec{a}})^{-1} = T_{-\vec{a}}, \quad T_{0} = 1 \end{array}\right\} \text{group law}$$

QM ~~rele~~ of 1 particle

in position basis

$\hat{T}_{\vec{a}} |\vec{x}\rangle = |\vec{x} + \vec{a}\rangle \Rightarrow$ group law ok

unitary $\hat{T}_{\vec{a}} = \left(\frac{\vec{a} \cdot \vec{p}}{i\hbar}\right)$

$$T_{\vec{a}} = \exp(\vec{a} \cdot \vec{p} / i\hbar).$$

$$= \exp\left(-\frac{i}{\hbar} a_x \hat{p}_x\right) \exp\left(-\frac{i}{\hbar} a_y \hat{p}_y\right) \exp\left(-\frac{i}{\hbar} a_z \hat{p}_z\right)$$

3 momenta $\hat{p}_x, \hat{p}_y, \hat{p}_z$ generate the Translation symmetry group

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QM of 2 particles $\psi(\vec{x}_1, \vec{x}_2)$

$\psi(x_1, y_1, z_1, x_2, y_2, z_2)$

$$\hat{T}_{\vec{a}} |\vec{x}_1, \vec{x}_2\rangle = |\underbrace{\vec{x}_1 + \vec{a}, \vec{x}_2 + \vec{a}}_{\text{same } \vec{a}}\rangle$$

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$$\exp\left(-\frac{i}{\hbar}\vec{a}\cdot\vec{p}_1\right)\psi(\vec{x}_1,\vec{x}_2) = \psi(\vec{x}_1-\vec{a},\vec{x}_2)$$

$$\exp\left(-\frac{i}{\hbar}\vec{a}\cdot\vec{p}_2\right)\psi(\vec{x}_1,\vec{x}_2) = \psi(\vec{x}_1,\vec{x}_2-\vec{a}).$$

$$\hat{T}_a = \exp\left(-\frac{i}{\hbar}\vec{a}\cdot\vec{P}_{net}\right)$$

$$\vec{P}_{net} = \vec{p}_1 + \vec{p}_2$$

$$\hat{T}_a\psi(\vec{x}_1,\vec{x}_2) = \psi(\vec{x}_1-\vec{a},\vec{x}_2-\vec{a})$$

$$\hat{T}_a|\hat{x}_1,\hat{x}_2\rangle = |\vec{x}_1+\vec{a},\vec{x}_2+\vec{a}\rangle.$$

Do the n particles

$$\hat{T}_a = \exp\left(-\frac{i}{\hbar}\vec{a}\cdot\vec{P}_{net}\right)$$

$$\vec{P}_{net} = \sum_{\text{particles}}\vec{p}_i$$

$$\hat{T}_a|\hat{x}_1,\hat{x}_2,\ldots\hat{x}_n\rangle = |\vec{x}_1+\vec{a},\vec{x}_2+\vec{a},\ldots,\vec{x}_n+\vec{a}\rangle$$

Time evolution

$|\psi\rangle$ @ time $= t_0 \rightarrow |\psi\rangle$ @ later time $= t$.

Coherent time evolution (between measurements) preserves superposition principle, also preserves probability overlaps $\rightarrow$ linear unitary operators

$\hat{U}(t,t_0)$ $\quad |\psi\rangle(t) = \hat{U}(t,t_0)|\psi\rangle(t_0)$

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$\forall\, t_2 > t_1 > t_0$

$$\hat{U}(t_2, t_0) = \hat{U}(t_2, t_1) \cdot \hat{U}(t_1, t_0)$$

(1)

$$\begin{tikzcd}
{|\psi\rangle(t_2)} & & {|\psi\rangle(t_0)} \arrow[ll, "U(t_2,t_0)"'] \arrow[dl, "U(t_1,t_0)"] \\
& {|\psi\rangle(t_1)} \arrow[ul, "U(t_2,t_1)"] &
\end{tikzcd}$$

For $t_1 = t_0$ $\quad \hat{U}(t_1 = t_0) = 1$

→ Implies Schroedinger eq-n

$$i\hbar \frac{\partial}{\partial t} \hat{U}(t, t_0) = \hat{H}(t)\, \hat{U}(t, t_0)$$

$$\hat{U}(t, t_0)\, |\psi(t_0)\rangle = |\psi(t)\rangle$$

$$\rightarrow\ i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}\, |\psi(t)\rangle$$

for some Hermitian operator $\hat{H}$

Physics: $\hat{H}$ is the Hamiltonian operator, measures the Energy

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When $\hat{H}$ is time independent.

→ may solve $i\hbar \frac{\partial}{\partial t} \hat{U}(t, t_0) = \hat{H}\, \hat{U}(t, t_0)$

$$U(t = t_0) = 1$$

$$\rightarrow\ \hat{U}(t, t_0) = \exp\left(\frac{-i}{\hbar}(t - t_0)\hat{H}\right)$$