Problem $\mathbf{1}$(a):

Let's begin by working out the indexology of eq. (2). Note that for all 9 combinations of $i, j = x, y, z$, there is only one non-vanishing term on the RHS of eq. (2), namely unit matrix I for $i = j$ or $\pm i\sigma_k$ for $i \neq j$ and $k \neq i, j$. Specifically,

$$\begin{aligned} \sigma_x\sigma_x &= I, & \sigma_x\sigma_y &= +i\sigma_z\,, & \sigma_x\sigma_z &= -i\sigma_y\,, \\ \sigma_y\sigma_x &= -i\sigma_z\,, & \sigma_y\sigma_y &= I, & \sigma_y\sigma_z &= +i\sigma_x\,, \\ \sigma_z\sigma_x &= +i\sigma_y\,, & \sigma_z\sigma_y &= -i\sigma_x & \sigma_z\sigma_z &= I. \end{aligned} \tag{S.1}$$

Actually multiplying the Pauli matrices (1) in all possible combinations verifies all nine of these equations and hence eq. (2).

Problem $\mathbf{1}$(b):

Take any 3–vector $\mathbf{v}$ whose components v_i are either ordinary numbers or else operators which have nothing to do with spin and therefore commute with the Pauli matrices. For all such vectors $\mathbf{v}$,

$$\left(\mathbf{v}\cdot\vec{\sigma}\right)^2 \;=\; (v_i\sigma_i)(v_j\sigma_j) \;=\; v_iv_j\left(\sigma_i\sigma_j \;=\; \delta_{ij}I + i\epsilon_{ijk}\sigma_k\right) \;=\; (v_iv_i)I \;+\; i\epsilon_{ijk}v_iv_j\sigma_k\,, \tag{S.2}$$

or in 3–vector notations

$$\left(\mathbf{v}\cdot\vec{\sigma}\right)^2 \;=\; (\mathbf{v}^2)I \;+\; i(\mathbf{v}\times\mathbf{v})\cdot\vec{\sigma}. \tag{S.3}$$

Moreover, if the components of $\mathbf{v}$ commute with each other, $v_iv_j = v_jv_i$, then $\mathbf{v}\times\mathbf{v} = 0$. Indeed,

$$\begin{aligned} \epsilon_{ijk}v_iv_j &= -\epsilon jikv_iv_j && \langle\!\langle \text{ by antisymmetry of } \epsilon_{ijk} \rangle\!\rangle \\ &= -\epsilon_{ijk}v_jv_i && \langle\!\langle \text{ re-labeling the summation indices } i \leftrightarrow j \rangle\!\rangle \\ &= \tfrac{1}{2}\epsilon_{ijk}[v_i,v_j] && \langle\!\langle \text{ averaging the LHS and the previous line} \rangle\!\rangle \\ &\to\; 0 \quad \text{for } [v_i,v_j] \;=\; 0. \end{aligned} \tag{S.4}$$

Consequently, eq. (S.3) reduces to its first term,

$$\big(\mathbf{v}\cdot\vec{\sigma}\big)^2 \;=\; (\mathbf{v}^2)I. \tag{S.5}$$

Problem $\mathbf{1}$(c):

Thanks to eq. (S.5), for any even $n=2m=0,2,4,6,\ldots$

$$\big(\mathbf{v}\cdot\vec{\sigma}\big)^{2m} \;=\; (\mathbf{v}^2)^m\,I \;=\; v^{2m}\,I \tag{S.6}$$

while for any odd $n=2m+1=1,3,5,\ldots$

$$\big(\mathbf{v}\cdot\vec{\sigma}\big)^{2m+1} \;=\; (\mathbf{v}^2)^m\big(\mathbf{v}\cdot\vec{\sigma}\big) \;=\; v^{2m+1}\,\frac{\mathbf{v}\cdot\vec{\sigma}}{v} \tag{S.7}$$

where $v=\sqrt{\mathbf{v}^2}$. Consequently,

$$\begin{aligned}
\exp\big(\mathbf{v}\cdot\vec{\sigma}\big) \;&=\; \sum_{n=0}^{\infty}\frac{1}{n!}\big(\mathbf{v}\cdot\vec{\sigma}\big)^n \\
&=\; \sum_{\text{even}\,n}\frac{1}{n!}\big(\mathbf{v}\cdot\vec{\sigma}\big)^n \;+\; \sum_{\text{odd}\,n}\frac{1}{n!}\big(\mathbf{v}\cdot\vec{\sigma}\big)^n \\
&=\; \sum_{\text{even}\,n}\frac{1}{n!}\,v^n\,I \;+\; \sum_{\text{odd}\,n}\frac{1}{n!}\,v^n\,\frac{\mathbf{v}\cdot\vec{\sigma}}{v} \\
&=\; \left(\sum_{\text{even}\,n}\frac{v^n}{n!}\right)I \;+\; \left(\sum_{\text{odd}\,n}\frac{v^n}{n!}\right)\frac{\mathbf{v}\cdot\vec{\sigma}}{v} \\
&=\; \cosh(v)\,I \;+\; \sinh(v)\,\frac{\mathbf{v}\cdot\vec{\sigma}}{v}\,.
\end{aligned} \tag{S.8}$$

Problem $\mathbf{1}$(d):

Classically, a magnetic moment $\mathbf{m}$ misaligned with the external magnetic field $\mathbf{B}$ precesses with the Larmor frequency $\omega=\mathcal{G}B$ where $\mathcal{G}$ is the gyromagnetic ratio of the magnetic moment to the angular momentum, $\mathcal{G}=m/J$. For the atomic magnetic moment due to orbital motion of the electrons $\mathcal{G} = e/2cM_e$ (in Gauss units) while for the magnetic moment due to electrons' spins $\mathcal{G} = 2 \times e/2cM_e$, thus precession frequency

$$\omega_L = \frac{eB}{cM_e}. \tag{S.9}$$

Now consider the quantum magnetic moment due to electron's spin. For the spin Hamiltonian (3), the time evolution operators is

$$\hat{U}(t, t_0) = \exp\bigl(-i(t - t_0)\hat{H}/\hbar\bigr) = \exp\left(-i(t - t_0)\frac{m_B}{\hbar}(\mathbf{B} \cdot \vec{\sigma})\right). \tag{S.10}$$

Or in terms of the magnitude B and the direction $\mathbf{n}$ of the magnetic field,

$$\hat{U}(t, t_0) = \exp\left(-i(t - t_0)\frac{Bm_B}{\hbar}(\mathbf{n} \cdot \vec{\sigma})\right) \tag{S.11}$$

where

$$\frac{Bm_B}{\hbar} = \frac{B}{\hbar} \times \frac{e\hbar}{2cM_e} = \frac{Be}{2cM_e} = \frac{\omega_L}{2}, \tag{S.12}$$

hence

$$\hat{U}(t, t_0) = \exp\left(-i\frac{\omega(t - t_0)}{2}(\mathbf{n} \cdot \vec{\sigma})\right). \tag{S.13}$$

To actually evaluate this matrix exponential, we use eq. (S.8) for $\mathbf{v}$ having direction $\mathbf{n}$ and imaginary magnitude $v = -i\omega(t - t_0)/2$, thus

$$\begin{aligned}\hat{U}(t, t_0) &= \cosh\left(-i\frac{\omega(t - t_0)}{2}\right) I + \sinh\left(-i\frac{\omega(t - t_0)}{2}\right)(\mathbf{n} \cdot \vec{\sigma}) \\ &= \cos\left(\frac{\omega(t - t_0)}{2}\right) I - i\sin\left(\frac{\omega(t - t_0)}{2}\right)(\mathbf{n} \cdot \vec{\sigma}),\end{aligned} \tag{4}$$

quod erat demonstrandum.

Problem $\mathbf{1}$(e):

Let

$$|\Psi\rangle \;=\; \alpha\,|Z+\rangle \;+\; \beta\,|Z-\rangle \tag{S.14}$$

for some complex coefficients α, β such that $|\alpha|^2 + |\beta|^2 = 1$ (so that $\langle\Psi|\Psi\rangle = 1$); or in column-vector notations

$$|\Psi\rangle \;=\; \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{S.15}$$

In this state

$$\begin{aligned} \langle\Psi|\,\hat{\sigma}_x\,|\Psi\rangle \;&=\; \alpha^*\beta \;+\; \beta^*\alpha \;=\; 2\,\mathrm{Re}(\alpha^*\beta), \\ \langle\Psi|\,\hat{\sigma}_y\,|\Psi\rangle \;&=\; -i\alpha^*\beta \;+\; i\beta^*\alpha \;=\; 2\,\mathrm{Im}(\alpha^*\beta), \\ \langle\Psi|\,\hat{\sigma}_z\,|\Psi\rangle \;&=\; \alpha^*\alpha \;-\; \beta^*\beta \;=\; |\alpha|^2 \;-\; |\beta|^2, \end{aligned} \tag{S.16}$$

and therefore

$$\langle m_x\rangle \;=\; -m_B\times 2\,\mathrm{Re}(\alpha^*\beta), \quad \langle m_y\rangle \;=\; -m_B\times 2\,\mathrm{Im}(\alpha^*\beta), \quad \langle m_z\rangle \;=\; -m_B(|\alpha|^2-|\beta|^2). \tag{S.17}$$

Now consider the time evolution of the state Ψ — and hence of the magnetic moment (S.17) in a constant uniform magnetic field $\vec{N}$. As we saw in part (d), the evolution operator $\hat{U}(t,t_0)$ is spelled out in eq. (4). In particular, for the magnetic field in z direction, the matrix (4) becomes diagonal

$$\begin{aligned} \hat{U}(t,t_0) \;&=\; \cos\left(\frac{\omega(t-t_0)}{2}\right)\,I \;-\; i\sin\left(\frac{\omega(t-t_0)}{2}\right)\,\sigma_z \\ &=\; \cos\left(\frac{\omega(t-t_0)}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;-\; i\sin\left(\frac{\omega(t-t_0)}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &=\; \begin{pmatrix} \cos\left(\frac{\omega(t-t_0)}{2}\right) - i\sin\left(\frac{\omega(t-t_0)}{2}\right) & 0 \\ 0 & \cos\left(\frac{\omega(t-t_0)}{2}\right) + i\sin\left(\frac{\omega(t-t_0)}{2}\right) \end{pmatrix} \\ &=\; \begin{pmatrix} \exp\left(-i\frac{\omega(t-t_0)}{2}\right) & 0 \\ 0 & \exp\left(+i\frac{\omega(t-t_0)}{2}\right) \end{pmatrix}. \end{aligned} \tag{S.18}$$

Consequently, if at the time t_0

$$|\Psi(t_0)\rangle \;=\; \begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix}, \tag{S.19}$$

then at a later time $t > t_0$

$$|\Psi(t)\rangle \;=\; \hat{U}(t,t_0)\,|\Psi(t_0)\rangle \;=\; \begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix} \tag{S.20}$$

for

$$\alpha(t) \;=\; \exp\left(-i\frac{\omega(t-t_0)}{2}\right)\times\alpha_0\,, \quad \beta(t) \;=\; \exp\left(+i\frac{\omega(t-t_0)}{2}\right)\times\beta_0\,. \tag{S.21}$$

For the expectation value (S.17) of the electron's magnetic moment, this means that

$$\langle m_z\rangle(t) \;=\; -m_B\big(|\alpha|^2-|\beta|^2\big) \;=\; -m_B\big(|\alpha_0|^2-|\beta_0|^2\big) \;=\; \text{const} \qquad \langle\!\langle\, \textit{i.e.}, \text{ time-independent}\,\rangle\!\rangle, \tag{S.22}$$

while

$$\begin{aligned}\langle m_x\rangle(t) \;+\; i\,\langle m_y\rangle(t) \;&=\; -m_B\times 2\alpha^*(t)\beta(t)\\ &=\; -m_B\times 2\alpha_0\beta_0\times\exp\big(+i\omega(t-t_0)\big)\\ &=\; -m_B\times 2\,|\alpha_0|\;|\beta_0|\times\exp\big(i\omega(t-t_0)+i\phi_0\big)\end{aligned} \tag{S.23}$$

for some constant phase $\phi_0 = \arg(\alpha_0^*\beta_0)$. Consequently,

$$\begin{aligned}\langle m_x\rangle(t) \;&=\; -m_B\times 2\,|\alpha_0|\;|\beta_0|\times\cos\big(\omega(t-t_0)+\phi_0\big),\\ \langle m_y\rangle(y) \;&=\; -m_B\times 2\,|\alpha_0|\;|\beta_0|\times\sin\big(\omega(t-t_0)+\phi_0\big),\end{aligned} \tag{S.24}$$

which in 2d terms describes a rotation of the vector $(\langle m_x\rangle\,,\langle m_y\rangle)$ with frequency ω. In 3d terms, this rotation of the x and y components while the z components stays constant means *precession* around the z axis of the magnetic field.

The bottom line is, in quantum mechanics the expectation value $\langle\mathbf{m}\rangle$ of the electron's magnetic moment behaves exactly like a classical magnetic moment $\mathbf{m}$: it precesses around the magnetic field's direction with the classical Larmor frequency ω.

Problem **2**(a):

Suppose the Hermitian operator $\hat{H}$ has a non-degenerate eigenvalue spectrum. In this case, any other operator which commutes with $\hat{H}$ must be a function of $\hat{H}$, thus $\hat{A}_1 = f_1(\hat{H})$ and $\hat{A}_2 = f_2(\hat{H})$. But any two functions of the same operator $\hat{H}$ must commute with each other, $[f_1(\hat{H}), f_2(\hat{H})] = 0$. So if the $\hat{A}_1$ and the A_2 do not commute with each other, some of the above assumptions must be wrong.

The problem insists on Hermiticity of all 3 operators $\hat{H}, \hat{A}_1, \hat{A}_2$ and on $[\hat{A}_1, \hat{H}] = [\hat{A}_2, \hat{H}] = 0$, so the only extra assumption I have made above is the non-degenerate spectrum of $\hat{H}$. Thus, if we must have $[\hat{A}_1, \hat{A}_2] \neq 0$ then that particular assumption must be wrong. In other words, $\hat{H}$ must have a degenerate eigenvalue spectrum.

Problem **2**(b):

Any non-degenerate eigenstate $|\Psi\rangle$ of $\hat{H}$ must also be an eigenstate of both $\hat{A}_1$ and $\hat{A}_2$,

$$\hat{A}_1 |\Psi\rangle = a_1 |\Psi\rangle, \quad \hat{A}_2 |\Psi\rangle = a_2 |\Psi\rangle. \tag{S.25}$$

Consequently,

$$\hat{A}_1 \hat{A}_2 |\Psi\rangle = \hat{A}_1\big(\hat{A}_2 |\Psi\rangle = a_2 |\Psi\rangle\big) = a_2 \hat{A}_1 |\Psi\rangle = a_2 a_1 |\Psi\rangle \tag{S.26}$$

and likewise

$$\hat{A}_2 \hat{A}_1 |\Psi\rangle = a_1 a_2 |\Psi\rangle, \tag{S.27}$$

hence

$$\hat{A}_2 \hat{A}_1 |\Psi\rangle = \hat{A}_1 \hat{A}_2 |\Psi\rangle \tag{S.28}$$

and therefore

$$[\hat{A}_1, \hat{A}_2] |\Psi\rangle = 0. \tag{S.29}$$

Thus, *if the operator $\hat{H}$ happens to have a non-degenerate eigenvalue, then the corresponding eigenstate must be annihilated by the commutator $[\hat{A}_1, \hat{A}_2]$.*

Problem **2**(c):

Suppose in some orthonormal basis $\hat{H}$ has diagonal matrix (6), where the first 2 eigenvalues are equal, $h_1 = h_2$, but the third eigenvalue is different, $h_3 \neq h_{1,2}$. In the same basis, the operators $\hat{A}_1$ and $\hat{A}_2$ which commute with $\hat{H}$ must have block-diagonal matrices

$$A_1 = \begin{pmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & \star \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & \star \end{pmatrix}, \tag{S.30}$$

where the stars denote matrix elements which do not have to vanish. In particular, the upper left 2×2 blocks of the respective A_1 and A_2 matrices could be any 2×2 matrices in their own right, and these two matrices do not have to commute with each other. For example, we may have the 2×2 block of the A_1 be proportional to the σ_x Pauli matrix while the similar block of the A_2 is proportional to the σ_y,

$$A_1 = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & \star \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & -i\beta & 0 \\ +i\beta & 0 & 0 \\ 0 & 0 & \star \end{pmatrix} \tag{S.31}$$

for some real α, β, then both A_1 and A_2 commute with H but do not commute with each other,

$$[A_1, A_2] = i \begin{pmatrix} +\alpha\beta & 0 & 0 \\ 0 & -\alpha\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0. \tag{S.32}$$

PS: According to part (b) of this problem, this commutator must annihilate the $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ eigenvector of $\hat{H}$ corresponding to the non-degenerate eigenvalue h_3. And the zeros all over the third row and third column of the matrix (S.32) tell us that it indeed annihilates that eigenvector.

Problem **3**:

Let's start with the Leibniz rules:

$$\begin{aligned}\hat{B}[\hat{A},\hat{C}] + [\hat{A},\hat{B}]\hat{C} &= \hat{B}\big(\hat{A}\hat{C}-\hat{C}\hat{A}\big) + \big(\hat{A}\hat{B}-\hat{B}\hat{A}\big)\hat{C} \\ &= \cancel{\hat{B}\hat{A}\hat{C}} - \hat{B}\hat{C}\hat{A} + \hat{A}\hat{B}\hat{C} - \cancel{\hat{B}\hat{A}\hat{C}} \\ &= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = [\hat{A},\hat{B}\hat{C}],\end{aligned} \tag{S.33}$$

and likewise

$$\begin{aligned}\hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B} &= \hat{A}\big(\hat{B}\hat{C}-\hat{C}\hat{B}\big) + \big(\hat{A}\hat{C}-\hat{C}\hat{A}\big)\hat{B} \\ &= \hat{A}\hat{B}\hat{C} - \cancel{\hat{A}\hat{C}\hat{B}} + \cancel{\hat{A}\hat{C}\hat{B}} - \hat{C}\hat{A}\hat{B} \\ &= \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} = [\hat{A}\hat{B},\hat{C}].\end{aligned} \tag{S.34}$$

And now, the Jacobi identity:

$$\begin{aligned}[\hat{A},[\hat{B},\hat{C}]] &= [\hat{A},(\hat{B}\hat{C}-\hat{C}\hat{B})] = \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} - \hat{B}\hat{C}\hat{A} + \hat{C}\hat{B}\hat{A}, \\ [\hat{B},[\hat{C},\hat{A}]] &= [\hat{B},(\hat{C}\hat{A}-\hat{A}\hat{C})] = \hat{B}\hat{C}\hat{A} - \hat{B}\hat{A}\hat{C} - \hat{C}\hat{A}\hat{B} + \hat{A}\hat{C}\hat{B}, \\ [\hat{C},[\hat{A},\hat{B}]] &= [\hat{C},(\hat{A}\hat{B}-\hat{B}\hat{A})] = \hat{C}\hat{A}\hat{B} - \hat{C}\hat{B}\hat{A} - \hat{A}\hat{B}\hat{C} + \hat{B}\hat{A}\hat{C},\end{aligned} \tag{S.35}$$

hence

$$\begin{aligned}&[\hat{A},[\hat{B},\hat{C}]] + [\hat{B},[\hat{C},\hat{A}]] + [\hat{C},[\hat{A},\hat{B}]] = \\ &\quad = \cancel{\hat{A}\hat{B}\hat{C}} - \cancel{\hat{A}\hat{C}\hat{B}} - \cancel{\hat{B}\hat{C}\hat{A}} + \cancel{\hat{C}\hat{B}\hat{A}} \\ &\qquad + \cancel{\hat{B}\hat{C}\hat{A}} - \cancel{\hat{B}\hat{A}\hat{C}} - \cancel{\hat{C}\hat{A}\hat{B}} + \cancel{\hat{A}\hat{C}\hat{B}} \\ &\qquad + \cancel{\hat{C}\hat{A}\hat{B}} - \cancel{\hat{C}\hat{B}\hat{A}} - \cancel{\hat{A}\hat{B}\hat{C}} + \cancel{\hat{B}\hat{A}\hat{C}} \\ &\quad = 0,\end{aligned}$$

quod erat demonstrandum.

Problem **4**(a):

Lemma: eq. (9) holds true for $f(p) = p^n$, thus

$$\big[\hat{X}, \hat{P}^n\big] \;=\; i\hbar \times n\hat{P}^{n-1}. \tag{S.36}$$

Proof by induction: we are given $[\hat{X}, \hat{P}] = i\hbar$, so the Lemma holds for $n = 1$. Also, it trivially holds for $n = 0$. Now suppose it holds for some n, then for $n + 1$ we have

$$\begin{aligned} \big[\hat{X}, \hat{P}^{n+1}\big] &= \big[\hat{X}, \hat{P} \times \hat{P}^n\big] \\ &\quad\langle\!\langle \text{ by the Leibniz rule } \rangle\!\rangle \\ &= \big[\hat{X}, \hat{P}\big]\hat{P}^n \;+\; \hat{P}\big[\hat{X}, \hat{P}^n\big] \\ &= i\hbar \times \hat{P}^n \;+\; \hat{P} \times \big(i\hbar \times n\hat{P}^{n-1}\big) \\ &= i\hbar(1+n)\hat{P}^n, \end{aligned} \tag{S.37}$$

so if the Lemma holds for n then it also holds for $n + 1$. So by induction, the Lemma (S.36) holds for any integer n.

Now, take any analytic function

$$f(p) \;=\; \sum_{n=0}^{\infty} c_n p^n, \tag{S.38}$$

hence

$$f'(p) \;=\; \frac{df}{dp} \;=\; \sum_{n=1}^{\infty} n c_n p^{n-1}. \tag{S.39}$$

For the corresponding function

$$f(\hat{P}) \;=\; \sum_{n=0}^{\infty} c_n \hat{P}^n \tag{S.40}$$

of the operator $\hat{P}$ we have

$$\begin{aligned} \big[\hat{X}, f(\hat{P})\big] &= \left[\hat{X}, \sum_{n=0}^{\infty} c_n \hat{P}^n\right] \;=\; \sum_{n=0}^{\infty} c_n \big[\hat{X}, \hat{P}^n\big] \\ \langle\!\langle \text{ by the Lemma (S.36) } \rangle\!\rangle \quad &= \sum_{n=0}^{\infty} c_n \times i\hbar n \hat{P}^{n-1} \\ &= i\hbar \sum_n n c_n \hat{P}^{n-1} \;=\; i\hbar \times f'(\hat{P}). \end{aligned} \tag{S.41}$$

Problem **4**(b):

Similar to part (a), $[\hat{X},\hat{P}]=i\hbar$ implies

$$\big[\hat{X}^n,\hat{P}\big] \;=\; i\hbar n\hat{X}^{n-1} \tag{S.42}$$

for any integer $n=0,1,2,\ldots$. The proof by induction in n is so similar to the Lemma in part (a) that I don't need to repeat it here.

Now let $g(x)$ be any analytic function

$$g(x) \;=\; \sum_{n=0}^{\infty} c_n x^n \quad\Longrightarrow\quad g'(x) \;=\; \frac{dg}{dx} \;=\; \sum_{n=1}^{\infty} nc_n x^{n-1} \tag{S.43}$$

and let

$$g(\hat{X}) \;=\; \sum_{n=0}^{\infty} c_n\hat{X}^n, \qquad g'(\hat{X}) \;=\; \sum_{n=1}^{\infty} nc_n\hat{X}^{n-1} \tag{S.44}$$

be the corresponding functions of the operator $\hat{X}$. Similar to part (a),

$$\begin{aligned}\big[g(\hat{X}),\hat{P}\big] \;&=\; \left[\sum_{n=0}^{\infty} c_n\hat{X}^n,\hat{P}\right] \;=\; \sum_{n=0}^{\infty} c_n\big[\hat{X}^n,\hat{P}\big]\\ &=\; \sum_{n=0}^{\infty} c_n\times i\hbar n\hat{X}^{n-1}\\ &=\; i\hbar\sum_n nc_n\hat{X}^{n-1} \;=\; i\hbar g'(\hat{X}).\end{aligned} \tag{S.45}$$

Problem **4**(c):

Let

$$g(x) \;=\; \exp(ikx) \quad\Longrightarrow\quad g'(x) \;=\; ik\exp(ikx) \;=\; ik\times g(x). \tag{S.46}$$

Applying eq. (10) from part (b) to this $g(x)$, we get

$$\begin{aligned}&\big[\exp(ik\hat{X}),\hat{P}\big] \;=\; i\hbar\times ik\exp(ik\hat{X}) \;=\; -\hbar k\exp(ik\hat{X})\\ &\qquad\Big\|\\ &\exp(ik\hat{X})\hat{P} \;-\; \hat{P}\exp(ik\hat{X})\end{aligned} \tag{S.47}$$

and therefore

$$\hat{P} \times \exp(ik\hat{X}) \;=\; \exp(ik\hat{X}) \times (\hat{P} + \hbar k). \tag{S.48}$$

Likewise, let

$$f(p) \;=\; \exp(i\lambda p) \quad\Longrightarrow\quad f'(p) \;=\; i\lambda\exp(i\lambda p) \;=\; i\lambda f(p). \tag{S.49}$$

Applying eq. (9) from part (a) to this $f(p)$, we get

$$\begin{array}{ccc} \big[\hat{X},\exp(i\lambda\hat{P})\big] & = & i\hbar \times i\lambda\exp(i\lambda\hat{P}) \;=\; -\hbar\lambda\exp(i\lambda\hat{P}) \\ \Big\| & & \\ \hat{X}\exp(i\lambda P) \;-\; \exp(i\lambda P)\hat{X} & & \end{array} \tag{S.50}$$

and therefore

$$\hat{X} \times \exp(i\lambda\hat{P}) \;=\; \exp(i\lambda\hat{P}) \times (\hat{X} - \hbar\lambda). \tag{S.51}$$

Problem **4**(d):
Multiplying both sides of eq. (S.48) from part (c) by the $\exp(-ik\hat{X})$ from the left, we get

$$\exp(-ik\hat{X})\hat{P}\exp(+ik\hat{X}) \;=\; (\hat{P} + \hbar k). \tag{S.52}$$

Consequently,

$$\begin{aligned} \exp(-ik\hat{X})\hat{P}^2\exp(+ik\hat{X}) \;&=\; \big(\exp(-ik\hat{X})\hat{P}\big) \times \big(1 \;=\; \exp(+ik\hat{X})\exp(-ik\hat{X})\big) \\ &\qquad \times \big(\hat{P}\exp(+ik\hat{X})\big) \\ &=\; \big(\exp(-ik\hat{X})\hat{P}\exp(+ik\hat{X})\big) \times \big(\exp(-ik\hat{X})\hat{P}\exp(+ik\hat{X})\big) \\ &=\; (\hat{P} + \hbar k) \times (\hat{P} + \hbar k) \;=\; (\hat{P} + \hbar k)^2, \end{aligned} \tag{S.53}$$

and likewise for any integer n

$$\exp(-ik\hat{X}) \times \hat{P}^n \times \exp(+ik\hat{X}) \;=\; (\hat{P} + \hbar k)^n. \tag{S.54}$$

The formal proof of eq. (S.54) is by induction in n: if it's true for some n, then it's also true for $n+1$ because

$$\begin{aligned}
\exp(-ik\hat{X})\hat{P}^{n+1}\exp(+ik\hat{X}) &= \exp(-ik\hat{X})\hat{P}^{n}\times\hat{P}\exp(+ik\hat{X}) \\
&= \big(\exp(-ik\hat{X})\hat{P}^{n}\exp({}_ik\hat{X})\big)\times\big(\exp(-ik\hat{X})\hat{P}\exp({}_ik\hat{X})\big) \\
&= (\hat{P}+\hbar k)^{n}\times(\hat{P}+\hbar k) \;=\; (\hat{P}+\hbar k)^{n+1}.
\end{aligned} \tag{S.55}$$

Anyhow, given eq. (S.54) for any integer $n=0,1,2,3,\ldots$, we may then apply it to any power series in $\hat{P}$ and hence to any analytic function

$$f(\hat{P}) \;=\; \sum_{n=0}^{\infty} c_n\hat{P}^n. \tag{S.56}$$

Indeed,

$$\begin{aligned}
\exp(-ik\hat{X})f(\hat{P})\exp(+ik\hat{X}) &= \exp(-ik\hat{X})\times\sum_{n=0}^{\infty}c_n\hat{P}^n\times\exp(+ik\hat{X}) \\
&= \sum_{n=0}^{\infty}c_n\times\exp(-ik\hat{X})\hat{P}^n\exp(+ik\hat{X}) \\
&= \sum_{n=0}^{\infty}c_n\times(\hat{P}+\hbar k)^n \;=\; f(\hat{P}+\hbar k).
\end{aligned} \tag{11.a}$$

Likewise, eq. (11,b) follows from eq. (S.51) in part (c). Multiplying both sides of *that* equation by $\exp(-i\lambda\hat{P})$ from the left, we get

$$\exp(-i\lambda\hat{P})\hat{X}\exp(+i\lambda\hat{P}) \;=\; \hat{X} \;-\; \hbar\lambda. \tag{S.57}$$

Consequently, by for any integer $n=0,1,2,\ldots$

$$\exp(-i\lambda\hat{P})\hat{X}^n\exp(+i\lambda\hat{P}) \;=\; (\hat{X}-\hbar\lambda)^n, \tag{S.58}$$

which follows from eq. (S.57) by induction in n. Indeed, eq. (S.58) is trivially true for $n=0$ while eq. (S.57) provides the induction base for $n=1$. As to the induction base, suppose eq. (S.58)holds true for some n, then it also holds for $n+1$ because

$$\begin{aligned}\exp(-i\lambda\hat{P})\hat{X}^{n+1}\exp(+i\lambda\hat{P}) &= \exp(-i\lambda\hat{P})\hat{X}^{n}\times\hat{X}\exp(+i\lambda\hat{P})\\ &= \big(\exp(-i\lambda\hat{P})\hat{X}^{n}\exp(+i\lambda\hat{P})\big)\times\big(\exp(-i\lambda\hat{P})\hat{X}\exp(+i\lambda\hat{P})\big)\\ &= (\hat{X}-\hbar\lambda)^{n}\times(\hat{X}-\hbar\lambda) \;=\; (\hat{X}-\hbar\lambda)^{n+1}.\end{aligned}\tag{S.59}$$

Now, given eq. (S.58)for all integer n, we may extend it to all power series in $\hat{X}$, *i.e.* for all analytic functions

$$g(\hat{X}) = \sum_{n=0}^{\infty} c_n\hat{X}^n. \tag{S.60}$$

Indeed,

$$\begin{aligned}\exp(-i\lambda\hat{P})g(\hat{X})\exp(+i\lambda\hat{P}) &= \exp(-i\lambda\hat{P})\times\sum_{n=0}^{\infty}c_n\hat{X}^n\times\exp(+i\lambda\hat{P})\\ &= \sum_{n=0}^{\infty}c_n\times\exp(-i\lambda\hat{P})\hat{X}^n\exp(+i\lambda\hat{P}) \qquad (11.b)\\ &= \sum_{n=0}^{\infty}c_n\times(\hat{X}-\hbar\lambda)^n \;=\; g(\hat{X}-\hbar\lambda).\end{aligned}$$

Problem **4**(e):

Consider the matrix element $\langle x_1|\,[\hat{X},\exp(-i\lambda P)]\,|x_2\rangle$. On one hand,

$$\begin{aligned}\langle x_1|\,\big[\hat{X},\exp(-i\lambda P)\big]\,|x_2\rangle &= \langle x_1|\,\hat{X}\exp(-i\lambda P)\,|x_2\rangle \;-\; \langle x_1|\exp(-i\lambda P)\hat{X}\,|x_2\rangle\\ &\quad\langle\!\langle\text{ where } \langle x_1|\,\hat{X} = (\hat{X}\,|x_1\rangle)^\dagger = (x_1\,|x_1\rangle)^\dagger = x_1\,\langle x_1|\,\rangle\!\rangle\\ &\quad\langle\!\langle\text{ and } \hat{X}\,|x_2\rangle = x_2\,|x_2\rangle\,\rangle\!\rangle\\ &= x_1\times\langle x_1|\exp(-i\lambda P)\,|x_2\rangle \;-\; x_2\times\langle x_1|\exp(-i\lambda P)\,|x_2\rangle\\ &= (x_1-x_2)\times\langle x_1|\exp(-i\lambda P)\,|x_2\rangle\,.\end{aligned}\tag{S.61}$$

On the other hand,

$$\big[\hat{X},\exp(-i\lambda P)\big] = i\hbar\times -i\lambda\exp(-i\lambda\hat{P}) = \hbar\lambda\exp(-i\lambda\hat{P}) \tag{S.62}$$

and hence

$$\langle x_1| \left[\hat{X}, \exp(-i\lambda P)\right] |x_2\rangle \;=\; \hbar\lambda \times \langle x_1| \exp(-i\lambda P) |x_2\rangle \,. \tag{S.63}$$

Reconciling eqs. (S.61) and(S.63) for the same matrix element requires

$$(x_1 - x_2) \times \langle x_1| \exp(-i\lambda P) |x_2\rangle \;=\; \hbar\lambda \times \langle x_1| \exp(-i\lambda P) |x_2\rangle \,, \tag{S.64}$$

and therefore

$$\text{either}\quad x_1 \;-\; x_2 \;=\; \hbar\lambda, \quad \text{or} \quad \langle x_1| \exp(-i\lambda P) |x_2\rangle \;=\; 0, \quad \text{or both.} \tag{S.65}$$

In other words,

$$\langle x_1| \exp(-i\lambda P) |x_2\rangle \;=\; 0 \quad \text{unless} \quad x_1 \;-\; x_2 \;=\; \hbar\lambda. \tag{12}$$

Problem **4**(f):
Eq. (12) for $\lambda = a/\hbar$ means that

$$\hat{T}_a \;=\; \exp\left(-i\frac{a}{\hbar}\hat{P}\right) \tag{S.66}$$

acts as a translation operator: renaming $x_2 \to y$, $x_1 \to x$ we have

$$\langle x| \hat{T}_a |y\rangle \;=\; 0 \quad \text{unless} \quad x \;=\; y \;+\; a, \tag{S.67}$$

so for the state $|y\rangle$ of a particle localized at a specific point y, the translated state $T_a |x\rangle$ is localized at $x = y + a$ because its wave-function $\langle x| \hat{T}_a |x\rangle$ vanishes for $x \neq y + a$.

Now, let normalize the eigenstates $|x\rangle$ of $\hat{X}$ and choose their phases such that

$$\langle x|\,\hat{T}_a\,|y\rangle \;=\; \delta(x-y-a) \tag{13}$$

without any extra phase factors. Then for some generic state $|\Psi\rangle$ with the coordinate-basis wave function $\psi(x)=\langle x|\Psi\rangle$, the state $\hat{T}_a\,|\Psi\rangle$ has wave function

$$(\hat{T}_a\psi)(x) \;=\; \langle x|\,\hat{T}_a\,|\Psi\rangle \;=\; \int\!dy\;\langle x|\,\hat{T}_a\,|y\rangle\times\langle y|\Psi\rangle \;=\; \int\!dy\,\delta(x-y-a)\times\langle y|\Psi\rangle \;=\; \langle x-a|\Psi\rangle\,. \tag{S.68}$$

Let's take a derivative of this wave function WRT a and then take the $a\to 0$ limit. On one hand,

$$\lim_{a\to 0}\frac{\partial}{\partial a}\langle x-a|\Psi\rangle \;=\; -\frac{\partial}{\partial x}\langle x|\Psi\rangle\,. \tag{S.69}$$

On the other hand,

$$\lim_{a\to 0}\frac{\partial}{\partial a}\big(\langle x|\,\hat{T}_a\,|\Psi\rangle\big) \;=\; \langle x|\left(\lim_{a\to 0}\frac{\partial}{\partial a}\hat{T}_a\right)|\Psi\rangle \tag{S.70}$$

where

$$\frac{\partial}{\partial a}\exp(-ia\hat{P}/\hbar) \;=\; (-i/\hbar)\hat{P}\times\exp(-ia\hat{P}/\hbar) \;\xrightarrow[a\to 0]{}\; (-i/\hbar)\hat{P}, \tag{S.71}$$

hence

$$\lim_{a\to 0}\frac{\partial}{\partial a}\big(\langle x|\,\hat{T}_a\,|\Psi\rangle\big) \;=\; (-i/\hbar)\,\langle x|\,\hat{P}\,|\Psi\rangle\,. \tag{S.72}$$

In light of eq. (S.68), the left hand sides of eqs. (S.69) and (S.72) are equal to each other, so their right hand sides should also be equal, thus

$$(-i/\hbar)\,\langle x|\,\hat{P}\,|\Psi\rangle \;=\; -\frac{\partial}{\partial x}\langle x|\Psi\rangle \tag{S.73}$$

and hence

$$\langle x|\,\hat{P}\,|\Psi\rangle \;=\; -i\hbar\frac{\partial}{\partial x}\langle x|\Psi\rangle\,. \tag{14.a}$$

Or in the wave-function language

$$\hat{P}\psi(x) \;=\; -i\hbar\frac{\partial}{\partial x}\psi(x). \tag{14.b}$$