Problem **1**(a):

Let $\pm$ denote the sign of QB, then

$$\hat{a} = \sqrt{\frac{c}{2\hbar|QB|}}(\hat{\pi}_x \pm i\hat{\pi}_y), \quad \hat{a}^\dagger = \sqrt{\frac{c}{2\hbar|QB|}}(\hat{\pi}_x \mp i\hat{\pi}_y), \tag{3}$$

hence

$$\begin{aligned}
[\hat{a}, \hat{a}^\dagger] &= \frac{c}{2\hbar|QB|}\big[(\hat{\pi}_x \pm i\hat{\pi}_y), (\hat{\pi}_x \mp i\hat{\pi}_y)\big] = \frac{c}{2\hbar|QB|}\Big(\pm i[\hat{\pi}_y, \hat{\pi}_x] \mp i[\hat{\pi}_x, \hat{\pi}_y]\Big) \\
&= \frac{c}{2\hbar|QB|}\left(\pm i \times \frac{-iQB\hbar}{c} \mp i \times \frac{+iQB\hbar}{c} = \pm\frac{2QB\hbar}{c}\right) \\
&= \frac{\pm QB}{|QB|} = +1 \quad \text{for} \quad \pm = \operatorname{sign}(QB).
\end{aligned} \tag{S.1}$$

Problem **1**(b):

In light of eqs. (3),

$$\hat{a} + \hat{a}^\dagger = 2\sqrt{\frac{c}{2\hbar|QB|}}\,\hat{\pi}_x\,, \qquad \hat{a} - \hat{a}^\dagger = \pm 2i\sqrt{\frac{c}{2\hbar|QB|}}\,\hat{\pi}_y\,, \tag{S.2}$$

hence

$$\hat{\pi}_x = \sqrt{\frac{\hbar|QB|}{2c}}\,(\hat{a} + \hat{a}^\dagger), \qquad \hat{\pi}_y = \sqrt{\frac{\hbar|QB|}{2c}}\,(\mp i\hat{a} \pm i\hat{a}^\dagger), \tag{S.3}$$

and therefore

$$\begin{aligned}
\hat{H} &= \frac{\hat{\pi}_x^2 + \hat{\pi}_y^2}{2M} = \frac{1}{2M}\,\frac{\hbar|QB|}{2c}\Big((\hat{a} + \hat{a}^\dagger)^2 + (\mp i\hat{a} \pm i\hat{a}^\dagger)^2\Big) \\
&= \frac{\hbar|QB|}{4Mc} \times \Big(\big(\hat{a}^2 + \hat{a}^{\dagger 2} + \{\hat{a}, \hat{a}^\dagger\}\big) + \big(-\hat{a}^2 - \hat{a}^{\dagger 2} + \{\hat{a}, \hat{a}^\dagger\}\big)\Big) \\
&= \frac{\hbar\Omega}{4} \times 2\{\hat{a}, \hat{a}^\dagger\}
\end{aligned} \tag{S.4}$$

where Ω is defined by eq. (5); I'll explain its relation to the classical cyclotron frequency in the next part.

Now let $\hat{n} = \hat{a}^\dagger\hat{a}$ exactly as we have defined in class for the harmonic oscillator. Consequently $\hat{a}\hat{a}^\dagger = \hat{n} + 1$, hence

$$\{\hat{a}, \hat{a}^\dagger\} \;=\; \hat{a}\hat{a}^\dagger \;+\; \hat{a}^\dagger\hat{a} \;=\; (\hat{n}+1) \;+\; \hat{n} \;=\; 2\hat{n} \;+\; 1, \tag{S.5}$$

so the bottom line of eq. (S.4) becomes

$$\hat{H} \;=\; \frac{\hbar\Omega}{2} \times \{\hat{a}, \hat{a}^\dagger\} \;=\; \hbar\Omega(\hat{n} + \tfrac{1}{2}). \tag{S.6}$$

The energy spectrum of the charged particle follows from this formula. As we saw in class for the harmonic oscillator, the commutator $[\hat{a}, \hat{a}^\dagger] = 1$ is enough to show that the spectrum of the $\hat{n} = \hat{a}^\dagger\hat{a}$ operator comprises all the non-negative integers $n = 0, 1, 2, \ldots$. Consequently, the energy spectrum of the Hamiltonian (S.6) is made from the discrete Landau levels

$$E_n \;=\; \hbar\Omega(n + \tfrac{1}{2}) \;=\; \frac{\hbar|QB|}{Mc}(n + \tfrac{1}{2}), \quad n = 0, 1, 2, \ldots. \tag{S.7}$$

Problem **1**(c):

In a uniform magnetic field $\mathbf{B}$, a classical charged particle moves in a circle at constant speed. The angular frequency $\vec{\Omega}$ of this motion follows from

$$\mathbf{a} \;=\; \vec{\Omega} \times \mathbf{v} \quad \text{and also} \quad \mathbf{a} \;=\; \frac{\mathbf{F}_{\text{Lorentz}}}{M} \;=\; \frac{QB}{Mc}\,\mathbf{v} \times \mathbf{B}, \tag{S.8}$$

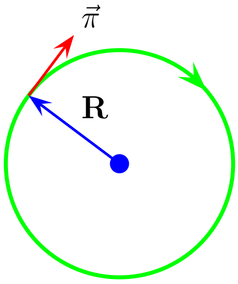
hence

$$\vec{\Omega} \;=\; -\frac{Q}{Mc}\,\mathbf{B} \tag{S.9}$$

In other words, the magnitude of this angular frequency is as in eq. (5) while the direction is $-z$ for $QB_z > 0$ and $+z$ for $QB_z < 0$. Or from the xy plane point of view, the circular motion is clockwise for $QB > 0$ and counterclockwise for $QB < 0$.

In the clockwise case, the velocity vector (and hence the momentum vector) points 90° to the right of the radius vector $\mathbf{R} = \mathbf{x} - \mathbf{x}_c$ (from the center of the circle)

$\boxed{QB > 0}$



$$\begin{aligned} \vec{\pi} &= M\Omega\,\mathrm{Rot}(\mathbf{R}, 90^\circ \text{ right}), \\ & \quad \text{or in components} \\ \pi_x &= +M\Omega(R_y = y - y_c), \\ \pi_y &= -M\Omega(R_x = x - x_c). \end{aligned} \tag{S.10}$$

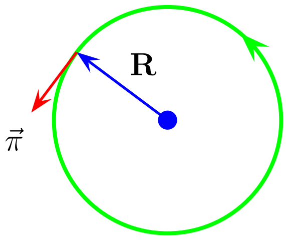
Also, for $QB > 0$ we have $M\Omega = +QB/c$, hence for this case

$$\begin{aligned} x_c &= x + \frac{\pi_y}{M\Omega} = x + \frac{c}{QB}\pi_y\,, \\ y_c &= y - \frac{\pi_x}{M\Omega} = y - \frac{c}{QB}\pi_x\,, \end{aligned} \tag{S.11}$$

precisely as in eq. (6).

In the opposite case of $QB < 0$, the particle's motion is counterclockwise, so its momentum vector points 90° to the left of the radius vector $\mathbf{R} = \mathbf{x} - \mathbf{x}_c$, thus

$\boxed{QB < 0}$



$$\begin{aligned} \vec{\pi} &= M\Omega\,\mathrm{Rot}(\mathbf{R}, 90^\circ \text{ left}), \\ & \quad \text{or in components} \\ \pi_x &= +M\Omega(R_y = y - y_c), \\ \pi_y &= -M\Omega(R_x = x - x_c). \end{aligned} \tag{S.12}$$

Also, for $QB < 0$ we have $M\Omega = |QB|/c = -QB/c$, hence

$$\begin{aligned} x_c &= x - \frac{\pi_y}{M\Omega} = x + \frac{c}{QB}\pi_y\,, \\ y_c &= y + \frac{\pi_x}{M\Omega} = y - \frac{c}{QB}\pi_x\,, \end{aligned} \tag{S.13}$$

again precisely as in eq. (6). *Quod erat demonstrandum.*

Problem $\mathbf{1}$(d):

The quantum analogues of the Larmor circle's center's coordinates (6) are the operators

$$\hat{x}_c \;=\; \hat{x} \;+\; \frac{c}{QB}\,\hat{\pi}_y\,, \quad \hat{y}_c \;=\; \hat{y} \;-\; \frac{c}{QB}\,\hat{\pi}_x\,. \tag{S.14}$$

Since $\hat{x}$ commutes with $\hat{\pi}_y$ and $\hat{y}$ commutes with $\hat{\pi}_x$, it immediately follows that

$$[\hat{x}_c, \hat{\pi}_y] \;=\; [\hat{y}_c, \hat{\pi}_x] \;=\; 0. \tag{S.15}$$

Less obviously

$$[\hat{x}_c, \hat{\pi}_x] \;=\; [\hat{x}, \hat{\pi}_x] \;+\; \frac{c}{QB}\,[\hat{\pi}_y, \hat{\pi}_x] \;=\; i\hbar \;+\; \frac{c}{QB}\left(-i\frac{\hbar QB}{c}\right) \;=\; i\hbar \;-\; i\hbar \;=\; 0, \tag{S.16}$$

and likewise,

$$[\hat{y}_c, \hat{\pi}_y] \;=\; [\hat{y}, \hat{\pi}_y] \;-\; \frac{c}{QB}\,[\hat{\pi}_x, \hat{\pi}_y] \;=\; i\hbar \;-\; \frac{c}{QB}\left(+i\frac{\hbar eB}{c}\right) \;=\; i\hbar \;-\; i\hbar \;=\; 0. \tag{S.17}$$

Thus, each of the center's coordinates $\hat{x}_c$ and $\hat{y}_c$ commutes with both $\hat{\pi}_x$ and $\hat{\pi}_y$

$$\begin{aligned} [\hat{x}_c, \hat{\pi}_x] \;&=\; [\hat{x}_c, \hat{\pi}_y] \;=\; 0, \\ [\hat{y}_c, \hat{\pi}_x] \;&=\; [\hat{y}_c, \hat{\pi}_y] \;=\; 0. \end{aligned} \tag{S.18}$$

Consequently, each of $\hat{x}_c$ and $\hat{y}_c$ commutes with both $\hat{\pi}_x^2$ and $\hat{\pi}_y^2$ and hence with the Hamiltonian

$$\hat{H} \;=\; \frac{\hat{\pi}_x^2 + \hat{\pi}_y^2}{2m}\,, \tag{15}$$

$$[\hat{x}_c, \hat{H}] \;=\; [\hat{y}_c, H] \;=\; 0. \tag{S.19}$$

In other words, both $\hat{x}_c$ and $\hat{y}_c$ are conserved operators.

Physically, $\hat{x}_c$ and $\hat{y}_c$ being conserved operators is the quantum analog of the classical center of the Larmor circle staying at the same time-independent point (6) while the particle moves around the circle.

Problem **1**(e):

$$\begin{aligned}[\hat{x}_c,\hat{y}_c] &= [\hat{x},\hat{y}] + \frac{c}{QB}[\hat{\pi}_y,\hat{y}] - \frac{c}{QB}[\hat{x},\hat{\pi}_x] - \left(\frac{c}{QB}\right)^2[\hat{\pi}_y,\hat{\pi}_x]\\ &= 0 + \frac{c}{QB}\times(-i\hbar) - \frac{c}{QB}\times(+i\hbar) - \left(\frac{c}{QB}\right)^2\times\left(-\frac{i\hbar QB}{c}\right)\\ &= -\frac{i\hbar c}{QB} - \frac{i\hbar c}{QB} + \frac{i\hbar c}{QB}\\ &= -\frac{i\hbar c}{QB}\,.\end{aligned} \tag{7}$$

Problem **1**(f):

Both operators $\hat{\pi}_x$ and $\hat{\pi}_y$ commute with the Hamiltonian $\hat{H}$ but they do not commute with each other. Back in homework set#3 (problem 2) we saw that this guarantees that $\hat{H}$ has degenerate eigenvalues. Moreover, since the commutator (7) never vanishes, all the eigenvalues of $\hat{H}$ must be degenerate.

To see that the degeneracy must be infinite, consider the block-diagonal matrices of the operators $\hat{x}_c$ and $\hat{y}_c$ in the eigenbasis of the Hamiltonian:

$$\langle m,\alpha|\,\hat{x}_c\,|n,\beta\rangle = \delta_{mn}\times X_{\alpha,\beta}(n), \qquad \langle m,\alpha|\,\hat{y}_c\,|n,\beta\rangle = \delta_{mn}\times Y_{\alpha,\beta}(n), \tag{S.20}$$

where α,β label independent eigenstates for the same degenerate eigenvalue E_n. The matrices $X_{\alpha\beta}(n)$ and $Y_{\alpha\beta}(n)$ do not commute; instead, eq. (7) requires

$$[X(n),Y(n)] = -\frac{i\hbar c}{QB}\times(\text{unit matrix}). \tag{S.21}$$

Now suppose there is a finite number N of independent eigenstates for some Landau level n_0, so the matrices $X_{\alpha\beta}(n_0)$ and $Y_{\alpha\beta}(n_0)$ are finite $N\times N$ matrices. In this case, the trace of a unit matrix is $\mathrm{tr}(1)=N$ hence

$$\mathrm{tr}\big([X(n_0),Y(n_0)]\big) = -\frac{i\hbar c}{QB}\times\mathrm{tr}(1_{N\times N}) = -\frac{i\hbar c}{QB}\times N \neq 0. \tag{S.22}$$

On the other hand, the trace of a commutator of any 2 *finite* matrices must vanish:

$$\mathrm{tr}(XY) \;=\; \sum_\alpha (XY)_{\alpha,\alpha} \;=\; \sum_{\alpha,\beta} X_{\alpha,\beta} Y_{\beta,\alpha} \;=\; \sum_\beta (YX)_{\beta,\beta} \;=\; \mathrm{tr}(YX), \tag{S.23}$$

hence

$$\mathrm{tr}\big([X,Y]\big) \;=\; \mathrm{tr}(XY) \;-\; \mathrm{tr}(YX) \;=\; 0. \tag{S.24}$$

Note: this zero-trace-of-a-commutator rule works only for finite matrices. For infinite matrices, if the double sum over α and β in eq. (S.23) is not absolutely convergent, the order of summation may affect the result, so we might have $\mathrm{tr}(XY) \neq \mathrm{tr}(YX)$ and hence $\mathrm{tr}([X,y]) \neq 0$. Thus, eq. (S.22) is impossible for finite matrices, but it might hold true for infinite matrices. And that's why N must be infinite, thus infinite number of independent eigenstates $|n_0,\alpha\rangle$ for any n_0. In other words, each Landau level has an infinite number of degenerate states.

Alternative solution:

The commutator (7) is proportional to a unit matrix, so there are linear combinations of $\hat{x}_c$ and $\hat{y}_c$ which act as rising and lowering operators of a harmonic oscillator, namely

$$\hat{b} \;\stackrel{\text{def}}{=}\; \sqrt{\frac{|QB|}{2\hbar c}}\,(\hat{x}_c \,-\, i\,\mathrm{sign}(QB)\hat{y}_c), \quad \hat{b}^\dagger \;=\; \sqrt{\frac{|QB|}{2\hbar c}}\,(\hat{x}_c \,+\, i\,\mathrm{sign}(QB)\hat{y}_c), \quad [\hat{b},\hat{b}^\dagger] \;=\; 1. \tag{S.25}$$

Indeed,

$$\begin{aligned}
[\hat{b},\hat{b}^\dagger] \;&=\; \frac{|QB}{2\hbar c}\,[(\hat{x}_c \mp i\hat{y}_c),(\hat{x}_c \pm i\hat{y}_c)] \;=\; \frac{|QB}{2\hbar c}\Big(\mp i[\hat{y}_c,\hat{x}_c] \pm i[\hat{x}_c,\hat{y}_c]\Big) \\
&=\; \frac{|QB}{2\hbar c} \times \left(\pm 2i[x_c,y_c] \;=\; \pm 2i \times \frac{-i\hbar c}{QB}\right) \\
&=\; \frac{\pm|QB|}{QB} \;=\; 1 \quad \text{for } \pm = \mathrm{sign}(QB).
\end{aligned} \tag{S.26}$$

Consequently, the operator $\hat{n}_c$ has spectrum comprised of all non-negative integers $n_c = 0, 1, 2, \ldots$.

Moreover, since $\hat{x}_c$ and $\hat{y}_c$ commute with both $\hat{\pi}_x$ and $\hat{\pi}_y$, they also commute with the $\hat{a}$ and $\hat{a}^\dagger$ operators, and therefore the $\hat{b}$ and $\hat{b}^\dagger$ operators also commute with $\hat{a}$ and $\hat{a}^\dagger$. Thus, we have a two-oscillator system

$$[\hat{a},\hat{b}] \;=\; [\hat{a}^\dagger,\hat{b}] \;=\; [\hat{a},\hat{b}^\dagger] \;=\; [\hat{a}^\dagger,\hat{b}^\dagger] \;=\; 0, \quad [\hat{a},\hat{a}^\dagger] \;=\; [\hat{b},\hat{b}^\dagger] \;=\; 1. \qquad \text{(S.27)}$$

so we may simultaneously diagonalize the $\hat{n} = \hat{a}^\dagger\hat{a}$ and the $\hat{n}_c = \hat{b}^\dagger\hat{b}$ operators to get a basis $\{|n,n_c\rangle\}$ where n and n_c run over non-negative integers *independently from each other.* Thus, for every n there is an infinite number of the $|n,n_c\rangle$ states.

Finally, the Hamiltonian is

$$\hat{H} \;=\; \hbar\omega(\hat{n}+\tfrac{1}{2}) \qquad \text{(S.28)}$$

regardless of the $\hat{b},\hat{b}^\dagger$ system, hence

$$\hat{H}\,|n,n_c\rangle \;=\; \hbar\Omega(n+\tfrac{1}{2})\,|n,n_c\rangle \quad \text{for all } n_c = 0,1,2,\ldots. \qquad \text{(S.29)}$$

Thus, every Landau level $E_n = \hbar\Omega(n+\frac{1}{2})$ is infinitely degenerate!

Problem **2**(a):

Note: Since the operators $\hat{a}$ and $\hat{a}^\dagger$ cannot be diagonalized, the functions of these operators must be defined algebraically. That is, expand the analytic function $f(\xi)$ in the Taylor series

$$f(\xi) \;=\; \sum_{n=0}^{\infty} f_n \times \xi^n, \qquad f'(\xi) \;=\; \sum_{n=0\,\mathrm{r}\,1}^{\infty} n f_n \times \xi^{n-1}, \qquad \text{(S.30)}$$

then

$$f(\hat{a}) \;=\; \sum_{n=0}^{\infty} f_n \times \hat{a}^n, \qquad f(\hat{a}^\dagger) \;=\; \sum_{n=0}^{\infty} f_n \times \hat{a}^{\dagger n}, \qquad \text{(S.31)}$$

and likewise for the derivatives $f'(\hat{a})$ and $f'(\hat{a}^\dagger)$.

Now let's prove by induction that

$$\left[\hat{a}, \hat{a}^{\dagger n}\right] \;=\; n\hat{a}^{\dagger n-1}. \tag{S.32}$$

Indeed, this is trivially true for $n = 0$ and we know that this is true for $n = 1$; this is the induction base. For the induction step, suppose (S.32) is true for some $n \geq 1$, then it also holds for $n + 1$, indeed

$$\left[\hat{a}, (\hat{a}^{\dagger n+1} = \hat{a}^{\dagger}\hat{a}^{\dagger n})\right] \;=\; \hat{a}^{\dagger}\times\left[\hat{a}, \hat{a}^{\dagger n}\right] + [\hat{a}, \hat{a}^{\dagger}]\times\hat{a}^{\dagger n} \;=\; \hat{a}^{\dagger}\times n\hat{a}^{\dagger n-1} + 1\times\hat{a}^{\dagger} \;=\; (n+1)\times\hat{a}^{\dagger n}. \tag{S.33}$$

By induction, this means that eq. (S.32) holds for every $n = 0, 1, 2, 3, \ldots$.

Consequently, for any analytic function $f(\hat{a}^{\dagger})$ we have

$$[\hat{a}, f(\hat{a}^{\dagger})] \;=\; \left[\hat{a}, \sum_n f_n \hat{a}^{\dagger n}\right] \;=\; \sum_n f_n \left[\hat{a}, \hat{a}^{\dagger n}\right] \;=\; \sum_n f_n \times n\hat{a}^{\dagger n-1} \;=\; f'(\hat{a}^{\dagger}). \tag{8.a}$$

Likewise, for any $n = 0, 1, 2, 3, \ldots$

$$\left[\hat{a}^{\dagger}, \hat{a}^{n}\right] \;=\; -n\hat{a}^{n-1}. \tag{S.34}$$

The proof is by induction in n: this is trivially true for $n = 0$ and we know this is true for $n = 1$; this is the induction base. For the induction step, suppose eq. (S.34) holds true for some $n \geq 1$, then it should also hold for $n + 1$, indeed

$$\left[\hat{a}^{\dagger}, (\hat{a}^{n+1} = \hat{a}\hat{a}^{n})\right] \;=\; [\hat{a}^{\dagger}, \hat{a}]\times\hat{a}^{n} + \hat{a}\times\left[\hat{a}^{\dagger}, \hat{a}^{n}\right] \;=\; (-1)\times\hat{a}^{n} + \hat{a}\times\left(-n\hat{a}^{n-1}\right) \;=\; -(n+1)\times\hat{a}^{n}. \tag{S.35}$$

By induction, this means that eq. (S.34) holds for every $n = 0, 1, 2, 3, \ldots$.

Consequently, for any analytic function $f(\hat{a})$ we have

$$[\hat{a}^{\dagger}, f(\hat{a})] \;=\; \left[\hat{a}^{\dagger}, \sum_n f_n \times \hat{a}^{n}\right] \;=\; \sum_n f_n \times \left[\hat{a}^{\dagger}, \hat{a}^{n}\right] \;=\; \sum_n f_n \times \left(-n\hat{a}^{n-1}\right) \;=\; -f'(\hat{a}). \tag{8.b}$$

Problem $\mathbf{2}$(b):

We have learned in class that

$$\hat{a}^{\dagger n}|0\rangle = \sqrt{n!}\,|n\rangle\,. \tag{S.36}$$

Consequently,

$$\exp(\xi\hat{a}^\dagger)|0\rangle = \sum_{n=0}^{\infty}\frac{\xi^n}{n!}\,\hat{a}^{\dagger n}|0\rangle = \sum_{n=0}^{\infty}\frac{\xi^n}{n!}\times\sqrt{n!}\,|n\rangle = \sum_{n=0}^{\infty}\frac{\xi^n}{\sqrt{n!}}\,|n\rangle\,, \tag{S.37}$$

hence

$$|\xi\rangle = e^{-|\xi|^2/2}\times\exp(\xi\hat{a}^\dagger)|0\rangle = e^{-|\xi|^2/2}\times\sum_{n=0}^{\infty}\frac{\xi^n}{\sqrt{n!}}\,|n\rangle\,, \tag{S.38}$$

and therefore

$$\langle n|\xi\rangle = e^{-|\xi|^2/2}\times\sum_{m=0}^{\infty}\frac{\xi^m}{\sqrt{m!}}\times\langle n|m\rangle = e^{-|\xi|^2/2}\times\frac{\xi^n}{\sqrt{n!}} \tag{S.39}$$

because $\langle n|m\rangle = \delta_{n,m}$.

Next, the normalization of the coherent state $|\xi\rangle$. Since $\{|n\rangle : n = 0, 1, 2, \ldots\}$ is a complete orthonormal basis of the Hilbert space,

$$\begin{aligned}
\langle\xi|\xi\rangle &= \sum_{n=0}^{\infty}\langle\xi|n\rangle\,\langle n|\xi\rangle = \sum_{n=0}^{\infty}|\langle n|\xi\rangle|^2 \\
&= \sum_{n=0}^{\infty}e^{-|\xi|^2}\times\frac{|\xi^n|^2}{n!} \\
&= e^{-|\xi|^2}\times\sum_{n=0}^{\infty}\frac{(|\xi|^2)^n}{n!} = e^{-|\xi|^2}\times e^{+|\xi|^2} \\
&= 1.
\end{aligned} \tag{S.40}$$

Quod erat demonstrandum.

Problem **2**(c):

Let $f(\hat{a}^\dagger) = \exp(\xi\hat{a}^\dagger)$, then according to eq. (8.a)

$$\left[\hat{a}, \left(f(\hat{a}^\dagger) = \exp(\xi\hat{a}^\dagger)\right)\right] \;=\; f'(\hat{a}^\dagger) \;=\; \xi\exp(\xi\hat{a}^\dagger), \tag{S.41}$$

hence

$$\begin{aligned} \hat{a}\times\exp(\xi\hat{a}^\dagger) \;&=\; \exp(\xi\hat{a}^\dagger)\times\hat{a} \;+\; \left[\hat{a}, \exp(\xi\hat{a}^\dagger)\right] \\ &=\; \exp(\xi\hat{a}^\dagger)\times\hat{a} \;+\; \xi\exp(\xi\hat{a}^\dagger) \\ &=\; \exp(\xi\hat{a}^\dagger)\times(\hat{a}+\xi). \end{aligned} \tag{S.42}$$

Now let's apply the operators on the two sides of this equation to the ground state $|0\rangle = |n=0\rangle$:

$$\hat{a}\times\exp(\xi\hat{a}^\dagger)\,|0\rangle \;=\; \exp(\xi\hat{a}^\dagger)\times(\hat{a}+\xi)\,|0\rangle\,. \tag{S.43}$$

On the RHS here $\hat{a}$ kills the ground state, $\hat{a}\,|0\rangle = 0$ hence $(\hat{a}+\xi)\,|0\rangle = \xi\,|0\rangle$. Therefore,

$$\hat{a}\times\exp(\xi\hat{a}^\dagger)\,|0\rangle \;=\; \exp(\xi\hat{a}^\dagger)\times\xi\,|0\rangle \;=\; \xi\times\exp(\xi\hat{a}^\dagger)\,|0\rangle\,, \tag{S.44}$$

which means that $\exp(\xi\hat{a}^\dagger)\,|0\rangle$ is an eigen-ket of the lowering operator $\hat{a}$ for the eigenvalue $= \xi$. And since $|\xi\rangle = (\text{number})\times\exp(\xi\hat{a}^\dagger)\,|0\rangle$, it follows that the coherent state $|\xi\rangle$ is indeed an eigen-ket of the lowering operator $|a\rangle$ for the eigenvalue $= \xi$,

$$\hat{a}\,|\xi\rangle \;=\; \xi\,|\xi\rangle\,. \tag{S.45}$$

By Hermitian conjugation,

$$\langle\xi|\,\hat{a}^\dagger \;=\; \left(\hat{a}\,|\xi\rangle\right)^\dagger \;=\; \left(\xi\,|\xi\rangle\right)^\dagger \;=\; \xi^*\,\langle\xi|\,, \tag{S.46}$$

so $\langle\xi|$ is the eigen-bra of the raising operator $\hat{a}^\dagger$ for the eigenvalue $= \xi^*$.

Problem **2**(d):

In class we saw that $\hat{a}^\dagger|\psi\rangle \neq 0$ for any $|\psi\rangle \neq 0$, so for any would-be eigen-ket of the $\hat{a}^\dagger$ operator, $\hat{a}^\dagger|\psi\rangle = \lambda|\psi\rangle$, the eigenvalue $\lambda \neq 0$. Now consider the Dirac products between $|\psi\rangle$ and the Hamiltonian eigenstates $|n\rangle$:

$$\begin{aligned} \lambda|\psi\rangle &= \hat{a}^\dagger|\psi\rangle \\ &\Downarrow \\ \lambda\langle n|\psi\rangle &= \langle n|\hat{a}^\dagger|\psi\rangle = \sqrt{n}\,\langle n-1|\psi\rangle \\ \text{because}\quad \langle n|\hat{a}^\dagger &= (\hat{a}|n\rangle)^\dagger = (\sqrt{n}\,|n-1\rangle)^\dagger = \sqrt{n}\,\langle n|\,. \end{aligned} \tag{S.47}$$

Since $\lambda \neq 0$, we may rewrite this formula as

$$\begin{aligned} \langle n|\psi\rangle &= \frac{\sqrt{n}}{\lambda}\,\langle n-1|\psi\rangle\,, \\ &\text{likewise for } n-1,\ n-2,\ \textit{etc.}, \\ \langle n-1|\psi\rangle &= \frac{\sqrt{n-1}}{\lambda}\,\langle n-2|\psi\rangle\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \langle 1|\psi\rangle &= \frac{\sqrt{1}}{\lambda}\,|0,\lambda\rangle\,, \end{aligned} \tag{S.48}$$

hence altogether

$$\langle n|\psi\rangle = \frac{\sqrt{n}}{\lambda}\times\frac{\sqrt{n-1}}{\lambda}\times\cdots\times\frac{1}{\lambda}\,\langle 0|\psi\rangle = \frac{\sqrt{n!}}{\lambda^n}\,\langle 0|\psi\rangle\,. \tag{S.49}$$

Consequently, the probability overlap

$$|\langle n|\psi\rangle|^2 = |\langle 0|\psi\rangle|^2 \times \frac{n!}{|\lambda|^{2n}} \tag{S.50}$$

starts growing with n once $n > |\lambda|^2$. And this leads to a badly divergent norm of any would-be eigen-ket $|\psi\rangle$ of the $\hat{a}^\dagger$,

$$\langle\psi|\psi\rangle = \sum_{n=0}^{\infty}|\langle n|\psi\rangle|^2 = |\langle 0|\psi\rangle|^2 \times \sum_{n=0}^{\infty}\frac{n!}{|\lambda|^{2n}} = \infty. \tag{S.51}$$

This infinite norm means that the would-be eigen-ket $|\psi\rangle$ does not belong to the Hilbert space of the harmonic oscillator. Worse, it does not even belong to the extended Hilbert space (which allows some un-normalizable states like $|x\rangle$ or $|p\rangle$) because the $n!$ divergence of the series (S.51) is so bad that we cannot even treat this $|\psi\rangle$ as meaningful asymptotic limit of normalizable states.

Thus, for any element $|\psi\rangle \neq 0$ of the extended Hilbert space,

$$\hat{a}^\dagger |\psi\rangle \;\neq\; \lambda |\psi\rangle \quad \text{for any complex number } \lambda. \tag{S.52}$$

In other words, the raising operator $\hat{a}^\dagger$ does not have any eigen-kets at all.

Consequently, the lowering operator $\hat{a}$ — which is the hermitian conjugate of the raising operator $\hat{a}^\dagger$ — does not have any eigen-bras, because the conjugate of any would-be eigen-bra of $\hat{a}$ would be and eigen-ket of $\hat{a}^\dagger$:

$$\textbf{if} \quad \langle\psi|\,\hat{a} \;=\; \lambda\,\langle\psi| \quad \textbf{then} \quad \hat{a}^\dagger\,|\psi\rangle \;=\; \lambda^*\,|\psi\rangle\,, \tag{S.53}$$

and since we have established that $\hat{a}^\dagger$ has no eigen-kets then $\hat{a}$ should likewise have no eigen-bras.

Problem **2**(e):

For a coherent state $|\xi\rangle$, $\hat{a}\,|\xi\rangle = \xi\,|\xi\rangle$ (*cf.* part (c)), so by Hermitian conjugation $\langle\xi|\,\hat{a}^\dagger = \xi^*\,\langle\xi|$. Consequently, for any integers $n, m \geq 0$,

$$\hat{a}^n\,|\xi\rangle \;=\; \xi^n\,|\xi\rangle\,, \quad \langle\xi|\,\hat{a}^{\dagger m} \;=\; \xi^{*m}\,\langle\xi|\,, \tag{S.54}$$

and therefore for any normal-ordered operator product $\hat{a}^{\dagger m}\hat{a}^n$, its expectation value in a coherent state $|\xi\rangle$ is simply

$$\langle\xi|\,\hat{a}^{\dagger m}\hat{a}^n\,|\xi\rangle \;=\; (\langle\xi|\,\hat{a}^{\dagger m})(\hat{a}^n\,|\xi\rangle) \;=\; (\xi^{*m}\,\langle\xi|)(\xi^n\,|\xi\rangle) \;=\; \xi^{*m}\xi^n \times \langle\xi|\xi\rangle \;=\; \xi^{*m}\xi^n. \tag{11}$$

Problem **2**(f):

The Hamiltonian of a harmonic oscillator is $\hat{H} = \hbar\omega(\hat{n} + \frac{1}{2})$, so the expectation value and the uncertainty of energy in any quantum state $|\psi\rangle$ are related to the expectation values of the operators $\hat{n} = \hat{a}^\dagger\hat{a}$ and $\hat{n}^2$ in the same quantum state:

$$\langle E\rangle \;=\; \hbar\omega \times (\langle n\rangle + \tfrac{1}{2}), \qquad (\Delta E) \;=\; \hbar\omega \times (\Delta n). \tag{S.55}$$

Now let's calculate

$$\langle n\rangle \;=\; \langle\xi|\,\hat{n}\,|\xi\rangle\,, \quad \langle n^2\rangle \;=\; \langle\xi|\,\hat{n}^2\,|\xi\rangle\,, \quad \text{hence} \quad (\Delta n)^2 \;=\; \langle n^2\rangle \;-\; \langle n\rangle^2 \tag{S.56}$$

in a coherent state $|\xi\rangle$. Since the number-of-quanta operator $\hat{n} = \hat{a}^\dagger\hat{a}$ is normal ordered, eq. (11) tells us that

$$\langle n\rangle \;=\; \langle\xi|\,\hat{a}^\dagger\hat{a}\,|\xi\rangle \;=\; \xi^*\xi \;=\; |\xi|^2. \tag{S.57}$$

On the other hand, the $\hat{n}^2 = \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}$ operator product is not normal ordered, but it can be re-expressed as a sum of normal-ordered terms:

$$\hat{n}^2 \;=\; \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} \;=\; \hat{a}^\dagger(\hat{a}^\dagger\hat{a} + [\hat{a},\hat{a}^\dagger])\hat{a} \;=\; \hat{a}^\dagger(\hat{a}^\dagger\hat{a} + 1)\hat{a} \;=\; \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \;\hat{a}^\dagger\hat{a}. \tag{S.58}$$

Consequently, in a coherent state

$$\langle\xi|\,\hat{n}^2\,|\xi\rangle \;=\; \langle\xi|\,\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\,|\xi\rangle \;+\; \langle\xi|\,\hat{a}^\dagger\hat{a}\,|\xi\rangle \;=\; \xi^{*2}\xi^2 \;+\; \xi^*\xi \;=\; |\xi|^4 \;+\; |\xi|^2. \tag{S.59}$$

Or in terms of $\langle n\rangle = |\xi|^2$,

$$\langle n^2\rangle \;=\; \langle\xi|\,\hat{n}^2\,|\xi\rangle \;=\; |\xi|^4 \;+\; |\xi|^2 \;=\; \langle n\rangle^2 \;+\; \langle n\rangle\,. \tag{S.60}$$

Therefore,

$$(\Delta n)^2 \;=\; \langle n^2\rangle \;-\; \langle n\rangle^2 \;=\; \langle n\rangle \quad\Longrightarrow\quad \Delta n \;=\; \sqrt{\langle n\rangle} \;=\; |\xi|. \tag{S.61}$$

Or in terms of average energy and its uncertainty,

$$\langle E \rangle \;=\; \hbar\omega(\langle n \rangle + \tfrac{1}{2}) \;=\; \hbar\omega(|\xi|^2 + \tfrac{1}{2}) \tag{S.62}$$

while

$$\Delta E \;=\; \hbar\omega \times \Delta n \;=\; \hbar\omega \times \sqrt{\langle n \rangle} \;=\; \hbar\omega \times |\xi|. \tag{S.63}$$

Finally, the relative energy uncertainty $\Delta E / \langle E \rangle$, or rather $\Delta E / (\langle E \rangle - E_0)$ decreases with average energy,

$$\frac{\Delta E}{\langle E \rangle - E_0} \;=\; \frac{\hbar\omega\sqrt{\langle n \rangle}}{\hbar\omega\,\langle n \rangle} \;=\; \frac{1}{\sqrt{\langle n \rangle}} \;=\; \frac{1}{|\xi|} \;=\; \sqrt{\frac{\hbar\omega}{\langle E \rangle - E_0}}\,. \tag{S.64}$$

Thus, the highly excited coherent state of the harmonic oscillator with $\langle E \rangle \gg \hbar\omega$ have small relative uncertainties of the energy

Problem **2**(g):

In part (b) we saw that for a coherent state $|\xi\rangle$,

$$\langle n|\xi\rangle \;=\; e^{-|\xi|^2/2} \times \frac{\xi^n}{\sqrt{n!}}. \tag{S.39}$$

Hence, for any 2 coherent states $|\xi\rangle$ and $|\eta\rangle$, their Dirac product is

$$\begin{aligned} \langle \eta|\xi\rangle \;&=\; \sum_{n=0}^{\infty} \langle \eta|n\rangle\,\langle n|\xi\rangle \;=\; \sum_{n=0}^{\infty} \langle n|\eta\rangle^*\,\langle n|\xi\rangle \\ &=\; e^{-|\xi|^2/2} e^{-|\eta|^2/2} \times \sum_{n=0}^{\infty} \frac{\eta^{*n}\xi^n}{n!} \\ &=\; e^{-|\xi|^2/2} e^{-|\eta|^2/2} \times e^{\eta^*\xi}, \end{aligned} \tag{S.65}$$

or in other words,

$$\langle \eta|\xi\rangle \;=\; \exp(\mathcal{E}) \quad \text{for} \quad \mathcal{E} \;=\; -\tfrac{1}{2}|\xi|^2 \;-\; \tfrac{1}{2}|\eta|^2 \;+\; \eta^*\xi. \tag{S.66}$$

Consequently, the probability overlap is

$$|\langle\eta|\xi\rangle|^2 \;=\; \exp(\mathcal{E}) \times \big(\exp\mathcal{E}\big)^* \;=\; \exp(\mathcal{E}) \times \exp(\mathcal{E}^*) \;=\; \exp(\mathcal{E}+\mathcal{E}^*) \tag{S.67}$$

where

$$\mathcal{E} \;+\; \mathcal{E}^* \;=\; -|\xi|^2 \;-\; |\eta|^2 \;+\; \eta^*\xi \;+\; \eta\xi^* \;=\; -\,|\xi-\eta|^2\,. \tag{S.68}$$

Altogether,

$$|\langle\eta|\xi\rangle|^2 \;=\; \exp\Big(|\xi-\eta|^2\Big)\,, \tag{S.69}$$

which indeed rapidly decreases for $|\xi-\eta| \gg 1$.

<u>Problem **2**(h)</u>:

Let

$$\hat{S} \;=\; \int\frac{d^2\xi}{\pi}\;|\xi\rangle\,\langle\xi|\;; \tag{S.70}$$

our task is to show that $\hat{S}=\hat{1}$, and we shall do it by calculating its matrix elements between the Hamiltonian's eigenstates and showing that $\langle m|\,\hat{S}\,|n\rangle = \delta_{m,n}$.

First, in light of

$$\langle n|\xi\rangle \;=\; e^{-|\xi|^2/2} \times \frac{\xi^n}{\sqrt{n!}} \tag{S.39}$$

we have calculated in part (b),

$$\langle m|\xi\rangle \times \langle\xi|n\rangle \;=\; e^{-|\xi|^2} \times \frac{\xi^m}{\sqrt{m!}} \times \frac{\xi^{*n}}{\sqrt{n!}}\,. \tag{S.71}$$

Next, we should integrate this formula over the whole complex plane, so let's use the polar coordinates

$$\xi \;=\; \rho e^{i\phi}, \qquad \int d^2\xi \;=\; \int_0^\infty \rho\,d\rho \int_0^{2\pi} d\phi. \tag{S.72}$$

In terms of these coordinates ρ and ϕ, eq. (S.71) becomes

$$\langle m|\xi\rangle \times \langle\xi|n\rangle \;=\; e^{-\rho^2} \times \frac{\rho^{n+m} e^{i(m-n)\phi}}{\sqrt{m!\,n!}}\,, \tag{S.73}$$

hence

$$\begin{aligned}\langle m|\,\hat{S}\,|n\rangle \;&=\; \int \frac{d^2\xi}{\pi}\,\langle m|\xi\rangle \times \langle\xi|n\rangle\\ &=\int_0^\infty \frac{\rho\,d\rho}{\pi}\,\frac{e^{-\rho^2}\rho^{m+n}}{\sqrt{m!\,n!}} \times \int_0^{2\pi} d\phi\, e^{i(m-n)\phi}.\end{aligned} \tag{S.74}$$

The $\int d\phi$ integral here evaluates to

$$\int_0^{2\pi} d\phi\, e^{i(m-n)\phi} \;=\; 2\pi\delta_{m,n} \tag{S.75}$$

hence $\langle m|\,\hat{S}\,|n\rangle = 0$ unless $m = n$. As to the diagonal matrix elements of $\hat{S}$, we get

$$\langle n|\,\hat{S}\,|n\rangle \;=\; \int_0^\infty 2\rho\,d\rho \times \frac{e^{-\rho^2}\,\rho^{2n}}{n!} \;=\; \frac{1}{n!}\int_0^\infty d(\rho^2)\,e^{-\rho^2} \times (\rho^2)^n \;=\; \frac{1}{n!} \times n! \;=\; 1. \tag{S.76}$$

Altogether,

$$\langle m|\,\hat{S}\,|n\rangle \;=\; \delta_{m,n}\,, \tag{S.77}$$

and since the states $\{|n\rangle : n = 0, 1, 2, \ldots\}$ form a complete orthonormal basis of the Hilbert space, it follows that $\hat{S}$ is a unit operator,

$$\hat{S} \;=\; \int \frac{d^2\xi}{\pi}\,|\xi\rangle\,\langle\xi| \;=\; \hat{1}, \tag{12}$$

quod erat demonstrandum.

Problem **3**(a):

For the $\xi(t) = \xi_0 \exp(-i\omega t)$ we have

$$\frac{d\xi}{dt} = -i\omega\xi(t), \tag{S.78}$$

hence

$$\frac{d}{dt}\exp(\xi(t)\hat{a}^\dagger) = \frac{d\xi}{dt} \times \frac{d}{d\xi}\exp(\xi\hat{a}^\dagger) = -i\omega\xi \times \hat{a}^\dagger \exp(\xi\hat{a}^\dagger). \tag{S.79}$$

Also, the normalization factor $e^{-|\xi|^2/2}$ of the coherent state $|\xi\rangle$ remains time independent, hence

$$\begin{aligned} \frac{d}{dt}|\xi(t)\rangle &= \frac{d}{dt}\Big(e^{-|\xi|^2/2}\exp(\xi(t)\hat{a}^\dagger)|0\rangle\Big) \\ &= e^{-|\xi|^2/2} \times -i\omega\xi(t) \times \hat{a}^\dagger \exp(\xi(t)\hat{a}^\dagger)|0\rangle \\ &= -i\omega\xi\hat{a}^\dagger |\xi(t)\rangle\,. \end{aligned} \tag{S.80}$$

Moreover, the coherent state $|\xi\rangle$ obeys $\hat{a}|\xi\rangle = \xi|\xi\rangle$, *cf.* problem 2(c), hence

$$\hat{a}^\dagger\hat{a}|\xi\rangle = \hat{a}^\dagger(\xi|\xi\rangle) = \xi\hat{a}^\dagger|\xi\rangle\,. \tag{S.81}$$

Consequently, we may rewrite eq. (S.80) as

$$\frac{d}{dt}|\xi(t)\rangle = -i\omega\xi(t)\hat{a}^\dagger|\xi(t)\rangle = -i\omega\hat{a}^\dagger\hat{a}|\xi(t)\rangle\,. \tag{S.82}$$

If we then multiply the coherent state $|\xi(t)\rangle$ by the time-dependent phase $e^{-i\omega t/2}$ as in eq. (14), we then have

$$\begin{aligned} \frac{d}{dt}|\psi\rangle(t) &= \frac{d}{dt}\Big(e^{-i\omega t/2}|\xi(t)\rangle\Big) = -\frac{i\omega}{2}e^{-i\omega t/2} \times |\xi(t)\rangle + e^{-i\omega t/2} \times \frac{d}{dt}|\xi(t)\rangle \\ &= -\frac{i\omega}{2}e^{-i\omega t/2} \times |\xi(t)\rangle + e^{-i\omega t/2} \times -i\omega\hat{a}^\dagger\hat{a}|\xi(t)\rangle \\ &= -\frac{i\omega}{2}|\psi\rangle(t) - i\omega\hat{a}^\dagger\hat{a}|\psi\rangle(t) \\ &= -i\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})|\psi\rangle(t), \end{aligned} \tag{S.83}$$

and therefore

$$i\hbar\frac{d}{dt}|\psi\rangle = +\hbar\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})|\psi\rangle = \hat{H}\psi. \tag{S.84}$$

Thus, the state (14) indeed obeys the time-dependent Schrödinger equation.

Problem **3**(b):

We saw in class that in terms of the lowering and raising operators $\hat{a}$ and $\hat{a}^\dagger$,

$$\hat{q} \;=\; \sqrt{\frac{\hbar}{2\omega m}}\,(\hat{a}+\hat{a}^\dagger), \qquad \hat{p} \;=\; \sqrt{\frac{\hbar\omega m}{2}}\,(-i\hat{a}+i\hat{a}^\dagger). \tag{S.85}$$

In a coherent state $|\xi\rangle$,

$$\langle\xi|\,\hat{a}\,|\xi\rangle \;=\; \xi, \qquad \langle\xi|\,\hat{a}^\dagger\,|\xi\rangle \;=\; \xi^*, \tag{S.86}$$

cf. eq. (11), hence

$$\langle\xi|\,\hat{q}\,|\xi\rangle \;=\; \sqrt{\frac{\hbar}{2\omega m}}\times\big(\xi+\xi^* \;=\; 2\,\mathrm{Re}\,\xi\big), \qquad \langle\xi|\,\hat{p}\,|\xi\rangle \;=\; \sqrt{\frac{\hbar\omega m}{2}}\times\big(-i\xi+i\xi^* \;=\; 2\,\mathrm{Im}\,\xi\big). \tag{S.87}$$

In particular, in the time-dependent coherent state with

$$\xi(t) \;=\; \xi_0\times e^{-i\omega t} \;=\; \rho e^{i\phi}\times e^{-i\omega t} \tag{13$'$}$$

we have

$$\mathrm{Re}\,\xi(t) \;=\; \rho\times\cos(\omega t-\phi), \qquad \mathrm{Im}\,\xi(t) \;=\; -\rho\times\sin(\omega t-\phi), \tag{S.88}$$

hence

$$\langle q\rangle\,(t) \;=\; \rho\sqrt{\frac{2\hbar}{\omega m}}\times\cos(\omega t-\phi), \qquad \langle p\rangle\,(t) \;=\; -\rho\sqrt{2\hbar\omega m}\times\sin(\omega t-\phi). \tag{S.89}$$

These are classical harmonic oscillations

$$\langle x\rangle\,(t) \;=\; A\times\cos(\omega t-\phi), \qquad \langle p\rangle\,(t) \;=\; -\omega m A\times\sin(\omega t-\phi) \tag{S.90}$$

with amplitude

$$A \;=\; \rho\times\sqrt{\frac{2\hbar}{\omega m}} \;=\; |\xi_0|\times\sqrt{\frac{2\hbar}{\omega m}}\,. \tag{S.91}$$

Problem **3**(c):

In light of eqs. (S.85),

$$\hat{q}^2 = \frac{\hbar}{2\omega m}(\hat{a}+\hat{a}^\dagger)^2, \qquad \hat{p}^2 = \frac{\hbar\omega m}{2}(-i\hat{a}+i\hat{a}^\dagger)^2, \tag{S.92}$$

where

$$\begin{aligned}(\hat{a}+\hat{a}^\dagger)^2 &= \hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger = \hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}^\dagger\hat{a} + 1,\\ (-i\hat{a}+i\hat{a}^\dagger)^2 &= -\hat{a}^2 - \hat{a}^{\dagger 2} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger = -\hat{a}^2 - \hat{a}^{\dagger 2} + 2\hat{a}^\dagger\hat{a} + 1.\end{aligned} \tag{S.93}$$

Consequently, in a coherent state $|\xi\rangle$,

$$\begin{aligned}\langle\xi|\,(\hat{a}+\hat{a}^\dagger)^2\,|\xi\rangle &= \xi^2 + \xi^{*2} + 2\xi\xi^* + 1 = (\xi+\xi^*)^2 + 1\\ &= (2\,\mathrm{Re}\,\xi)^2 + 1,\\ \langle\xi|\,(-\hat{a}+i\hat{a}^\dagger)^2\,|\xi\rangle &= -\xi^2 - \xi^{*2} + 2\xi\xi^* + 1 = (-i\xi+i\xi^*)^2 + 1\\ &= (2\,\mathrm{Im}\,\xi)^2 + 1,\end{aligned} \tag{S.94}$$

hence

$$\begin{aligned}\left\langle\hat{q}^2\right\rangle &= \frac{\hbar}{2\omega m}\times\big((2\,\mathrm{Re}\,\xi)^2 + 1\big)\\ &= \langle\hat{q}\rangle^2 + \frac{\hbar}{2\omega m},\\ \left\langle\hat{p}^2\right\rangle &= \frac{\hbar\omega m}{2}\times\big((2\,\mathrm{Im}\,\xi)^2 + 1\big)\\ &= \langle\hat{q}\rangle^2 + \frac{\hbar\omega m}{2}.\end{aligned} \tag{S.95}$$

and therefore

$$(\Delta q)^2 = \left\langle\hat{q}^2\right\rangle - \langle\hat{q}\rangle^2 = \frac{\hbar}{2\omega m}, \qquad (\Delta p)^2 = \left\langle\hat{p}^2\right\rangle - \langle\hat{p}\rangle^2 = \frac{\hbar\omega m}{2}. \tag{S.96}$$

In particular,

$$\Delta q\times\Delta p = \sqrt{\frac{\hbar}{2\omega m}}\times\sqrt{\frac{\hbar\omega m}{2}} = \frac{\hbar}{2}, \tag{S.97}$$

the minimum allowed by the Heisenberg uncertainty principle.

Problem **3**(d):

First, let's relate $\bar{q} = \langle\xi|\,\hat{q}\,|\xi\rangle$ and $\bar{p} = \langle\xi|\,\hat{p}\,|\xi\rangle$ to the complex ξ: According to eqs. (S.87),

$$\bar{q} \;=\; \sqrt{\frac{2\hbar}{\omega m}}\times\mathrm{Re}\,\xi,\quad \bar{p} \;=\; \sqrt{2\hbar\omega m}\times\mathrm{Im}\,\xi, \tag{S.98}$$

hence

$$\xi \;=\; \frac{m\omega\bar{q}+i\bar{p}}{\sqrt{2\hbar\omega m}}\,. \tag{S.99}$$

Next, in terms of the coordinate basis wave functions $\psi(q)$,

$$\hat{q}\psi(q) \;=\; q\times\psi(q),\quad \hat{p}\psi(q) \;=\; -i\hbar\frac{d}{dq}\psi(q), \tag{S.100}$$

hence

$$\hat{a}\psi(q) \;=\; \frac{m\omega\hat{q}+i\hat{p}}{\sqrt{2\hbar\omega m}}\,\psi(x) \;=\; \frac{1}{\sqrt{2\hbar\omega m}}\left(m\omega q\,+\,\hbar\frac{d}{dq}\right)\psi(x). \tag{S.101}$$

Consequently, the eigenstate equation $\hat{a}\,|\psi\rangle = \xi\,|\psi\rangle$ for the coherent state $|\psi\rangle = |\xi\rangle$ becomes

$$\frac{1}{\sqrt{2\hbar\omega m}}\left(m\omega q\,+\,\hbar\frac{d}{dq}\right)\psi(x) \;=\; \xi\times\psi(x) \;=\; \frac{m\omega\bar{q}+i\bar{p}}{\sqrt{2\hbar\omega m}}\times\psi(x) \tag{S.102}$$

and hence

$$\frac{d\psi}{dq} \;=\; -\frac{m\omega}{\hbar}\times q\times\psi \;+\; \frac{m\omega\bar{q}+i\bar{p}}{\hbar}\times\psi. \tag{S.103}$$

This is a first order linear differential equation, so it has a unique solution modulo the overall constant coefficient. To solve it, we re-write it as

$$\frac{d\psi}{\psi} \;=\; dq\times\left(-\frac{m\omega}{\hbar}\times q\,+\,\frac{m\omega\bar{x}+i\bar{p}}{\hbar}\right) \;=\; d\left(-\frac{m\omega}{2\hbar}\times(q-\bar{q})^2\,+\,i\frac{\bar{p}}{\hbar}\times q\right) \tag{S.104}$$

where on the LHS

$$\frac{d\psi}{\psi} \;=\; d\log\psi(q), \tag{S.105}$$

hence

$$\log\psi(q) \;=\; -\frac{m\omega}{2\hbar}\times(q-\bar{q})^2 \;+\; i\frac{\bar{p}}{\hbar}\times q \;+\; \text{const} \tag{S.106}$$

and therefore

$$\psi(q) \;=\; C\times\exp\left(-\frac{m\omega}{2\hbar}\times(q-\bar{q})^2 \;+\; i\frac{\bar{p}}{\hbar}\times q\right) \tag{16}$$

for some constant C. In other words, the wave-function of the coherent state is indeed a Gaussian wave packet.

<u>Problem **3**(e)</u>:
This was done in class on 9/14, but let's repeat the solutions here.

Since $\hat{q}$ and $\hat{p}$ are linear combinations of the raising and lowering operators $\hat{a}^\dagger$ and $\hat{a}$, it follows that

$$\langle m|\,\hat{q}\,|n\rangle \;=\; 0 \quad\text{and}\quad \langle m|\,\hat{p}\,|n\rangle \;=\; 0 \quad \text{unless } m=n\pm 1, \tag{S.107}$$

in particular

$$\langle n|\,\hat{q}\,|n\rangle \;=\; \langle n|\,\hat{p}\,|n\rangle \;=\; 0. \tag{18}$$

Next, we saw in part (c) that

$$\begin{aligned} \hat{q}^2 &= \frac{\hbar}{2\omega m}\Big(\hat{a}^2+\hat{a}^{\dagger 2}+2\hat{a}^\dagger\hat{a}+1\Big),\\ \hat{p}^2 &= \frac{\hbar\omega m}{2}\Big(-\hat{a}^2-\hat{a}^{\dagger 2}+2\hat{a}^\dagger\hat{a}+1\Big). \end{aligned} \tag{S.108}$$

Since $\hat{a}$ always lowers n by 1 while $\hat{a}^\dagger$ raises n by 1, it follows that

$$\begin{aligned} \langle m|\,\hat{a}^2\,|n\rangle &= 0 \quad \text{unless} \quad m \;=\; n\;-\;2,\\ \langle m|\,\hat{a}^{\dagger 2}\,|n\rangle &= 0 \quad \text{unless} \quad m \;=\; n\;+\;2, \end{aligned} \tag{S.109}$$

and in particular

$$\langle n|\,\hat{a}^2\,|n\rangle \;=\; \langle n|\,\hat{a}^{\dagger 2}\,|n\rangle \;=\; 0. \tag{S.110}$$

On the other hand,

$$\langle n|\,(2\hat{a}^\dagger\hat{a}+1)\,|n\rangle \;=\; \langle n|\,(2\hat{n}+1)\,|n\rangle \;=\; 2n \;+\; 1, \tag{S.111}$$

hence

$$\langle n|\,\hat{q}^2\,|n\rangle \;=\; \frac{\hbar}{2\omega m}\times(2n+1) \quad\text{and}\quad \langle n|\,\hat{p}^2\,|n\rangle \;=\; \frac{\hbar\omega m}{2}\times(2n+1). \tag{S.112}$$

Therefore, in the Hamiltonian eigenstate $|n\rangle$, the values of q and p have uncertainties[2]

$$\begin{aligned}(\Delta q)^2 \;&=\; \left\langle\hat{q}^2\right\rangle \;-\; \langle\hat{q}\rangle^2 \;=\; \frac{\hbar}{2\omega m}\times(2n+1) \;-\; 0,\\ (\Delta p)^2 \;&=\; \left\langle\hat{p}^2\right\rangle \;-\; \langle\hat{p}\rangle^2 \;=\; \frac{\hbar\omega m}{2}\times(2n+1) \;-\; 0.\end{aligned} \tag{19}$$

Note that both uncertainties grow with n as $\sqrt{2n+1}$:

$$\Delta q \;=\; \sqrt{\frac{\hbar}{2\omega m}}\times\sqrt{2n+1}, \qquad \Delta p \;=\; \sqrt{\frac{\hbar\omega m}{2}}\times\sqrt{2n+1}, \tag{S.113}$$

so for the highly excited states with $n\gg 1$ they become very large. In particular,

$$\Delta q\times\Delta p \;=\; \frac{\hbar}{2}\times(2n+1) \;=\; \frac{E}{\omega}\,, \tag{S.114}$$

which can become classically large for $E\gg\hbar\omega$.