Problem **1**(a):

Note: by symmetry of the potential, $V(-x) = V(+x)$, we have

$$x_2 = -x_3 \quad \text{and} \quad x_1 = -x_4 . \tag{S.1}$$

Also, the WKB approximate solutions (5) for the middle of the barrier are symmetric or antisymmetric WRT $x \to -x$,

$$\Psi_+(-x) = +\Psi_+(+x), \qquad \Psi_-(-x) = -\Psi_-(+x), \tag{S.2}$$

so they should have the same anti/symmetry at all x and not just inside the barrier. In particular, $\Psi_+(x)$ should be symmetric and $\Psi_-(x)$ antisymmetric between the two potential wells, and this makes eqs. (6) and (7) completely equivalent to each other. Thus, I am going to derive eqs. (6) — for $\delta\phi$ as in eq. (8) — for the right well, and then eq. (7) for the left well will follow from the symmetry.

So let's start with eq. (5) for the WKB functions in the classically forbidden zone. Inside each exponential, we have

$$\int_0^x \kappa(x')\,dx' = \int_0^{x_3} \kappa(x')\,dx' - \int_x^{x_3} \kappa(x')\,dx' \tag{S.3}$$

where

$$\int_0^{x_3} \kappa(x')\,dx' = \frac{1}{2}\int_{x_2}^{x_3} \kappa(x')\,dx' = \frac{w}{2} . \tag{S.4}$$

Consequently, from the x_3 point of view, the wave-functions (5) become

$$\Psi_\pm(x) = \frac{Ce^{+w/2}}{\sqrt{\kappa(x)}} \times \exp\left(-\int_x^{x_3} \kappa(x')\,dx'\right) \pm \frac{Ce^{-w/2}}{\sqrt{\kappa(x)}} \exp\left(+\int_x^{x_3} \kappa(x')\,dx'\right) . \tag{S.5}$$

For x near the x_3 turning point, the first term here looks like the regular Airy function while the second term looks like the irregular Airy function, so analytically continuing x into the right well between x_3 and x_4, we get

$$\Psi_\pm(x) \;=\; \frac{2Ce^{+w/2}}{\sqrt{k(x)}} \times \sin\left(\frac{\pi}{4} + \int_{x_3}^{x} k(x')\,dx'\right) \;\pm\; \frac{Ce^{-w/2}}{\sqrt{k(x)}} \times \cos\left(\frac{\pi}{4} + \int_{x_3}^{x} k(x')\,dx'\right), \quad \text{(S.6)}$$

which a bit of trigonometry turns to

$$\Psi_\pm(x) \;=\; \frac{D}{\sqrt{k(x)}} \times \sin\left(\frac{\pi}{4} + \int_{x_3}^{x} k(x')\,dx' \pm \delta\phi\right), \quad (6)$$

where

$$D \;=\; C \times \sqrt{4e^{+w} + e^{-w}} \quad \text{(S.7)}$$

and

$$\tan\delta\phi \;=\; \frac{e^{-w/2}}{2e^{+w/2}} \;=\; \frac{e^{-w}}{2}\,, \quad \text{(S.8)}$$

exactly as in eq. (8).

Problem **1**(b):

Now look at the WKB-approximated wave-functions in the right well $x_3 < x < x_4$ from the x_3 and x_4 points of view. Looking from the x_3 end of the well, we have

$$\Psi_\pm(x) \;=\; \frac{D}{\sqrt{k(x)}} \times \sin\left(\frac{\pi}{4} \pm \delta\phi + \int_{x_3}^{x} k(x')\,dx'\right) \quad (6)$$

where $\delta\phi$ stems from the tunneling to/from the left well. OOH, looking from the x_4 end we get

$$\Psi_\pm \;=\; \frac{D'}{\sqrt{k(x)}} \times \sin\left(\frac{\pi}{4} + \int_{x}^{x_4} k(x')\,dx'\right), \quad \text{(S.9)}$$

and there is no $\delta\phi$-like correction here because the particle has no where to tunnel to the right of the right well. To keep eqs. (6) and (S.9) consistent, we need either

$$\frac{\pi}{4} \pm \delta\phi + \int_{x_3}^{x} k(x') = \frac{\pi}{4} + \int_{x}^{x_4} k(x')\,dx' + n\pi \tag{S.10}$$

or

$$\frac{\pi}{4} \pm \delta\phi + \int_{x_3}^{x} k(x') = -\frac{\pi}{4} - \int_{x}^{x_4} k(x')\,dx' + (n+1)\pi \tag{S.11}$$

for some integer n. Furthermore, eq. (S.10) is x-dependent and possibly hold for all x between x_3 and x_4, while eq. (S.11) is x-independent and amounts to

$$\int_{x_3}^{x_4} k(x')\,dx' = (n+\tfrac{1}{2})\pi \mp \delta\phi. \tag{S.12}$$

Or in terms of the bounce action for the well,

$$S_{\text{bounce}}^{\text{well}} \stackrel{\text{def}}{=} \oint_{\text{period}} p\,dx = 2\hbar \int_{x_3}^{x_4} k(x')\,dx', \tag{S.13}$$

for the bound state energy $E = E_n \mp \delta E_n$ we should have

$$S_{\text{bounce}}^{\text{well}}(E) = 2\pi\hbar(n+\tfrac{1}{2}) \mp 2\hbar\delta\phi. \tag{9}$$

Quod erat demonstrandum.

Problem **1**(b):

Assume $\delta\phi \ll 1$ and hence $\delta E_n \ll E_n$. Consequently, for the bounce action of the right well we approximate

$$S(E_n \mp \delta E_n) \approx S(E_n) \mp \frac{dS}{dE} \times \delta E_n = S(E_n) \mp T(E_n) \times \delta E_n\,, \tag{S.14}$$

and plugging this bounce action into eq. (9) we get

$$S(E_n) \mp T(E_n) \times \delta E_n = 2\pi\hbar(n+\tfrac{1}{2}) \mp 2\hbar\delta\phi. \tag{S.15}$$

At the same time, for the standalone right well (without tunneling to/from the left well), Bohr–Sommerfeld rule gives us

$$S(E_n) \;=\; 2\pi\hbar(n+\tfrac{1}{2}), \tag{2}$$

so taking the difference between the two equations we immediately obtain

$$T(E_n)\times\delta E_n \;=\; 2\hbar\delta\phi \tag{S.16}$$

and hence

$$\delta E_n \;=\; \frac{2\hbar}{T(E_n)}\times\delta\phi(E_n). \tag{11}$$

Quod erat demonstrandum.

Problem **2**(a):

First, a point about covariant derivatives: for a field $\phi(\mathbf{x})$ which transforms under gauge symmetries $\mathbf{A}(\mathbf{x})\to\mathbf{A}(\mathbf{x})+\nabla\Lambda(\mathbf{x})$ as $\phi(\mathbf{x})\to\phi(\mathbf{x})\times\exp\bigl(+i\frac{q}{\hbar c}\Lambda(\mathbf{x})\bigr)$, the covariant derivative of this field acts as

$$\mathbf{D}\phi(\mathbf{x}) \;=\; \nabla\phi(\mathbf{x}) \;+\; \frac{iq}{\hbar c}\mathbf{A}(\mathbf{x})\phi(\mathbf{x}). \tag{S.17}$$

In particular, for the Landau–Ginzburg field of the Cooper-pair BEC or the covariant derivatives of this field,

$$\mathbf{D}\Psi(\mathbf{x}) \;=\; \nabla\Psi(\mathbf{x}) \;-\; \frac{2ie}{\hbar c}\mathbf{A}(\mathbf{x})\Psi(\mathbf{x}),\quad D_iD_j\Psi(\mathbf{x}) \;=\; \left(\nabla_i \;-\; \frac{2ie}{\hbar c}A_i(\mathbf{x})\right)D_j\Psi(\mathbf{x}),\;\ldots, \tag{S.18}$$

but for the complex-conjugate field $\Psi^*(\mathbf{x})$ (or its covariant derivatives), we have

$$\mathbf{D}\Psi^*(\mathbf{x}) \;=\; \nabla\Psi^*(\mathbf{x}) + \frac{2ie}{\hbar c}\mathbf{A}(\mathbf{x})\Psi^*(\mathbf{x}), \tag{S.19}$$

hence $\mathbf{D}\Psi^*(\mathbf{x})=\bigl(\mathbf{D}\Psi(\mathbf{x})\bigr)^*$.

Now let $\Psi_1(\mathbf{x})$ be $\Psi(\mathbf{x})$ or any of its covariant derivatives $D_i\Psi(\mathbf{x})$, $D_iD_j\Psi(\mathbf{x})$, *etc.*, while $\Psi_2^*(\mathbf{x})$ is $\Psi^*(\mathbf{x})$ or any of its covariant derivatives. Consequently,

$$D_i\Psi_1 \;=\; \nabla_i\Psi_1 \;-\; \frac{2ie}{\hbar c}\mathbf{A}\Psi_1\,, \qquad D_i\Psi_2^* \;=\; \nabla_i\Psi_2^* \;+\; \frac{2ie}{\hbar c}\mathbf{A}\Psi_2^*\,, \tag{S.20}$$

and hence

$$\begin{aligned}(D_i\Psi_2^*)\Psi_1 \;+\; \Psi_2^*(D_i\Psi_1) \;&=\; (\nabla_1\Psi_2^*)\Psi_1 \;+\; \frac{2ie}{\hbar c}\mathbf{A}\Psi_2^*\Psi_1 \;+\; \Psi_2^*(\nabla_i\Psi_1) \;-\; \Psi_2^*\frac{2ie}{\hbar c}\mathbf{A}\Psi_1 \\ &=\; (\nabla_i\Psi_2^*)\Psi_1 \;+\; \Psi_2^*(\nabla_i\Psi_1) \;=\; \nabla(\Psi_2^*\Psi_1).\end{aligned} \tag{S.21}$$

In particular, for $\Psi_2^* = \Psi^*$ and $\Psi_1 = D_j\Psi$, we have

$$\nabla_i(\Psi^* D_j\Psi) \;=\; (D_i\Psi^*)(D_j\Psi) \;+\; \Psi^*(D_iD_j\Psi) \tag{S.22}$$

and hence (after summing over $i = j = x, y, z$)

$$\nabla\cdot(\Psi^*\mathbf{D}\Psi) \;=\; (\mathbf{D}\Psi^*)\cdot(\mathbf{D}\Psi) \;+\; \Psi^*(\mathbf{D}^2\Psi). \tag{S.23}$$

Note that the first term on the RHS here is real since $\mathbf{D}\Psi^* = (\mathbf{D}\Psi)^*$. Consequently, taking the imaginary parts of both sides of this equation we arrive at

$$\nabla\cdot\mathrm{Im}(\Psi^*\mathbf{D}\Psi) \;=\; 0 \;+\; \mathrm{Im}\big(\Psi^*\mathbf{D}^2\Psi\big). \tag{S.24}$$

Now let's apply this result to the supercurrent

$$\mathbf{J}_s \;=\; \frac{-2e\hbar}{M}\,\mathrm{Im}(\Psi^*\mathbf{D}\Psi) \tag{SC.9}$$

and its divergence:

$$\nabla\cdot\mathbf{J}_s \;=\; \frac{-2e\hbar}{M}\nabla\cdot\mathrm{Im}(\Psi^*\mathbf{D}\Psi) \;=\; \frac{-2e\hbar}{M}\,\mathrm{Im}\big(\Psi^*\mathbf{D}^2\Psi\big). \tag{S.25}$$

For the Landau–Ginzburg field obeying its Schrödinger-like equation of motion

$$i\hbar D_t\Psi \;=\; -\frac{\hbar^2}{2M}\mathbf{D}^2\Psi \;+\; (\lambda|\Psi|^2-\mu)\Psi, \tag{SC.2}$$

we have

$$i\hbar\Psi^* D_t\Psi \;=\; -\frac{\hbar^2}{2M}\,\Psi^*\mathbf{D}^2\Psi \;+\; (\lambda|\Psi|^2-\mu)\,|\Psi^2| \tag{S.26}$$

where the second term on the RHS is real, hence taking the imaginary parts of both sides gives us

$$\mathrm{Im}\big(i\hbar\Psi^* D_t\Psi\big) \;=\; -\frac{\hbar^2}{2M}\,\mathrm{Im}\big(\Psi^*\mathbf{D}^2\Psi\big), \tag{S.27}$$

or equivalently

$$\mathrm{Re}\big(\Psi^* D_t\Psi\big) \;=\; -\frac{\hbar}{2M}\,\mathrm{Im}\big(\Psi^*\mathbf{D}^2\Psi\big). \tag{S.28}$$

Plugging this formula into eq. (S.25), we arrive at

$$\nabla\cdot\mathbf{J}_s \;=\; +4e\,\mathrm{Re}\big(\Psi^* D_t\Psi\big). \tag{S.29}$$

Now consider the superconducting charge density $\rho_s = -2e|\Psi|^2$ and its time derivative. The covariant time derivative D_t of a product $\Psi^*\Psi$ works similarly to eq. (S.21) for the covariant space derivatives:

$$\begin{aligned}(D_t\Psi^*)\Psi \;+\; \Psi^*(D_t\Psi) \;&=\; \frac{\partial\Psi^*}{\partial t}\Psi \;-\; \frac{2ie}{\hbar}\Psi^*\Psi \;+\; \Psi^*\frac{\partial\Psi}{\partial t} \;+\; \Psi^*\,\frac{2ie}{\hbar}\Psi\\ &=\; \frac{\partial\Psi^*}{\partial t}\Psi \;+\; \Psi^*\frac{\partial\Psi}{\partial t} \;+\; 0 \;=\; \frac{\partial}{\partial t}(\Psi^*\Psi).\end{aligned} \tag{S.30}$$

Consequently,

$$\begin{aligned}\frac{\partial\rho_s}{\partial t} \;&=\; -2e\frac{\partial}{\partial t}(\Psi^*\Psi) \;=\; -2e\Big((D_t\Psi^*)\Psi \;+\; \Psi^*(D_t\Psi)\Big)\\ &=\; -2e\Big(2\,\mathrm{Re}(\Psi^* D_t\Psi)\Big) \;=\; -4e\,\mathrm{Re}(\Psi^* D_t\Psi)\\ &=\; -\nabla\cdot\mathbf{J}_s \qquad \langle\!\langle \text{ by eq. (S.29) } \rangle\!\rangle.\end{aligned} \tag{S.31}$$

Quod erat demonstrandum.

Problem **2**(b):

For the Landau–Ginzburg field as in eq. (SC.13),

$$\nabla\Psi \;=\; \sqrt{n_s}e^{iS/\hbar}\left(\frac{\nabla n_s}{2n_s}\,+\,\frac{i}{\hbar}\nabla S\right), \tag{S.32}$$

$$\mathbf{D}\Psi \;=\; \nabla\Psi\,+\,\frac{2ei}{\hbar c}\mathbf{A}\Psi \;=\; \sqrt{n_s}e^{iS/\hbar}\left(\frac{\nabla n_s}{2n_s}\,+\,\frac{i}{\hbar}\nabla S\,+\,\frac{2ie}{\hbar c}\mathbf{A}\right), \tag{S.33}$$

$$\Psi^*\mathbf{D}\Psi \;=\; n_s\left(\frac{\nabla n_s}{2n_s}\,+\,\frac{i}{\hbar}\nabla S\,+\,\frac{2ie}{\hbar c}\mathbf{A}\right), \tag{S.34}$$

where inside the $(\cdots)$ on the last line, the first term is real while the other 2 terms are imaginary. Therefore,

$$\mathrm{Im}\big(\Psi^*\mathbf{D}\Psi\big) \;=\; \frac{n_s}{\hbar}\left(\nabla S\,+\,\frac{2e}{c}\mathbf{A}\right), \tag{S.35}$$

and hence the electric current of the Cooper pair BEC

$$\mathbf{J}_s \;=\; \frac{-2e\hbar}{M}\,\mathrm{Im}\big(\Psi^*\mathbf{D}\Psi\big) \;=\; -\frac{2en_s}{M}\left(\nabla S\,+\,\frac{2e}{c}\mathbf{A}\right). \tag{SC.14}$$

Problem **2**(c):

For a time-independent magnetic field in a non-magnetic material, the relevant Maxwell equations are

$$\nabla\cdot\mathbf{B} \;=\; 0 \quad\text{and}\quad \nabla\times\mathbf{B} \;=\; \frac{4\pi}{c}\mathbf{J} \tag{S.36}$$

(in Gauss units). Taking a curl of both sides of the second equation here, we get

$$\frac{4\pi}{c}\nabla\times\mathbf{J} \;=\; \nabla\times\nabla\times\mathbf{B} \;=\; \nabla(\nabla\cdot\mathbf{B})\;-\;\nabla^2\mathbf{B} \;=\; 0\;-\;\nabla^2\mathbf{B}. \tag{S.37}$$

Inside a superconductor (with un-quenched superconductivity), we have

$$\mathbf{J}_{\rm net} \;=\; \mathbf{J}_s \;=\; -\frac{2en_s}{M}\left(\nabla S\,+\,\frac{2e}{c}\mathbf{A}\right), \tag{S.38}$$

and hence

$$\frac{4\pi}{c}\nabla\times\mathbf{J} \;=\; -\frac{8\pi e}{Mc}\,(\nabla n_s)\times\left(\nabla S\,+\,\frac{2e}{c}\mathbf{A}\right)\;-\;\frac{8\pi e n_s}{Mc}\,\nabla\times\left(\nabla S\,+\,\frac{2e}{c}\mathbf{A}\right). \tag{S.39}$$

On the RHS here, the first term vanishes in a *bulk* superconductor with a uniform $n_s = \text{const}$

(because of $\nabla n_s = 0$), while in the second term

$$\nabla\times\left(\nabla S + \frac{2e}{c}\mathbf{A}\right) \;=\; \nabla\times\nabla S \;+\; \frac{2e}{c}\,\nabla\mathbf{A} \;=\; 0 \;+\; \frac{2e}{c}\,\mathbf{B}, \tag{S.40}$$

hence

$$\frac{4\pi}{c}\nabla\times\mathbf{J} \;=\; -\frac{16\pi e^2 n_s}{Mc^2}\,\mathbf{B}. \tag{S.41}$$

Comparing this formula to eq. (S.37), we immediately obtain

$$\nabla^2\mathbf{B} \;=\; \frac{16\pi e^2 n_s}{Mc^2}\,\mathbf{B}, \tag{S.42}$$

or

$$\bigl(\nabla^2 - \ell^{-2}\bigr)\mathbf{B}(\mathbf{x}) \;=\; 0, \tag{2.11}$$

where

$$\ell \;=\; \sqrt{\frac{Mc^2}{16\pi e^2\, n_s}} \tag{S.43}$$

is the *London's penetration depth.* A typical solution of eq. (2.11) would be $\mathbf{B}$ exponentially decreasing as one moves inside the superconductor,

$$B_y(x) \;=\; B_y^{\text{surface}} \times e^{-x/\ell}, \tag{S.44}$$

and that's why ℓ is called the penetration depth: the magnetic field penetrates into the superconductor's bulk to the depth $O(\ell)$ and not much deeper than that.

BTW, eq. (S.43) for ℓ is in the Gauss units; in the MKSA units, it becomes

$$\ell \;=\; \sqrt{\frac{M}{4e^2\mu_0\, n_s}}\,. \tag{S.45}$$

Problem **2**(d):

In the absence of a magnetic field, let's use the gauge where $\mathbf{A}=0$. Consequently, eq. (SC.20) for the Landau–Ginzburg field in (or immediately around) a Josephson junction becomes

$$-\frac{\hbar^2}{2m}\nabla^2\Psi\;+\;\lambda|\Psi|^2\times\Psi\;-\;\mu\Psi\;+\;\Delta\mathcal{V}(\mathbf{x})\Psi\;=\;0 \tag{S.46}$$

where

$$\Delta\mathcal{V}(\mathbf{x})\;=\;\begin{cases}0 & \text{inside either SC wire, 1 or 2,}\\ +V_0 & \text{between the SC wires,}\end{cases} \tag{S.47}$$

for some large positive constant V_0. To solve the non-linear differential equation (S.46), let's split the (relevant part of) 3D space into 3 regions:

(A) the interior and the immediate vicinity of the SC wire #1;

(B) the interior and the immediate vicinity of the SC wire #1;

(C) space between the two SC wires and not too close to either wire.

In the region (A), the influence of the second SC wire is negligible, so we may simplify the boundary conditions for this region as

$$\begin{aligned}&\Psi(\mathbf{x})\;\to\;\sqrt{n_0}\,e^{i\phi_1}\quad\text{for}\quad\mathbf{x}\;\to\;\text{the middle of the wire\#1,}\\ &\Psi(\mathbf{x})\;\to\;0\quad\text{for}\quad\mathbf{x}\;\to\;\text{far away from the wire\#1.}\end{aligned} \tag{S.48}$$

Eq. (S.46) is a complicated non-linear differential equation, but all its coefficients are real, so for $\phi_1=0$ it has a real solution $\Psi_1(\mathbf{x})$ obeying the boundary conditions (S.47). Moreover, eq. (S.46) has a symmetry

$$\Psi(\mathbf{x})\;\to\;e^{i\theta}\times\Psi(\mathbf{x}),\quad\langle\!\langle\,\text{same }\theta\text{ for all }\mathbf{x}\,\rangle\!\rangle, \tag{S.49}$$

so if it has a solution $\Psi_1(\mathbf{x})$ obeying boundary conditions (S.48) for $\phi_1=0$, then

$$\Psi_A(\mathbf{x})\;=\;e^{i\phi_1}\times\Psi_1(\mathbf{x}) \tag{S.50}$$

is another solution obeying the boundary conditions (S.48) for the given phase ϕ_1.

Please note: for our purposes, we do not care about any details of the Landau–Ginzburg field in the region (A), only that it has form (S.50) for a real function $\Psi_1(x)$.

In the region (B), we have a similar situation: the influence of the first wire is negligible, so we may simplify the boundary conditions for this region as

$$\begin{aligned} \Psi(\mathbf{x}) \ &\to\ \sqrt{n_0}\, e^{i\phi_2} \quad \text{for} \quad \mathbf{x}\ \to\ \text{the middle of the wire\#2}, \\ \Psi(\mathbf{x}) \ &\to\ 0 \quad \text{for} \quad \mathbf{x}\ \to\ \text{far away from the wire\#2}. \end{aligned} \tag{S.51}$$

Again, the solution of eq. (S.46) with these boundary conditions has general form

$$\Psi_B(\mathbf{x}) \ =\ e^{i\phi_2} \times \Psi_2(\mathbf{x}) \tag{S.52}$$

for some real function $\Psi_2(\mathbf{x})$. Again, we do not care for the specifics of the $\Psi_2(\mathbf{x})$ function but only that it's real.

Finally, in the middle region (C) we expect to find much lower Cooper pair density than in either SC wire, thus $|\Psi(\mathbf{x})|^2 \ll n_0$. Consequently, the non-linear term $\lambda|\Psi|^2 \times \Psi$ in eq. (S.46) becomes much smaller then the linear terms, so we may neglect it and solve the linear equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi \ -\ \mu\Psi \ +\ \Delta\mathcal{V}(\mathbf{x})\Psi \ =\ 0. \tag{S.53}$$

By linearity, a sum of two solutions is a solution, so if we analytically continue the solutions $\Psi_1(\mathbf{x})$ and $\Psi_2(\mathbf{x})$ in from the respective regions (A) and (B) into the middle region (C), then

$$\Psi(\mathbf{x}) \ =\ e^{i\phi_1} \times \Psi_1(\mathbf{x}) \ +\ e^{i\phi_2} \times \Psi_2(\mathbf{x}) \tag{S.54}$$

is a good solution of eq. (S.53) and hence a good approximate solution of eq. (S.46). Moreover this $\Psi_C(\mathbf{x})$ smoothly continues to the $\Psi_A(\mathbf{x})$ in region (A) or to the $\Psi_B(\mathbf{x})$ in region (B), which is precisely what we want for the middle region. And that's why the Landau–Ginzburg field in the middle region indeed has form

$$\Psi_C(\mathbf{x}) \ =\ e^{i\phi_1} \times \Psi_1(\mathbf{x}) \ +\ e^{i\phi_2} \times \Psi_2(\mathbf{x}) \tag{S.55}$$

for some real functions $\Psi_1(\mathbf{x})$ and $\Psi_2(\mathbf{x})$.

Finally, we may extend the solution (S.55) from the middle region (C) back to the regions (A) and (B), thus

$$\Psi(\mathbf{x}) \;=\; e^{i\phi_1}\times\Psi_1(\mathbf{x}) \;+\; e^{i\phi_2}\times\Psi_2(\mathbf{x}) \quad \forall\mathbf{x}. \tag{SC.21}$$

Indeed, in region (A) we have $\Psi_2 \ll \Psi_1$, which allows us to neglect the second term in eq. (SC.21) and get

$$\Psi_A(\mathbf{x}) \;\approx\; e^{i\phi_1}\Psi_2(\mathbf{x}), \tag{S.56}$$

while in region (B) we have $\Psi_1 \ll \Psi_2$, hence neglecting the first term in eq. (SC.21) we recover

$$\Psi_B(\mathbf{x}) \;\approx\; e^{i\phi_2}\times\Psi_2(\mathbf{x}). \tag{S.57}$$

The bottom line is, in the absence of magnetic field, the LG field through the whole Josephson junction has form (SC.21) for some real functions $\Psi_1(\mathbf{x})$ and $\Psi_2(\mathbf{x})$. Finding the actual form of these 2 functions would be a much harder exercise, but fortunately it is not a part of this homework.

Problem **2**(d)

:

Given eq. (SC.21) for the Landau–Ginzburg field, let's calculate the net current through the Josephson junction:

$$\begin{aligned}
\Psi(\mathbf{x}) &= e^{i\phi_1}\Psi_1(\mathbf{x}) \;+\; e^{i\phi_2}\Psi_2(\mathbf{x}), && \text{(SC.21)}\\
\mathbf{D}\Psi(\mathbf{x}) &= \nabla\Psi(\mathbf{x}) \qquad \langle\!\langle \text{ in the absence of magnetic field } \rangle\!\rangle \\
&= e^{i\phi_1}\nabla\Psi_1(\mathbf{x}) \;+\; e^{i\phi_2}\nabla\Psi_2(\mathbf{x}), && \text{(S.58)}\\
\Psi^*(\mathbf{x}) &= e^{-i\phi_1}\Psi_1(\mathbf{x}) \;+\; e^{-i\phi_2}\Psi_2(\mathbf{x}), && \text{(S.59)}
\end{aligned}$$

hence

$$\Psi^*\mathbf{D}\Psi \;=\; \Psi_1\nabla\Psi_1 \;+\; \Psi_2\nabla\Psi_2 \;+\; e^{i\phi_1-i\phi_2}\Psi_2\nabla\Psi_1 \;+\; e^{i\phi_2-i\phi_1}\Psi_1\nabla\Psi_2 \tag{S.60}$$

and therefore

$$\begin{aligned}\operatorname{Im}(\Psi^*\mathbf{D}\Psi) &= 0 + 0 + \sin(\phi_1-\phi_2)\Psi_2\nabla\Psi_1 + \sin(\phi_2-\phi_1)\Psi_1\nabla\Psi_2 \\ &= \sin(\phi_1-\phi_2)\big(\Psi_2\nabla\Psi_1 - \Psi_1\nabla\Psi_2\big).\end{aligned} \tag{S.61}$$

In terms of the supercurrent density through the junction, this formula means

$$\mathbf{J}_s = \frac{-2e\hbar}{M}\operatorname{Im}(\Psi^*\mathbf{D}\Psi) = \frac{-2e\hbar}{M}\sin(\phi_1-\phi_2)\big(\Psi_2\nabla\Psi_1 - \Psi_1\nabla\Psi_2\big), \tag{S.62}$$

which we may rewrite as

$$\mathbf{J}_s(\mathbf{x}) = \sin(\phi_1-\phi_2)\,\mathbf{J}_0(\mathbf{x}) \tag{S.63}$$

for

$$\mathbf{J}_0(\mathbf{x}) = -\frac{2e\hbar}{M}\big(\Psi_2(\mathbf{x})\nabla\Psi_1(\mathbf{x}) - \Psi_1(\mathbf{x})\nabla\Psi_2(\mathbf{x})\big). \tag{S.64}$$

For our purposes, we do not need to know any of the details of this supercurrent density. All we need to know is that everywhere in the middle region the supercurrent depends on the phases ϕ_1 and ϕ_2 as in eq. (S.63). Consequently, when we calculate the net current between the two SC wires by integrating

$$I = \int \mathbf{J}_s\cdot d^2\mathbf{area} \tag{S.65}$$

over some surface between the two wires, we may pick a surface going through the middle region to get

$$I_{\rm net} = I_0\times\sin(\phi_1-\phi_2) \tag{2.17}$$

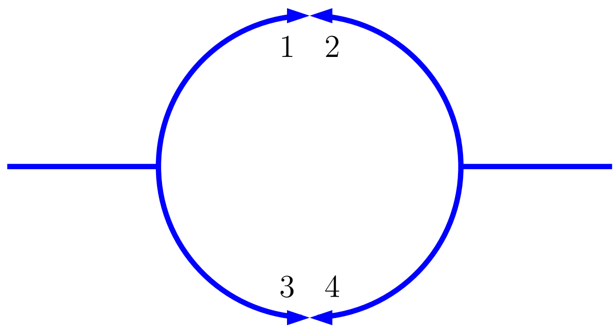
for some

$$I_0 = \int \mathbf{J}_0\cdot d^2\mathbf{area}\,. \tag{S.66}$$

Fortunately, we do not need to calculate this I_0 in this homework.

Problem **2**(f):

Consider the phases $\phi_1, \ldots, \phi_4$ of the Cooper pair condensate at 4 points of the SQUID: The two ends (1) and (2) of the top Josephson junction, and the two ends (3) and (4) of the bottom junction:



(S.67)

As we saw in the previous question, the currents through the each Josephson junction are

$$\begin{aligned} I^{\rm top} &= I_0^{\rm top} \times \sin(\phi_1 - \phi_2), \\ I^{\rm bot} &= I_0^{\rm bot} \times \sin(\phi_3 - \phi_4), \end{aligned} \tag{S.68}$$

so assuming the two junctions are of similar make (and hence similar I_0), the net current through the SQUID is

$$I^{\rm net} = I^{\rm top} + I^{\rm bot} = I_0 \times \Big(\sin(\phi_1 - \phi_2) + \sin(\phi_3 - \phi_4)\Big). \tag{S.69}$$

Now consider the left half of the SQUID, specifically the SC wire going from the left end (3) of the bottom junction to the left end (1) of the top junction. Assuming this wire is thick enough, the magnetic field and the supercurrent are expelled by the Meissner effect from the middle of the wire; instead, the supercurrent flows in a thin layer (of thickness $\ell =$ London's penetration depth) along the wire's surface. Thus, in the middle of the wire

$$\nabla S + \frac{2e}{c}\mathbf{A} = \mathbf{J}_s = 0, \tag{S.70}$$

hence

$$\text{along the wire's axis}: \quad dS = -\frac{2e}{c}\mathbf{A}(\mathbf{x}) \cdot d\mathbf{x} \tag{S.71}$$

and therefore

$$S(1) \; - \; S(3) \; = \; -\frac{2e}{c} \int_3^1 \mathbf{A} \cdot d\mathbf{x} \quad \text{along the left wire.} \tag{S.72}$$

Or in terms of the condensate's phase $\phi = S/\hbar$,

$$\phi_1 \; - \; \phi_3 \; = \; \frac{-2e}{\hbar c} \int_3^1 \mathbf{A} \cdot d\mathbf{x} \quad \text{along the left wire.} \tag{S.73}$$

Likewise, for the right half of the SQUID — specifically the SC wire going from the right end (4) of the bottom junction to the right end (2) of the top junction — we have a similar formula

$$\phi_2 \; - \; \phi_4 \; = \; \frac{-2e}{\hbar c} \int_3^1 \mathbf{A} \cdot d\mathbf{x} \quad \text{along the right wire.} \tag{S.74}$$

Now let's take a difference between eqs. (S.73) and (S.74) for the two halves of the SQUID:

$$(\phi_1 - \phi_3) \; - \; (\phi_2 - \phi_4) \; = \; \frac{-2e}{\hbar c} \left[\int\limits_{\substack{\text{left wire} \\ \text{from 3 to 1}}} \mathbf{A} \cdot d\mathbf{x} \; - \int\limits_{\substack{\text{right wire} \\ \text{from 4 to 2}}} \mathbf{A} \cdot d\mathbf{x} \right]. \tag{S.75}$$

Geometrically, the SQUID has a much larger size then each of the two Josephson junctions, so the distances between the two ends of the same junction — *i.e.*, between (1) and (2), or between (3) and (4), — are much shorter than the distance between the two junctions along either side of the SQUID. So for a non-singular vector potential $\mathbf{A}(\mathbf{x})$ — or at least not having singularities near either Josephson junction, — we may approximate the integrals in eq. (S.75) as integrals from the bottom junction to the top junction along 2 different paths.

Thus,

$$\begin{aligned}
\int_3^1 \mathbf{A}\cdot d\mathbf{x} \;&\approx \int_{\text{bottom JJ}}^{\text{topJJ}} \mathbf{A}\cdot d\mathbf{x} \quad \text{along the left wire,}\\
\int_4^2 \mathbf{A}\cdot d\mathbf{x} \;&\approx \int_{\text{bottom JJ}}^{\text{topJJ}} \mathbf{A}\cdot d\mathbf{x} \quad \text{along the right wire,}
\end{aligned} \tag{S.76}$$

so the difference between these two integrals is the closed loop integral along both halves of the SQUID,

$$\int_3^1 \mathbf{A}\cdot d\mathbf{x} \;-\; \int_4^2 \mathbf{A}\cdot d\mathbf{x} \;=\; \oint_{\text{whole SQUID}} \mathbf{A}\cdot d\mathbf{x}, \tag{S.77}$$

where the closed integration path runs up the left wire from the bottom JJ to the top JJ and then down the right wire from the top JJ down to the bottom JJ. In terms of this closed-loop integral, eq. (S.75) becomes

$$(\phi_1-\phi_3) \;-\; (\phi_2-\phi_4) \;=\; -\frac{2e}{\hbar c}\oint_{\text{SQUID}} \mathbf{A}\cdot d\mathbf{x}. \tag{S.78}$$

Moreover, by the Stokes theorem, the closed-loop integral on the RHS of eq. (S.78) is the magnetic flux through that loop, in our case through the SQUID's loop:

$$\oint_{\text{SQUID}} \mathbf{A}\cdot d\mathbf{x} \;=\; \int_{\text{SQUID}} \mathbf{B}\cdot d^2\text{area} \;=\; F[\text{SQUID}], \tag{S.79}$$

hence

$$(\phi_1-\phi_3) \;-\; (\phi_2-\phi_4) \;=\; -\frac{2e}{\hbar c}\,F[\text{SQUID}] \;=\; -2\pi\frac{F[\text{SQUID}]}{F_0} \tag{S.80}$$

where

$$F_0 \;=\; \frac{2\pi\hbar c}{e} \tag{2.13}$$

is a quantum of magnetic flux un-detectable by the Aharonov–Bohm effect.

Next, let's re-organize the LHS here in terms of differences $\phi_1-\phi_2$ and $\phi_3-\phi_4$ instead of $\phi_1-\phi_3$ and $\phi_2-\phi_4$, thus

$$(\phi_1-\phi_2)\ -\ (\phi_3-\phi_3)\ =\ (\phi_1-\phi_3)\ -\ (\phi_2-\phi_4)\ =\ \frac{-2\pi F}{F_0}\,. \tag{S.81}$$

Also, let θ denote the average between $\phi_1-\phi_2$ and $\phi_3-\phi_4$,

$$\theta\ =\ \tfrac{1}{2}(\phi_1-\phi_2)\ +\ \tfrac{1}{2}(\phi_3-\phi_4); \tag{S.82}$$

then in terms of this θ and the magnetic flux F through the SQUID,

$$(\phi_1-\phi_2)\ =\ \theta\ -\ \frac{\pi F}{F_0}\,,\qquad (\phi_3-\phi_4)\ =\ \theta\ +\ \frac{\pi F}{F_0}\,. \tag{S.83}$$

Finally, let's plug these phase difference across each Josephson junctions into eq. (S.69) for the net current through the SQUID:

$$\begin{aligned}\frac{I^{\text{net}}}{I_0}\ &=\ \sin(\phi_1-\phi_2)\ +\ \sin(\phi_3-\phi_4)\\ &=\ \sin\left(\theta\ -\ \frac{\pi\ F}{f_0}\right)\ +\ \sin\left(\theta\ +\ \frac{\pi\ F}{f_0}\right)\\ &=\ 2\sin\theta\times\cos\frac{\pi F}{f_0}\,,\end{aligned} \tag{S.84}$$

or equivalently

$$I^{\text{net}}\ =\ \left(2I_0\cos\frac{\pi F}{F_0}\right)\times\sin\theta. \tag{S.85}$$

Experimentally, we control the magnetic flux F through the SQUID but we have no direct control over the averaged phase difference θ. Instead, we control the net current through the SQUID while the θ adjusts itself to whatever it takes to carry the desired current. However, for any possible value of θ, its sine $\sin\theta$ ranges between -1 and $+1$ and cannot exceed these limits. Consequently, the net supercurrent through the SQUID varies in the range

$$-2I_0 \left|\cos\frac{\pi F}{F_0}\right| \;<\; I^{\rm net} \;<\; +2I_0 \left|\cos\frac{\pi F}{F_0}\right| \qquad \text{(S.86)}$$

but cannot get any stronger than this in either direction. In other words, *the maximal supercurrent through the SQUID* is

$$I_{\max} \;=\; 2I_0 \times \left|\cos\frac{\pi F}{F_0}\right|. \qquad \text{(SC.25)}$$

Quod erat demonstrandum.