MULTIPOLE EXPANSION

The electric multipole expansion is a useful tool for calculating the Coulomb potential $\Phi(\mathbf{x})$ (and hence the electric field $\mathbf{E}(\mathbf{x})$) due to some *compact* charge system at long distances from that system, $|\mathbf{x}| \gg$ (systemsize). For this geometry, we may expand the Coulomb potential

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \iiint\limits_{\text{system}} d^3\mathbf{y}\, \frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \tag{1}$$

into inverse powers of the distance $|\mathbf{x}|$,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{\mathcal{M}_\ell(\mathbf{n}_x)}{|\mathbf{x}|^{\ell+1}}\,. \tag{2}$$

As we shall see in these notes, the coefficients $\mathcal{M}_\ell$ of this expansion are the *electric multipole moments* of the charges system $\rho(\mathbf{y})$; or rather, each $\mathcal{M}_\ell(\mathbf{n}_x)$ is the component of the ℓ^{th} multipole tensor in the direction of $\mathbf{x}$.

A point of notation: In these notes, $\mathbf{x}$ always denotes the distant point where we measure the potential $\Phi(\mathbf{x})$. Hence, $r_x = |\mathbf{x}|$ is the distance to that point from the coordinate center, $\mathbf{n}_x = \mathbf{x}/r_x$ is the unit vector in the direction of $\mathbf{x}$, and (r_x, θ_x, ϕ_x) are the spherical coordinates of that point. On the other hand, $\mathbf{y}$ always denotes some point where a charge is located, hence $r_y = |\mathbf{y}|$, $\mathbf{n}_y = \mathbf{y}/r_y$, and (r_y, θ_y, ϕ_y) are the spherical coordinates of that charge. I assume the coordinate origin is located inside or nearby the compact charge system, thus for all the relevant observer points $\mathbf{x}$ and charge locations $\mathbf{y}$

$$r_y \simeq \text{system size} \ll r_x\,. \tag{3}$$

Also, α denotes the angle between the directions of $\mathbf{y}$ and $\mathbf{x}$, thus

$$\cos\alpha = \mathbf{n}_y \cdot \mathbf{n}_x. \tag{4}$$

Finally, sometimes I shall use the asterisk $*$ to emphasize the product of two scalars or of a scalar and a vector to make sure it does not get confused with the dot product or cross product of two vectors.

The key to the multipole expansion (2) is the following **theorem 1:** Take any two points with respective radius vectors $\mathbf{x}$ and $\mathbf{y}$ such that $r_x > r_y$; then the inverse distance between the two points

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} \;=\; \frac{1}{\sqrt{r_x^2 + r_y^2 = 2r_x r_y \cos\alpha}} \tag{5}$$

(where α is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$) can be expanded in powers of the ratio r_y/r_x as

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} \;=\; \sum_{\ell=0}^{\infty} \frac{r_y^\ell}{r_x^{\ell+1}} \times P_\ell(\cos\alpha) \tag{6}$$

where $P_\ell(c)$ is the Legendre polynomial of degree ℓ,

$$P_\ell(c) \;=\; \frac{1}{2^\ell\,\ell!}\,\frac{d^\ell}{dc^\ell}\,(c^2-1)^\ell. \tag{7}$$

Spelling out the first few Legendre polynomials explicitly, the expansion (6) becomes

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} \;=\; \frac{1}{r_x}\times 1 \;+\; \frac{r_y}{r_x^2}\times\cos\alpha \;+\; \frac{r_y^2}{r_x^3}\times\frac{3\cos^2\alpha-1}{2} \;+\; \frac{r_y^3}{r_x^4}\times\frac{5\cos^3\alpha-3\cos\alpha}{2} \;+\;\cdots. \tag{8}$$

The proof of the theorem (6) involves complex contour integration — a technique many students have not yet learned, — so I present it at as *optional* reading material in the separate set of notes.

Now let's apply the theorem (6) to the Coulomb potential

$$\Phi(\mathbf{x}) \;=\; \frac{1}{4\pi\epsilon_0}\iiint d^3\mathbf{y}\,\frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \tag{1}$$

of a compact charge distribution. Formally the integral here is over the whole space, but since the charge system in question is compact we presume

$$\rho(\mathbf{y}) \;=\; 0 \quad \text{unless } |\mathbf{y}| \;\leq\; \text{system size}, \tag{9}$$

so we may just as well limit the integration range to the compact system's volume. On the other hand, we are interested in measuring the potential $\Phi(\mathbf{x})$ only at distant points $\mathbf{x}$ where

$r_x \gg$ system size, so throughout the $\mathbf{y}$ integration range we have $r_y < r_x$. This allows us to apply the theorem (6) to the integrand of the Coulomb potential (1),

$$\Phi(\mathbf{x}) \;=\; \frac{1}{4\pi\epsilon_0} \iiint\limits_{\text{system}} d^3\mathbf{y}\,\rho(\mathbf{y}) * \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} \;=\; \sum_{\ell=0}^{\infty} \frac{r_y^\ell}{r_x^{\ell+1}} P_\ell(\cos\alpha) \right) \tag{10}$$

and since both the series and the integral (over the finite system's volume) converge absolutely, we may change their order and integrate before summing over ℓ, thus

$$\Phi(\mathbf{x}) \;=\; \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r_x^{\ell+1}} * \iiint\limits_{\text{system}} d^3\mathbf{y}\,\rho(\mathbf{y}) * r_y^\ell P_\ell(\cos\alpha). \tag{11}$$

Or in other words,

$$\Phi(\mathbf{x}) \;=\; \sum_{\ell=0}^{\infty} \frac{\mathcal{M}_\ell(\mathbf{n}_x)}{4\pi\epsilon_0\, r_x^{\ell+1}}\,, \tag{12}$$

exactly as in eq. (2), where the multipole moments of the charge distribution $\rho(\mathbf{y})$ — or rather their components in the direction $\mathbf{n}_x$ — obtain as integrals

$$\mathcal{M}_\ell(\mathbf{n}_x) \;=\; \iiint r^\ell P_\ell(\mathbf{n}_y \cdot \mathbf{n}_x) * \rho(\mathbf{y})\, d^3\mathbf{y}. \tag{13}$$

Or for a system of N discrete point charges Q_ν located at $\mathbf{y}_\nu = (r_\nu, \mathbf{n}_\nu)$, as sums

$$\mathcal{M}_\ell(\mathbf{n}_x) \;=\; \sum_{\nu=1}^{N} r_\nu^\ell * P_\ell(\mathbf{n}_\nu \cdot \mathbf{n}_x) * Q_\nu\,. \tag{14}$$

The $\mathbf{n}_x$-dependence of the multipole moments can be described in terms of ℓ-index tensors, or alternatively in terms of spherical harmonics $Y_{\ell,m}(\Theta,\Phi)$. To see how this works — especially the tensor description, — let's start with the leading multipole moments for $\ell = 0, 1, 2, 3$.

The Monopole Moment for $\ell = 0$

The multipole moment for $\ell = 0$ is simply the net electric charge of the distribution. Indeed, $P_0(x) = 1$, hence $r^0 \times P_0(\mathbf{n}_y \cdot \mathbf{n}_x) = 1 \times 1 = 1$ and therefore

$$\mathcal{M}_0 \;=\; \iiint \rho(\mathbf{y})\, d^3\mathbf{y} \;=\; Q^{\text{net}} \tag{15}$$

regardless of the direction $\mathbf{n}_x$. Consequently, the $\ell = 0$ term in the potential is the isotropic Coulomb potential

$$\Phi_{\ell=0}(\mathbf{x}) \;=\; \frac{Q^{\text{net}}}{4\pi\epsilon_0\, r_x} \tag{16}$$

of a point charge Q^{net}.

If this net electric charge of the compact charge system does not vanish, then in the large distance limit $r_x \to \infty$ the leading $\ell = 0$ term in the multipole expansion dominates the whole expansion, so the whole potential far away can be approximated by the Coulomb potential (16) of the net charge. Effectively, from a long distance away, the whole distribution looks like a single point charge $Q = Q^{\text{net}}$. Such single point charges are sometimes called *electric monopoles*, so the $\ell = 0$ multipole moment — the net electric charge of the system — is also called the *monopole moment* of the system.

The Dipole Moment for $\ell = 1$

When the net electric charge of a system happens to vanish, the multipole expansion in the large distance limit is dominated by the next leading term for $\ell = 1$. And the $\ell = 1$ multipole moment of a charged system is its net electric dipole moment $\mathbf{p}^{\text{net}}$, or rather the projection of this dipole moment vector onto the direction of $\mathbf{x}$.

Before I show you this relation, let's remember how the net electric dipole moment works. A simple electric dipole comprises two opposite charges $\pm q$ located at vector displacement $\mathbf{a} = \mathbf{p}_+ - \mathbf{p}_-$ from each other. Consequently, the potential $\Phi(\mathbf{x})$ at $r_x \gg |\mathbf{y}_\pm|$ can be approximated as

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{+q}{4\pi\epsilon_0}\left(\frac{1}{\mathbf{x}-\mathbf{y}_+} \approx \frac{1}{r_x} + \frac{\mathbf{n}_x\cdot\mathbf{y}_+}{r_x^2}\right) + \frac{-q}{4\pi\epsilon_0}\left(\frac{1}{\mathbf{x}-\mathbf{y}_-} \approx \frac{1}{r_x} + \frac{\mathbf{n}_x\cdot\mathbf{y}_-}{r_x^2}\right)\\ &\approx \frac{q}{4\pi\epsilon_0}\left(\frac{\mathbf{n}_x\cdot\mathbf{y}_+}{r_x^2} - \frac{\mathbf{n}_x\cdot\mathbf{y}_-}{r_x^2}\right)\\ &= \frac{q\mathbf{n}_x\cdot(\mathbf{y}_+-\mathbf{y}_-)}{4\pi\epsilon_0\, r_x^2} = \frac{\mathbf{n}_x\cdot\mathbf{p}}{4\pi\epsilon_0\, r_x^2}\end{aligned} \tag{17}$$

where $\mathbf{p} = q(\mathbf{y}_+ - \mathbf{y}_-) = q\mathbf{a}$ is the dipole moment of the charge pair. For the future reference, let's also write this dipole moment as

$$\mathbf{p} = Q_+\mathbf{y}_+ + Q_-\mathbf{y}_- \quad \text{where} \quad Q_\pm = \pm q. \tag{18}$$

Now consider several discrete charges Q_ν located at near-by points $\mathbf{y}_\nu$, and let the net charge of this system be zero, $\sum_\nu Q_\nu = 0$. Then proceeding similar to the simple dipole and expanding each inverse distance $1/|\mathbf{x}-\mathbf{y}_\nu$ to the first order in $\mathbf{y}_\nu$, we have

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\sum_\nu Q_\nu\left(\frac{1}{|\mathbf{x}-\mathbf{y}_\nu|} \approx \frac{1}{r_x} + \frac{\mathbf{n}_x\cdot\mathbf{y}_\nu}{r_x^2}\right)\\ &= \frac{1}{4\pi\epsilon_0\, r_x}\sum_\nu Q_\nu + \frac{\mathbf{n}_x}{4\pi\epsilon_0\, r_x^2}\cdot\sum_\nu Q_\nu\mathbf{y}_\nu\\ &\qquad \langle\!\langle \text{ where the first sum is } \textstyle\sum_\nu Q_\nu = Q_{\text{net}} = 0 \,\rangle\!\rangle\\ &= 0 + \frac{\mathbf{n}_x}{4\pi\epsilon_0\, r_x^2}\cdot\mathbf{p}_{\text{net}}\end{aligned} \tag{19}$$

where

$$\mathbf{p}_{\text{net}} = \sum_\nu Q_\nu\mathbf{y}_\nu \tag{20}$$

— which clearly generalizes eq. (18) — is the *net dipole moment* of the charge system.

A useful **theorem** says that *if the net electric charge of the system is zero, then its electric dipole moment does not depend on the choice of the coordinate origin, but of* $Q_{\text{net}} \neq 0$ *then it does.*

Proof: let's displace the coordinate origin of the system by some vector $\mathbf{d}$, so that each charge's radius vector changes to

$$\mathbf{y}_\nu \;\to\; \mathbf{y}'_\nu \;=\; \mathbf{y}_\nu \;-\; \mathbf{d}. \tag{21}$$

Consequently, the net dipole moment (20) of the system changes to

$$\mathbf{p}'_{\rm net} \;=\; \sum_\nu Q_\nu(\mathbf{y}'_\nu = \mathbf{y}_\nu - \mathbf{d}) \;=\; \sum_\nu Q_\nu \mathbf{y}_\nu \;-\; \mathbf{d}\sum_\nu Q_\nu \;=\; \mathbf{p}_{\rm net} \;-\; \mathbf{d}Q_{\rm net}\,. \tag{22}$$

In particular, if $Q_{\rm net} = 0$ then the net dipole moment remains unchanged, but of $Q_{\rm net} \neq 0$ then it does change by $-Q_{\rm net}\mathbf{d}$. *Quod erat demonstrandum.*

Eq. (20) for the net dipole moment of a discrete charge system can be easily generalized to a continuous charge density $\rho(\mathbf{y})$, namely

$$\mathbf{p}_{\rm net} \;=\; \iiint\limits_{\rm system} d^3\mathbf{y}\,\rho(\mathbf{y}) * \mathbf{y}. \tag{23}$$

Again, for a system with a zero net charge this net dipole moment is independent on the choice of the coordinate origin, but for a system with $Q_{\rm net} \neq 0$ it is not.

Finally, after all these preliminaries, let's see the relation between the electric dipole moment (23) of a charged system and its first multipole moment

$$\mathcal{M}_1(\mathbf{n}_x) \;=\; \iiint\limits_{\rm system} d^3\mathbf{y}\,\rho(\mathbf{y}) * r_y P_1(\mathbf{n}_y\cdot\mathbf{n}_x) \tag{24}$$

in the expansion (2). The P_1 here is the first Legendre polynomial, $P_1(c) = c$, hence

$$r_y * P_1(\mathbf{n}_y\cdot\mathbf{n}_x) \;=\; r_y * (\mathbf{n}_y\cdot\mathbf{n}_x) \;=\; \mathbf{y}\cdot\mathbf{n}_x\,, \tag{25}$$

and therefore

$$\mathcal{M}_1(\mathbf{n}_x) \;=\; \iiint d^3\mathbf{y}\,\rho(\mathbf{y}) * \mathbf{y}\cdot\mathbf{n}_x = \;\mathbf{n}_x\cdot\iiint d^3\mathbf{y}\,\rho(\mathbf{y}) * \mathbf{y} = \;\mathbf{n}_x\cdot\mathbf{p}^{\rm net}. \tag{26}$$

Thus, the $\mathcal{M}_1(\mathbf{x})$ is indeed the component of the system's net electric dipole vector $\mathbf{p}_{\rm net}$ in the direction of the observer point $\mathbf{x}$. Consequently, the potential due to the $\ell = 1$ term is the dipole potential

$$\Phi_{\text{dipole}}(\mathbf{x}) \;=\; \frac{\mathbf{p}^{\text{net}} \cdot \mathbf{n}_x}{4\pi\epsilon_0\, r_x^2}\,. \tag{27}$$

In particular, for the net dipole moment pointing in the z direction,

$$\Phi_{\text{dipole}}(r_x, \theta_x, \not\phi_x) \;=\; \frac{p\,\cos\theta_x}{4\pi\epsilon_0\, r_x^2}\,. \tag{28}$$

Multipole Moment Tensors

The net electric charge of a system is a scalar while its net dipole moment is a vector, and both scalars and vectors are special case of tensors with respectively 0 indices or 1 index. And in a moment, we shall see that all the higher multipole moments $\ell \geq 2$ of a charge system are also tensors, specifically, totally symmetric tensors with ℓ indices. Thus, for $\ell = 2$ we have the 2-index *quadrupole moment tensor* $\mathcal{Q}^{ij} = \mathcal{Q}^{ji}$, and the $\mathcal{M}_2(\mathbf{n}_x)$ coefficient in the multipole expansion of the potential is the component of this tensor in the direction of $\mathbf{x}$:

$$\mathcal{M}_2(\mathbf{n}_x) \;=\; n_x^i n_x^j * \mathcal{Q}^{ij} \qquad \langle\!\langle\, \text{implicit sums over indices } i \text{ and } j\, \rangle\!\rangle. \tag{29}$$

Likewise, $\ell = 3$ we have the 3-index *octupole moment tensor* $\mathcal{O}^{ijk}$ which is totally symmetric in its 3 indices,

$$\mathcal{O}^{ijk} \;=\; \mathcal{O}^{jki} \;=\; \mathcal{O}^{kij} \;=\; \mathcal{O}^{jik} \;=\; \mathcal{O}^{ikj} \;=\; \mathcal{O}^{kji}. \tag{30}$$

And again, the $\mathcal{M}_3(\mathbf{n}_x)$ coefficient in the multipole expansion is the component of this 3-index tensor in the $\mathbf{x}$ direction,

$$\mathcal{M}_3(\mathbf{n}_x) \;=\; n_x^i n_x^j n_k^k * \mathcal{O}^{ijk} \qquad \langle\!\langle\, \text{implicit sums over indices } i,\ j, \text{ and } k\, \rangle\!\rangle. \tag{31}$$

In the same spirit, for every higher ℓ the system has a 2^ℓ–pole moment, which is a tensor with ℓ indices, totally symmetric WRT to all index permutations; and the $\mathcal{M}_\ell(\mathbf{n}_x)$ coefficient in the multipole expansion (2) is the component of this tensor in the direction of $\mathbf{x}$:

$$\mathcal{M}_\ell(\mathbf{n}_x) \;=\; n_x^{i_1} n_x^{i_2} \cdots n_x^{i_\ell} * \mathcal{M}_\ell^{i_1,i_2,\ldots,i_\ell} \qquad \langle\!\langle\, \text{implicit sum over all indices } i_1 \text{ through } i_\ell\, \rangle\!\rangle. \tag{32}$$

But before we look at the high-ℓ multipole moments, lets take a closer look at the quadrupole moment tensor for $\ell = 2$ and the octupole moment tensor for $\ell = 3$.

Quadrupole Moment for $\ell = 2$

When a charged system has zero net charge and also zero net dipole moment, then the dominant term in the multipole expansion (2) of the large-distance potential becomes the $\ell = 2$ term

$$\Phi_{\ell=2}(\mathbf{x}) \;=\; \frac{\mathcal{M}_2(\mathbf{n}_x)}{4\pi\epsilon_0\, r_x^3} \tag{33}$$

where

$$\mathcal{M}_2(\mathbf{n}_x) \;=\; \iiint\limits_{\text{system}} d^3\mathbf{y}\,\rho(\mathbf{y}) * r_y^2 P_2(\mathbf{n}_y\cdot\mathbf{n}_x). \tag{34}$$

Let's take a closer look at this moment and its dependence on the direction $\mathbf{n}_x$. The P_2 here is the second Legendre polynomial $P_2(c) = \frac{3}{2}c^2 - \frac{1}{2}$, hence

$$r_y^2 P_2(\mathbf{n}_y\cdot\mathbf{n}_x) \;=\; r_y^2\bigl(\tfrac{3}{2}(\mathbf{n}_y\cdot\mathbf{n}_x)^2 - \tfrac{1}{2}\bigr) \;=\; \tfrac{3}{2}(\mathbf{y}\cdot\mathbf{n}_x)^2 \;-\; \tfrac{1}{2}\mathbf{y}^2 \;=\; \tfrac{3}{2}(\mathbf{y}\cdot\mathbf{n}_x)^2 \;-\; \tfrac{1}{2}\mathbf{y}^2 * \mathbf{n}_x^2, \tag{35}$$

or in index notations (with implicit summation over twice repeated indices)

$$r_y^2 P_2(\mathbf{n}_y\cdot\mathbf{n}_x) \;=\; \tfrac{3}{2}(y^i n_x^i)(y^j n_x^j) \;-\; \tfrac{1}{2}\mathbf{y}^2\delta^{ij} * n_x^i n_x^j \;=\; \Bigl(\tfrac{3}{2}y^i y^j \;-\; \tfrac{1}{2}\mathbf{y}^2\delta^{ij}\Bigr) * n_x^i n_x^j\,. \tag{36}$$

Plugging in this formula into eq. (34), we arrive at

$$\begin{aligned} \mathcal{M}_2(\mathbf{n}_x) \;&=\; \iiint d^3\mathbf{y}\,\rho(\mathbf{y}) * \Bigl(\tfrac{3}{2}y^i y^j \;-\; \tfrac{1}{2}\mathbf{y}^2\delta^{ij}\Bigr) * n_x^i n_x^j \\ &=\; n_x^i n_x^j * \mathcal{Q}^{ij} \end{aligned} \tag{37}$$

— exactly as promised in eq. (29), — for

$$\mathcal{Q}^{ij} \;=\; \mathcal{Q}^{ji} \;=\; \iiint d^3\mathbf{y}\,\rho(\mathbf{y}) * \Bigl(\tfrac{3}{2}y^i y^j \;-\; \tfrac{1}{2}\mathbf{y}^2\delta^{ij}\Bigr) \tag{38}$$

This symmetric 2-index tensor is called the *electric quadrupole moment* for the reasons we shall see in a moment. And in terms of this tensor, the $\ell = 2$ term in the long-distance potential is

$$\Phi_{\ell=2}(\mathbf{x}) \;=\; \frac{\mathcal{Q}^{ij} n_x^i n_x^j}{4\pi\epsilon_0\, r_x^3}\,. \tag{39}$$

Note: there are different conventions for the overall normalization of the quadrupole moment tensor. My convention in eq. (38) is taken from the Griffith's textbook *"Introduction* *to Electrodynamics"*. However, the Jackson's textbook uses a different convention where

$$\mathcal{Q}^{ij}_{\text{Jackson}} \;=\; 2\mathcal{Q}^{ij}_{\text{Griffith}} \;=\; \iiint d^3\mathbf{y}\,\rho(\mathbf{y})*\Big(3y^iy^j\;-\;\mathbf{y}^2\delta^{ij}\Big) \tag{40}$$

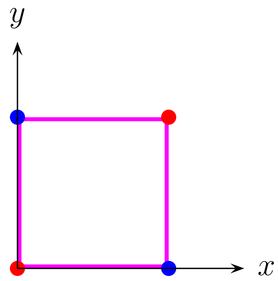
and hence

$$\Phi_{\ell=2}(\mathbf{x}) \;=\; \frac{\mathcal{Q}^{ij}n^i_x n^j_x}{8\pi\epsilon_0\,r^3_x}\,. \tag{41}$$

Please beware of this convention difference when comparing formulae from different sources.

EXAMPLE 1.

A good example of a quadrupole moment is a simple quadrupole — four alternating charges $\pm Q$ at the corners of a square, hence the name *quadrupole*,



$$\begin{aligned} +Q\ &@(0,0,0),\\ -Q\ &@(a,0,0),\\ -Q\ &@(0,a,0),\\ +Q\ &@(a,a,0). \end{aligned} \tag{42}$$

It is easy to see that this 4-charge system has zero net charge as well as zero net dipole moment, so the leading term in its potential at long distances is the quadrupole potential (39). Let's calculate the quadrupole moment tensor for this simple quadrupole.

For a discrete charge system, the integral (38) becomes a discrete sum

$$\mathcal{Q}^{ij} \;=\; \sum_{\nu=1}^{N} Q_\nu * \Big(\tfrac{3}{2}y^i_\nu y^j_\nu \;-\; \tfrac{1}{2}\mathbf{y}^2_\nu\,\delta^{ij}\Big). \tag{43}$$

For the simple quadrupole, the 4 terms in this sum amount to (in matrix notations): In matrix notations,

$$Q_1\big(\tfrac{3}{2}y^i_1y^j_1 \;-\; \tfrac{1}{2}\mathbf{y}^2_1\delta^{ij}\big) \;=\; +Q\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{44}$$

$$Q_2\left(\tfrac{3}{2}y_2^i y_2^j - \tfrac{1}{2}\mathbf{y}_2^2\delta^{ij}\right) = -Q\left(\frac{3}{2}\begin{pmatrix} a^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{\mathrm{a}^2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$
$$= +\frac{Qa^2}{2}\begin{pmatrix} -2 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix}, \tag{45}$$

$$Q_3\left(\tfrac{3}{2}y_3^i y_3^j - \tfrac{1}{2}\mathbf{y}_3^2\delta^{ij}\right) = -Q\left(\frac{3}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{\mathrm{a}^2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$
$$= +\frac{Qa^2}{2}\begin{pmatrix} +1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & +1 \end{pmatrix}, \tag{46}$$

$$Q_4\left(\tfrac{3}{2}y_4^i y_4^j - \tfrac{1}{2}\mathbf{y}_4^2\delta^{ij}\right) = +Q\left(\frac{3}{2}\begin{pmatrix} a^2 & a^2 & 0 \\ a^2 & a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{2\mathrm{a}^2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$
$$= +\frac{Qa^2}{2}\begin{pmatrix} +1 & +3 & 0 \\ +3 & +1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{47}$$

hence altogether

$$\mathcal{Q}^{ij} = \frac{Qa^2}{2}\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{48}$$

For this quadrupole moment tensor

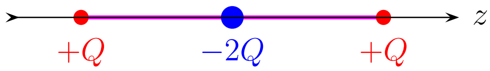
$$\mathcal{M}_2(\mathbf{n}_x) = \mathcal{Q}^{ij} * n_x^i n_x^j = \tfrac{3}{2}Qa^2 * 2n_x^1 n_x^2 = 3Qa^2 * \sin^2\theta_x \cos\phi_x \sin\phi_x\,, \tag{49}$$

hence the quadrupole potential

$$\Phi(r_x,\theta_x,\phi_x) = \frac{3Qa^2}{4\pi\epsilon_0} * \frac{\sin^2\theta_x \cos\phi_x \sin\phi_x}{r_x^3} = \frac{3Qa^2}{4\pi\epsilon_0} * \frac{x_1 x_2}{r_x^5}\,. \tag{50}$$

Example 2.

Another good example of a quadrupole moment is a linear quadrupole: a charge $-2Q$ in the middle, and two charges $+Q$ at its opposite sides (and at exactly the same distance a):



Again, this charge system has zero net charge and zero net dipole moment, so the leading term in the multipole expansion of its potential is the quadrupole term $\ell = 2$, but this time the quadrupole moment tensor

$$\mathcal{Q}^{ij} = \sum_{\nu=1}^{3} Q_\nu * \left(\tfrac{3}{2} y_\nu^i y_\nu^j - \tfrac{1}{2} \mathbf{y}_\nu^2 \delta^{ij}\right) \tag{51}$$

has a different form. Indeed, for the charges at hand

$$\sum_\nu Q_\nu y_\nu^i y_\nu^j = 2Q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a^2 \end{pmatrix} \tag{52}$$

while

$$\sum Q_\nu \mathbf{y}_\nu^2 = 2Qa^2, \tag{53}$$

hence

$$\mathcal{Q}^{ij} = Qa^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +2 \end{pmatrix}. \tag{54}$$

For this quadrupole moment,

$$\begin{aligned} \mathcal{M}_2(\mathbf{n}_x) &= Qa^2\big(-n_{x,1}^2 - n_{x,2}^2 + 2n_{x,3}^2\big) = Qa^2(-\sin^2\theta_x + 2\cos^2\theta_x) \\ &= Qa^2(3\cos^2\theta_x - 1) = 2Qa^2 * P_2(\cos\theta_x) \end{aligned} \tag{55}$$

where $P_2(x) = \frac{3}{2}c^2 - \frac{1}{2}$ is the second Legendre polynomial, and the quadrupole potential is

$$\Phi_{\text{quadrupole}} = \frac{2Qa^2}{4\pi\epsilon_0} * \frac{P_2(\cos\theta_x)}{r_x^3}. \tag{56}$$

Tracelessness.

In both of the above examples, the matrices of the quadrupole moment tensors (48) and (54) have zero traces,

$$\mathrm{tr}(\mathcal{Q}) \stackrel{\mathrm{def}}{=} \mathcal{Q}^{ii} \stackrel{\mathrm{def}}{=} \sum_{i=1,2,3} \mathcal{Q}^{ii} = 0. \tag{57}$$

Actually, this is a general property of the quadrupole moment tensor of any system. Indeed, by definition of the quadrupole moment tensor,

$$\mathcal{Q}^{ij} = \iiint d^3\mathbf{y}\,\rho(\mathbf{y}) * \left(\tfrac{3}{2}y^i y^j - \tfrac{1}{2}\mathbf{y}^2\delta^{ij}\right) \tag{38}$$

its trace is

$$\mathrm{tr}(\mathcal{Q}) = \iiint d^3\mathbf{y}\,\rho(\mathbf{y}) * \left(\tfrac{3}{2}y^i y^i - \tfrac{1}{2}\mathbf{y}^2\delta^{ii}\right) \tag{58}$$

where

$$y^i y^i = \mathbf{y}^2, \qquad \delta^{ii} = \sum_i (\delta^{ii} = 1) = 3, \tag{59}$$

hence

$$\tfrac{3}{2}y^i y^i - \tfrac{1}{2}\mathbf{y}^2\delta^{ii} = \tfrac{3}{2}\mathbf{y}^2 - \tfrac{3}{2}\mathbf{y}^2 = 0 \tag{60}$$

and therefore $\mathrm{tr}(\mathcal{Q}) = 0$.

Consequently, out of $3^3 = 9$ components of the 2-index quadrupole moment tensor, only 5 components are linearly independent: The symmetry $\mathcal{Q}^{ij} = \mathcal{Q}^{ji}$ of the tensor imposes 3 linear relations between the components, and the zero trace condition is another linear constraint, so only $9 - 3 - 1 = 5$ components are linearly independent.

Octupole Moment for $\ell = 3$

Now suppose the charge system in question has zero net charge, zero dipole moment, and even zero quadrupole moment, so the dominant term in the multipole expansion of the long-distance potential is the $\ell = 3$ term

$$\Phi_{\ell=3}(\mathbf{x}) \;=\; \frac{\mathcal{M}_3(\mathbf{n}_x)}{4\pi\epsilon_0\, r_x^4} \tag{61}$$

where

$$\mathcal{M}_3(\mathbf{x}) \;=\; \iiint\limits_{\text{system}} d^3\mathbf{y}\,\rho(\mathbf{y}) * r_y^3 P_3(\mathbf{n}_y\cdot\mathbf{n}_x). \tag{62}$$

The P_3 here is the third Legendre polynomial $P_c(c) = \frac{5}{2}c^3 - \frac{3}{2}c$, hence

$$\begin{aligned} r_y^3 P_3(\mathbf{n}_y\cdot\mathbf{n}_x) \;&=\; \tfrac{5}{2}r_y^3(\mathbf{n}_y\cdot\mathbf{n}_x)^3 \;-\; \tfrac{3}{2}r_y^3(\mathbf{n}_y\cdot\mathbf{n}_x) \;=\; \tfrac{5}{2}(\mathbf{y}\cdot\mathbf{n}_x)^3 \;-\; \tfrac{3}{2}\mathbf{y}^2(\mathbf{y}\cdot\mathbf{n}_x) \\ &\quad\langle\!\langle \text{ in index notations } \rangle\!\rangle \\ &=\; \tfrac{5}{2}(y^i n_x^i)(y_j n_x^j)(y^k n_x^k) \;-\; \tfrac{3}{2}\mathbf{y}^2(y^i n_x^i) * (1 = \delta^{jk} n_x^j n_x^k) \\ &=\; \left(\tfrac{5}{2}y^i y^j y^k \;-\; \tfrac{3}{2}\mathbf{y}^2 y^i \delta^{jk}\right) * n_x^i n_x^j n_x^k \\ &\quad\langle\!\langle \text{ Symmetrizing } (\cdots)^{ijk} \text{ in } i,\ j, \text{ and } k \,\rangle\!\rangle \\ &=\; \left(\tfrac{5}{2}y^i y^j y^k \;-\; \tfrac{1}{2}\mathbf{y}^2\big(y^i\delta^{jk} + y^j\delta^{ik} + y^k\delta^{ij}\big)\right) * n_x^i n_x^j n_x^k\,. \end{aligned} \tag{63}$$

Consequently, plugging this formula into the integrand of eq. (62) and pulling the $\mathbf{y}$-independent $n_x^i n_x^j n_x^k$ factor outside the integral, we arrive at

$$\mathcal{M}_3(\mathbf{n}_x) \;=\; \mathcal{O}^{ijk} * n_x^i n_x^j n_x^k \tag{31}$$

where

$$\mathcal{O}^{ijk} \;=\; \iiint\limits_{\text{system}} d^3\mathbf{y}\,\rho(\mathbf{y}) * \left(\tfrac{5}{2}y^i y^j y^k \;-\; \tfrac{1}{2}\mathbf{y}^2\big(y^i\delta^{jk} + y^j\delta^{ik} + y^k\delta^{ij}\big)\right) \tag{64}$$

are components of the 3-index *octupole moment tensor*. Or for a discrete charge system

$$\mathcal{O}^{ijk} \;=\; \sum_{\nu=1}^{N} Q_\nu \left(\tfrac{5}{2}y^i y^j y^k \;-\; \tfrac{1}{2}\mathbf{y}^2\big(y^i\delta^{jk} + y^j\delta^{ik} + y^k\delta^{ij}\big)\right)_\nu. \tag{65}$$

Anyway, in terms of the octupole moment tensor, the $\ell = 3$ term in the multipole expansion of the potential at large distances amounts to

$$\Phi_{\ell=3}(\mathbf{x}) \;=\; \frac{\mathcal{O}^{ijk}\, n_x^i n_x^j n_x^k}{4\pi\epsilon_0\, r_x^4}\,. \tag{66}$$

By construction (64), the octupole moment tensor is totally symmetric in its 3 indices,

$$\mathcal{O}^{\text{any permutation of } i,j,k} \;=\; \mathcal{O}^{i,j,k}. \tag{67}$$

It also obeys the generalized zero-trace condition

$$\mathcal{O}^{i,j,j} \;\langle\!\langle\, \text{implicit } \textstyle\sum_j \,\rangle\!\rangle \;=\; 0 \quad \forall i = 1,2,3 \tag{68}$$

or equivalently

$$\mathcal{O}^{ijk}\delta^{jk} \;\langle\!\langle\, \text{implicit } \textstyle\sum_{j,k} \,\rangle\!\rangle \;=\; 0. \tag{69}$$

Indeed, let's contract δ^{jk} with the (cubic polynomial in $\mathbf{y}$)^{ijk} in the integrand of eq. (64):

$$\begin{aligned} \delta^{jk} * &\Big(\tfrac{5}{2}y^i y^j y^k \;-\; \tfrac{1}{2}\mathbf{y}^2\big(y^i\delta^{jk} + y^j\delta^{ik} + y^k\delta^{ij}\big)\Big) \\ &=\; \tfrac{5}{2}y^i * \mathbf{y}^2 \;-\; \tfrac{1}{2}\mathbf{y}^2 * \big(y^i * 3 \,+\, y^i \,+\, y^i \;=\; 5y^i\big) \\ &=\; \tfrac{5}{2}\mathbf{y}^2 * y^i \;-\; \tfrac{5}{2}\mathbf{y}^2 * y^i \;=\; 0, \end{aligned} \tag{70}$$

hence

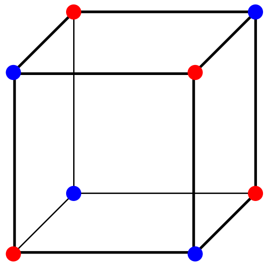
$$\delta^{jk}\mathcal{O}^{ijk} \;=\; \iiint d^3\mathbf{y}\,\rho(y) * \Big(\cdots\Big)^{ijk}\delta^{jk} \;=\; 0 \tag{71}$$

for any continuous charge distribution $\rho(\mathbf{y})$, and likewise for a discrete charge system

$$\delta^{jk}\mathcal{O}^{ijk} \;=\; \sum_\nu Q_\nu * \Big(\cdots\Big)_\nu^{ijk} * \delta^{jk} \;=\; 0. \tag{72}$$

EXAMPLES.

A good example of an octupole moment is made from 8 alternating charges $\pm Q$ — hence the name *octupole* — at the vertices of a cube:



(73)

For this cube,

$$\mathcal{O}^{ijk} \stackrel{\text{def}}{=} \sum_{\nu=1}^{8} Q_\nu \Big(\tfrac{5}{2} y_\nu^i y_\nu^j y_\nu^k \;-\; \tfrac{1}{2} \mathbf{y}_\nu^2 \big(y_\nu^i \delta^{jk} + y_\nu^j \delta^{ik} + y_\nu^k \delta^{ij} \big) \Big) \tag{74}$$

evaluates to

$$\mathcal{O}^{ijk} \;=\; \begin{cases} \frac{5}{2} Q a^3 & \text{for } (i,j,k) = (1,2,3) \text{ in some order,} \\ 0 & \text{otherwise.} \end{cases} \tag{75}$$

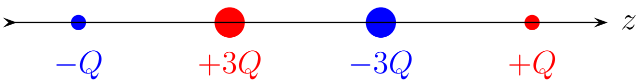
Consequently

$$\mathcal{O}^{ijk} * n_x^i n_x^j n_x^k \;=\; \frac{5Qa^3}{2} * 6 n_x^1 n_x^2 n_x^3 \;=\; 15Qa^3 * \frac{x^1 x^2 x^3}{r_x^3} \;=\; 15Qa^3 * \sin^2\theta_x \cos\theta_x \cos\phi_x \sin\phi_x \tag{76}$$

and therefore octupole potential

$$\Phi_{\text{octupole}} \;=\; \frac{15Qa^3}{4\pi\epsilon_0} * \frac{x^1 x^2 x^3}{r_x^7} \;=\; \frac{15Qa^3}{4\pi\epsilon_0} * \frac{\sin^2\theta_x \cos\theta_x \cos\phi_x \sin\phi_x}{r_x^4}\,. \tag{77}$$

Another example of the octupole moment is the linear octupole — 4 equidistant charges $-Q$, $+3Q$, $-3Q$, $+Q$ arranged in a line, say the z axis:



For this system, the octupole moment tensor evaluates to

$$\begin{aligned} \mathcal{O}^{z,z,z} &= +6Qa^3, \\ \mathcal{O}^{(x,x,z)} &= -3Qa^3, \\ \mathcal{O}^{(y,y,z)} &= -3Qa^3, \\ \text{all other } Q^{i,j,k} &= 0. \end{aligned} \tag{78}$$

Consequently,

$$\begin{aligned} n_x^i n_x^j n_x^k * \mathcal{O}^{i,j,k} &= 6Qa^3 * (n_x^3)^3 \;-\; 3Qa^3 * 3 * (n_x^3)\big((n_x^1)^2 + (n_x^2)^2\big) \\ &= 6Qa^3 * \cos^3\theta_x \;-\; 9Qa^3 * \cos\theta_x \sin^2\theta_x \\ &= 3Qa^3 * \Big(2\cos^3\theta_x \;-\; 3\cos\theta_s \sin^2\theta_c \;=\; 5\cos^3\theta_x \;-\; 3\cos\theta_x\Big) \\ &= 6Qa^3 * P_3(\cos\theta_x) \end{aligned} \tag{79}$$

where $P_3(c) = \frac{5}{2}c^3 - \frac{3}{2}c$ is the third Legendre polynomial, and therefore the octupole potential

$$\Phi_{\text{octupole}}(r_x, \theta_x, \not{\phi}_x) \;=\; \frac{6Qa^3}{4\pi\epsilon_0} * \frac{P_3(\cos\theta_x)}{r_x^4}\,. \tag{80}$$

Higher Multipole Moments

We saw that for $\ell = 0, 1, 2, 3$, the ℓ^{th} term in the multipole expansion is related to an ℓ-index tensor $\mathcal{M}_\ell^{i_i,\ldots,i_\ell}$ — called the 2^ℓ-pole moment — as

$$\mathcal{M}_\ell(\mathbf{n}_x) \;=\; \mathcal{M}_\ell^{i_i,\ldots,i_\ell} * n_x^{i_1}\cdots n_x^{i_\ell} \qquad \langle\!\langle \text{ implicit sum over all } \ell \text{ indices } i_1,\ldots,i_\ell \,\rangle\!\rangle$$

hence

$$\Phi_\ell(\mathbf{x}) \;=\; \frac{\mathcal{M}_\ell^{i_i,\ldots,i_\ell} * n_x^{i_1}\cdots n_x^{i_\ell}}{4\pi\epsilon_0 r_x^{\ell+1}}\,. \tag{81}$$

Also, the 2^ℓ-pole moment tensor itself obtains as an integral

$$\mathcal{M}_\ell^{i_i,\ldots,i_\ell} \;=\; \iiint\limits_{\text{system}} d^4\mathbf{y}\,\rho(\mathbf{y}) * F_\ell^{i_i,\ldots,i_\ell}(\mathbf{y}). \tag{82}$$

or a similar sum over discrete charges

$$\mathcal{M}_ell^{i_i,\ldots,i_\ell} \;= \sum_\nu Q_\nu * F_\ell^{i_i,\ldots,i_\ell}(\mathbf{y}_\nu) \tag{83}$$

where each component of the $F_\ell^{i_i,\ldots,i_\ell}(\mathbf{y})$ is a homogeneous polynomial of degree ℓ in the 3 components (y^1, y^2, y^3) of the vector $\mathbf{y}$. Specifically,

$$\begin{aligned}
F_0(\mathbf{y}) &= 1, \\
F_1^i(\mathbf{y}) &= y^i\,, \\
F_2^{i,j}(\mathbf{y}) &= \tfrac{3}{2}y^i y^j \;-\; \tfrac{1}{2}\mathbf{y}^2\delta^{i,j}\,, \\
F_3^{i,j,k}(\mathbf{y}) &= \tfrac{5}{2}y^i y^j y^k \;-\; \tfrac{1}{2}\mathbf{y}^2\big(y^i\delta^{j,k} + y^j\delta^{i,k} + y^k\delta^{i,j}\big), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{84}$$

At the higher $\ell > 3$ levels of the multipole expansion, we get exactly the same behavior for the higher-rank 2^ℓ-pole moment tensors with ℓ indices: Specifically,

$$\Phi_\ell(\mathbf{x}) \;=\; \frac{\mathcal{M}_\ell^{i_i,\ldots,i_\ell} * n_x^{i_1}\cdots n_x^{i_\ell}}{4\pi\epsilon_0 r_x^{\ell+1}} \tag{81}$$

for

$$\mathcal{M}_\ell^{i_i,\ldots,i_\ell} \;=\; \iiint\limits_{\text{system}} d^4\mathbf{y}\,\rho(\mathbf{y}) * F_\ell^{i_i,\ldots,i_\ell}(\mathbf{y}) \tag{82}$$

or

$$\mathcal{M}_\ell^{i_i,\ldots,i_\ell} \;= \sum_\nu Q_\nu * F_\ell^{i_i,\ldots,i_\ell}(\mathbf{y}_\nu) \tag{83}$$

only the polynomials $F_\ell^{i_i,\ldots,i_\ell}(\mathbf{y})$ become more complicated for higher ℓ. But fortunately, we are not going to need their explicit form in this class.

Instead, let me simply state that for any ℓ, the 2^ℓ-pole moment tensor $\mathcal{M}_\ell^{i_i,\ldots,i_\ell}$ is totally symmetric WRT to all permutations of its ℓ indices $i_1, \ldots, i_l$. Also, for any $\ell \geq 2$, it obeys the generalized zero-trace condition:

$$\forall i_3, \ldots, i_\ell : \quad \delta^{i_1,i_2} * \mathcal{M}_\ell^{i_1,i_2,i_3,\ldots,i_\ell} \;=\; 0. \tag{85}$$

Consequently, out of 3^ℓ components of the 2^ℓ-pole moment tensor, only $2\ell + 1$ components are linearly independent. The rest of the components follow from these by the permutation symmetries of the tensor's indices and by the zero-trace conditions (85). Note that $2\ell + 1$ is also the number of independent spherical harmonics $Y_{\ell,m}(\theta, \phi)$ for a given ℓ, and this is no coincidence. Instead, this allows us to re-express the angular dependence of all the 2^ℓ-pole terms in the potential in terms of the spherical harmonics, as we shall see in the next section.

Spherical Harmonic Expansion

Instead of describing the angular dependence of the multipole moments' components in the direction $\mathbf{n}_x$ in terms of symmetric multipole tensors, we may expand it in terms of spherical harmonics. The key to this expansion is the following **Theorem 2:** for any integer $\ell = 0, 1, 2, 3, \ldots$ and any two unit vectors $\mathbf{n}_x$ and $\mathbf{n}_y$, the Legendre polynomial of their dot product (*i.e.*, of the cosine of the angle α between these vectors) expands into products of spherical harmonics according to

$$P_\ell(\mathbf{n}_x \cdot \mathbf{n}_y) \;=\; \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\theta_x, \phi_x) Y^*_{\ell,m}(\theta_y, \phi_y). \tag{86}$$

Proving this lemma is best done in the quantum-mechanical language of Dirac brackets and projection operators. Since some students may be unfamiliar with this language, the proof is presented in the separate set of notes as *optional reading*.

Meanwhile, let's apply this Theorem to the vectors $\mathbf{n}_x$ and $\mathbf{n}_y$ in the context of eq. (13) for the multipole moment's component in the direction $\mathbf{n}_x$.

$$\begin{aligned} \mathcal{M}_\ell(\mathbf{n}_x) \;&=\; \iiint d^3\mathbf{y}\, \rho(\mathbf{y}) * r_y^\ell P_\ell(\mathbf{n}_y \cdot \mathbf{n}_x) \\ &=\; \iiint d^3\mathbf{y}\, \rho(\mathbf{y}) * r_y^\ell * \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{+\ell} Y^*_{\ell,m}(\mathbf{n}_y) Y_{\ell,m}(\mathbf{n}_x) \\ &=\; \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\mathbf{n}_x) * \iiint d^3\mathbf{y}\, \rho(\mathbf{y}) * r_y^\ell Y^*_{\ell,m}(\mathbf{n}_y). \end{aligned} \tag{87}$$

Hence, let's define the *spherical multipole momentss* according to

$$\mathcal{M}_{\ell,m} \;\stackrel{\rm def}{=}\; \sqrt{\frac{4\pi}{2\ell+1}} \iiint d^3\mathbf{y}\, \rho(\mathbf{y}) * r_y^\ell Y^*_{\ell,m}(\theta_y, \phi_y), \tag{88}$$

or for discrete charge systems

$$\mathcal{M}_{\ell,m} \;\stackrel{\rm def}{=}\; \sqrt{\frac{4\pi}{2\ell+1}} \sum_{\nu=1}^{N} Q_\nu * r_\nu^\ell Y^*_{\ell,m}(\theta_\nu, \phi_\nu). \tag{89}$$

Then, the 2^ℓ–pole potential has form

$$\Phi_{2^\ell-\text{pole}}(r_x,\theta_x,\phi_x) \;=\; \frac{1}{4\pi\epsilon_0}\sum_{m=-\ell}^{+\ell}\mathcal{M}_{\ell,m}\times\sqrt{\frac{4\pi}{2\ell+1}}\,\frac{Y_{\ell,m}(\theta_x,\phi_x)}{r_x^{\ell+1}} \tag{90}$$

and the entire potential expands into

$$\Phi_{\text{net}}(r_x,\theta_x,\phi_x) \;=\; \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\mathcal{M}_{\ell,m}\times\sqrt{\frac{4\pi}{2\ell+1}}\,\frac{Y_{\ell,m}(\theta_x,\phi_x)}{r_x^{\ell+1}}\,. \tag{91}$$

The advantage of the spherical multipole moments is that for each 2^ℓ pole there are only $2\ell+1$ moments — which are all independent — rather than a much larger number of tensor components interrelated by the symmetry and zero-trace conditions. This is particularly convenient for high $\ell\geq 3$. On the other hand, for low $\ell=0,1,2$ it is usually more convenient to work with the explicitly scalar net charge, vector dipole moment, and 2-index tensor quadrupole moment than convert them to the spherical moments.

Axial Symmetry

Another great advantage of the spherical multipole moments is that for any axially symmetric charged systemthere is only one non-zero spherical moment for each ℓ, namely

$$\mathcal{M}_\ell^{\text{axial}} \;\stackrel{\text{def}}{=}\; \mathcal{M}_{\ell,0}\,, \tag{92}$$

while all the other $\mathcal{M}_{\ell,m}$ with $m\neq 0$ happen to vanish. By axial symmetry of charges we meen that in suitable sphericaal coordinates

$$\rho(r_y,\theta_y,\phi_y) \;=\; \rho(r_y,\theta_y\text{ only}), \tag{93}$$

although more generally, we may also have axially symmetric surface charges, circular line charges, as well as line or point charges on the symmetry axis itself.

Indeed, for the axially symmetric charges, the integral (88) becomes in spherical coordinates

$$\begin{aligned} \mathcal{M}_{\ell,m} &= \sqrt{\frac{4\pi}{2\ell+1}} \int_0^\infty dr_y\, r_y^2 * \int_0^\pi d\theta_y\, \sin\theta_y * \int_0^{2\pi} d\phi_y\, \rho(r_y,\theta_y) * r_y^\ell Y^*_{\ell,m}(\theta_y,\phi_y) \\ &= \sqrt{\frac{4\pi}{2\ell+1}} \int_0^\infty dr_y\, r_y^{\ell+2} \int_0^\pi d\theta_y\, \sin\theta_y\, \rho(r_y,\theta_y \text{ only}) * \int_0^{2\pi} d\phi_y\, Y^*_{\ell,m}(\theta_y,\phi_y) \end{aligned} \tag{94}$$

where

$$Y^*_{\ell,m}(\theta_y,\phi_y) = \text{function}(\theta_y) * \exp(-im\phi_y). \tag{95}$$

Consequently, for $m \neq 0$, the ϕ_y integral in eq. (94) amounts to

$$\text{function}(\theta_y) * \int_0^{2\pi} d\phi_y\, \exp(-im\phi_y) = 0. \tag{96}$$

For the remaining $m = 0$ moments, the spherical harmonics $Y_{\ell,0}(\theta_y,\phi_y)$ are proportional to the Legendre polynomials,

$$\sqrt{\frac{4\pi}{2\ell+1}} \times Y_{\ell,0}(\theta_y,\phi_y) = P_\ell(\cos\theta), \tag{97}$$

hence

$$\begin{aligned} \mathcal{M}_\ell^{\text{axial}} \stackrel{\text{def}}{=} \mathcal{M}_{\ell,0} &= \iiint d^3\mathbf{y}\, \rho(\mathbf{y}) * r_y^\ell P_\ell(\cos\theta_y) \\ &= 2\pi \int_0^\infty dr_y\, r_y^{\ell+2} \int_0^\pi d\theta_y\, \sin\theta_y\, P_\ell(\cos\theta_y) * \rho(r_y,\theta_y). \end{aligned} \tag{98}$$

In terms of these 'axial' multipole components, the multipole expansion (91) of the potential reduces to

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^\infty \mathcal{M}_\ell^{\text{axial}} * \frac{P_\ell(\cos\theta_x)}{r_x^{\ell+1}}. \tag{99}$$

Note that this potential is automatically axially symmetric: it depends on the r_x and θ_x but not on the ϕ_x.

In terms of the multipole moment tensors $\mathcal{M}_\ell^{i_1,\ldots,i_\ell}$, the axial multipole moments (98) are the all-z components of the tensor,

$$\mathcal{M}_\ell^{\rm axial} = \mathcal{M}_\ell^{3,\ldots,3}: \quad \mathcal{M}_1^{\rm axial} = p^3_{\rm net}\,, \quad \mathcal{M}_2^{\rm axial} = \mathcal{Q}^{3,3}, \quad \mathcal{M}_3^{\rm axial} = \mathcal{O}^{3,3,3}, \quad etc. \tag{100}$$

Indeed, by construction of the tensors, their all-z components are

$$\begin{aligned}\mathcal{M}_\ell^{3,\ldots,3} &= \mathcal{M}_\ell(\mathbf{n}_x=(0,0,1)) = \iiint d^3\mathbf{y}\,\rho(\mathbf{y})*r_y^\ell P_\ell\big(\mathbf{n}_y\cdot(0,0,1)\big)\\ &\quad\langle\!\langle \text{ where } \mathbf{n}_y\cdot(0,0,1)=\cos\theta_y \rangle\!\rangle\\ &= \iiint d^3\mathbf{y}\,\rho(\mathbf{y})*r_y^\ell P_\ell(\cos\theta_y) = \mathcal{M}_\ell^{\rm axial}.\end{aligned} \tag{101}$$

Consequently, in terms of the multipole moment tensors, it's enough to calculate the all-z component of each 2^ℓ-pole moment to get the corresponding potential term,

$$\Phi_\ell(r_x,\theta_x) = \frac{\mathcal{M}_\ell^{3,\ldots,3}}{4\pi\epsilon_0}*\frac{P_\ell(\cos\theta_x)}{r_x^{\ell+1}}\,. \tag{102}$$

For example, for a simple dipole, a linear quadrupole, and a linear octupole — which all have axial symmetries —

$$\mathcal{M}_1^{\rm axial} = p^3 = Qa, \tag{103}$$
$$\Phi_{\rm dipole}(r_x,\theta_x) = \frac{Qa}{4\pi\epsilon_0}\,\frac{P_1(\cos\theta_x)}{r_x^2}\,, \tag{104}$$
$$\mathcal{M}_2^{\rm axial} = \mathcal{Q}^{3,3} = 2Qa^2, \tag{105}$$
$$\Phi_{\rm lin.quadrupole}(r_x,\theta_x) = \frac{2Qa^2}{4\pi\epsilon_0}\,\frac{P_2(\cos\theta_x)}{r_x^3}\,, \tag{106}$$
$$\mathcal{M}_3^{\rm axial} = \mathcal{O}^{3,3,3} = 6Qa^3, \tag{107}$$
$$\Phi_{\rm lin.octupole}(r_x,\theta_x) = \frac{6Qa^3}{4\pi\epsilon_0}\,\frac{P_3(\cos\theta_x)}{r_x^4}\,, \tag{108}$$

. . . .

Indeed, for any linear set of charges on the symmetry axis,

$$\cos\theta_y = \pm 1 = \mathrm{sign}(y^3) \implies P_\ell(\cos\theta_y) = \begin{cases} +1 & \text{for even } \ell,\\ \mathrm{sign}(y^3) & \text{for odd } \ell,\end{cases} \tag{109}$$

hence

$$r_y^\ell * P_\ell(\cos\theta_y) \;=\; (y^3)^\ell, \tag{110}$$

and therefore

$$\mathcal{M}_\ell^{\rm axial} \;=\; \sum_{\nu=1}^{N} Q_\nu (y_\nu^3)^\ell. \tag{111}$$

Applying this formula to the linear dipole, quadrupole, and octupole, we immediately reproduce eqs. (103), (105), and (107) that we have earlier derived through rather harder calculations.