Electric–Magnetic Duality and Magnetic Monopoles

Electric–Magnetic Duality

In vacuum, — and in the absence of any electric charges or currents, — the electric and the magnetic field behave in similar ways. Moreover, for $\rho = 0$ and $\mathbf{J} = 0$ there is an exact electric-magnetic symmetry $\mathbf{E} \leftrightarrow \mathbf{B}$, — or rather

$$\mathbf{B}' = +\mathbf{E}, \quad \mathbf{E}' = -\mathbf{B} \tag{1}$$

in Gauss units, — or more generally

$$\begin{aligned} \mathbf{B}' &= \cos\theta\,\mathbf{B} + \sin\theta\,\mathbf{E}, \\ \mathbf{E}' &= \cos\theta\,\mathbf{E} - \sin\theta\,\mathbf{B}. \end{aligned} \tag{2}$$

Indeed, any such *duality transform* preserves the Maxwell equations

$$\begin{aligned} \nabla\cdot\mathbf{E} &= 0 \\ \nabla\cdot\mathbf{B} &= 0 \\ \nabla\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\times\mathbf{B} &= +\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \end{aligned} \tag{3}$$

$$(\text{for } \rho = 0 \text{ and } \mathbf{J} = 0),$$

as well as the EM energy density

$$U = \frac{1}{8\pi}\big(\mathbf{E}^2 + \mathbf{B}^2\big), \tag{4}$$

the Poynting vector

$$\mathbf{S} = \frac{1}{4\pi}\,\mathbf{E}\times\mathbf{B} \tag{5}$$

governing the momentum density and the energy flux, and the EM stress-tensor

$$T^{ij} = \frac{1}{4\pi}\big(E^iE^j + B^iB^j\big) - \delta^{ij}U. \tag{6}$$

As written, the electric-magnetic symmetry (2) works in empty space but not in presence of any electric charges or currents. However, we may repair this duality symmetry in a generalized electromagnetic theory which has both electric and magnetic charges and currents.

Both kinds of charges must be locally conserved, thus continuity equations

$$\begin{aligned} \nabla \cdot \mathbf{J}_{\mathrm{el}} \;+\; \frac{\partial}{\partial t}\rho_{\mathrm{el}} \;&=\; 0, \\ \nabla \cdot \mathbf{J}_{\mathrm{mag}} \;+\; \frac{\partial}{\partial t}\rho_{\mathrm{mag}} \;&=\; 0, \end{aligned} \tag{7}$$

which make for consistent generalized Maxwell equations

$$\begin{aligned} \nabla \cdot \mathbf{E} \;&=\; 4\pi\rho_{\mathrm{el}}\,, \\ \nabla \cdot \mathbf{B} \;&=\; 4\pi\rho_{\mathrm{mag}}\,, \\ \nabla \times \mathbf{E} \;&=\; -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \;-\; \frac{4\pi}{c}\mathbf{J}_{\mathrm{mag}}\,, \\ \nabla \times \mathbf{B} \;&=\; +\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \;+\; \frac{4\pi}{c}\mathbf{J}_{\mathrm{el}}\,. \end{aligned} \tag{8}$$

These equations are invariant under electric-magnetic dualities (2) provided they act on the charges and currents similarly to how they act on the fields: In matrix notations,

$$\begin{aligned} \begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} \;&=\; \begin{pmatrix} +\cos\theta & -\sin\theta \\ +\sin\theta & +\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \\ \begin{pmatrix} \rho'_{\mathrm{el}} \\ \rho'_{\mathrm{mag}} \end{pmatrix} \;&=\; \begin{pmatrix} +\cos\theta & -\sin\theta \\ +\sin\theta & +\cos\theta \end{pmatrix} \begin{pmatrix} \rho_{\mathrm{el}} \\ \rho_{\mathrm{mag}} \end{pmatrix}, \\ \begin{pmatrix} \mathbf{J}'_{\mathrm{el}} \\ \mathbf{J}'_{\mathrm{mag}} \end{pmatrix} \;&=\; \begin{pmatrix} +\cos\theta & -\sin\theta \\ +\sin\theta & +\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{J}_{\mathrm{el}} \\ \mathbf{J}_{\mathrm{mag}} \end{pmatrix}, \end{aligned} \tag{9}$$

all for the same angle θ.

Microscopically, the electric charges and currents stem from the existence and motion of the electrically charged elementary particles such as electrons or protons. Likewise, the magnetic charges and currents should stem from the existence and motion of some not-yet-discovered magnetically charged elementary particles better known as *magnetic monopoles.*

A single monopole at rest generates a Coulomb-like magnetic field

$$\mathbf{B}(\mathbf{x}) \;=\; \frac{M}{r^2}\,\mathbf{n} \tag{10}$$

(in Gauss units), which looks like the field of a single North or South pole of a long thin bar magnet, hence the name. The magnetic monopoles play important roles in theoretical high-energy physics, but the specifics are way too complicated for this class. Let me simply say that most unified theories of the fundamental forces predict that the monopoles do exist in Nature, but they are much too heavy to be made by the present-day accelerators, while the primordial monopoles made at the Big Bang was subsequently diluted by the cosmic inflation to such an extent that today the nearest monopole is a few good light years away from Earth. And that's why the monopoles have not been detected yet, and are unlikely to be detected any time soon.

Magnetic Monopoles in Quantum Mechanics

Quantizing magnetic monopoles — let alone a quantum field theory including the monopoles — are subjects way beyond the scope of this class, even the extra lectures. Instead, let me discuss a much simpler problem: take a classical static monopole generating the magnetic field (10) and consider the quantum motion of an electrically charged particle in this field. This is a non-trivial problem because the particle's Hamiltonian operator

$$\hat{H} \;=\; \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{Q}{c}\mathbf{A}(\hat{\mathbf{x}})\right)^2 \;+\; Q\Phi(\hat{\mathbf{x}}) \tag{11}$$

involves not the magnetic field itself but the vector potential $\mathbf{A}(\mathbf{x})$, — *cf.* my notes on dynamics of a charged particle, — and any vector potential whose curl $\nabla\times\mathbf{A}$ is the monopole field (10) must be singular. Worse, such $\mathbf{A}(\mathbf{x})$ must be singular not only at the monopole itself but along a whole line stretching from the monopole to infinity, or perhaps along several lines. Indeed, there is a **Theorem:** if $\mathbf{B} = \nabla\times\mathbf{A}$ is the monopole field (10), then for any closed surface $\mathcal{S}$ surrounding the monopole, $\mathbf{A}(\mathbf{x})$ must be singular at at least one point on $\mathcal{S}$. In spherical coordinates, this means $\mathbf{A}(r,\theta,\phi)$ is singular for some $\theta(r)$ and $\phi(r)$ for each $r>0$, so from the 3D point of view such singularities form lines.

Proof by contradiction: I am going to show that if $\mathbf{A}(\mathbf{x})$ is regular everywhere on some closed orientable surface $\mathcal{S}$ then that surface does not surround the monopole. Indeed, let's divide the closed surface $\mathcal{S}$ into two parts $\mathcal{S}_1$ and $\mathcal{S}_2$ along some closed loop $\mathcal{L}$. If $\mathbf{A}(\mathbf{x})$ has no singularities on $\mathcal{S}_1$ and on $\mathcal{S}_2$, then the fluxes of $\mathbf{B} = \nabla \times \mathbf{A}$ through these surfaces obtain by the Stokes theorem as contour integrals of $\mathbf{A}$ along the boundary loop $\mathcal{L}$:

$$\begin{aligned} \iint\limits_{\mathcal{S}_1} \mathbf{B} \cdot d^2\mathbf{area} \;&=\; \pm \oint\limits_{\mathcal{L}} \mathbf{A} \cdot d\mathbf{x}, \\ \iint\limits_{\mathcal{S}_2} \mathbf{B} \cdot d^2\mathbf{area} \;&=\; \pm' \oint\limits_{\mathcal{L}} \mathbf{A} \cdot d\mathbf{x}, \end{aligned} \tag{12}$$

where the signs $\pm$ and $\pm'$ depend on the sense of the boundary loop $\mathcal{L}$ relative to the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. Furthermore, for the loop $\mathcal{L}$ that divides a closed orientable surface $\mathcal{S}$ into two 'halves' $\mathcal{S}_1$ and $\mathcal{S}_2$, the loop must run clockwise around one of the $\mathcal{S}_{1,2}$ and counterclockwise around the other. This leads to the opposite signs in eqs. (12) and hence

$$\iint\limits_{\mathcal{S}_1} \mathbf{B} \cdot d^2\mathbf{area} \;=\; - \iint\limits_{\mathcal{S}_2} \mathbf{B} \cdot d^2\mathbf{area}\,. \tag{13}$$

Consequently, the net magnetic flux through the whole closed surface $\mathcal{S}$ must vanish,

$$\iint\limits_{\text{whole}\,\mathcal{S}} \mathbf{B} \cdot d^2\mathbf{area} \;=\; 0. \tag{14}$$

On the other hand, by the Gauss Theorem, the net magnetic flux through a closed surface is $4\pi \times$ the net magnetic charge inside the surface, thus

$$Q_{\text{mag}}[\text{inside } \mathcal{S}] \;=\; \frac{1}{4\pi} \iint\limits_{\mathcal{S}} \mathbf{B} \cdot d^2\mathbf{area} \;=\; 0. \tag{15}$$

So here is the bottom line: if $\mathbf{A}(\mathbf{x})$ has no singularities anywhere on a closed surface $\mathcal{S}$, then the net magnetic charge surrounded by such surface is zero.

On the other hand, for any surface surrounding the magnetic monopole, the net magnetic charge inside such surface is the monopole's charge $M \neq 0$. Therefore, if

$$\nabla \times \mathbf{A}(\mathbf{x}) \;=\; \mathbf{B}(\mathbf{x}) \;=\; \frac{M}{r^2}\mathbf{n} \tag{16}$$

then $\mathbf{A}(\mathbf{x})$ must have at least one singular point on any closed surface $\mathcal{S}$ surrounding the monopole. *Quod erat demonstrandum.*

Dirac Monopole

In 1937, Paul A. M. Dirac found a way to quantize the charged particle's motion despite the singular $\mathbf{A}$ problem. Instead of using a single vector potential $\mathbf{A}(\mathbf{x})$ throughout the whole space surrounding the monopole, Dirac divided the space into two overlapping regions and used a different $\mathbf{A}$ in each region. However, the two potentials are related by a gauge transform and thus are physically equivalent to each other.⋆

Specifically, in the spherical coordinates (r, θ, ϕ), the Northern region (N) span latitudes $0 \leq \theta < \pi - \epsilon$ — everything except a small neighborhood of the South pole, — while the Southern region (S) spans $\epsilon < \theta \leq \pi$ — everything except a neighborhood of the North pole. The two regions overlap in a broad band around the equator. The vector potentials for the two regions are respectively:

$$\begin{aligned} \mathbf{A}_N(r,\theta,\phi) &= M\,\frac{+1-\cos\theta}{r\sin\theta}\,\mathbf{n}_\phi &= M\,(+1-\cos\theta)\,(\nabla\phi), \\ \mathbf{A}_S(r,\theta,\phi) &= M\,\frac{-1-\cos\theta}{r\sin\theta}\,\mathbf{n}_\phi &= M\,(-1-\cos\theta)\,(\nabla\phi), \end{aligned} \tag{17}$$

The two potentials are gauge-equivalent:

$$\mathbf{A}_N \;-\; \mathbf{A}_S \;=\; M\,\frac{2}{r\sin\theta}\,\mathbf{n}_\phi \;=\; 2M\,(\nabla\phi) \;=\; \nabla\,(2M\,\phi) \tag{18}$$

⋆ From the mathematical point of view, the Dirac monopole is a *gauge bundle*, a construction that generalizes multiple coordinate patches in Riemannian geometry. But Dirac himself did not use the bundle language, and you do not need it to understand my notes.

so they lead to the same magnetic field, namely (10). Indeed,

$$\begin{aligned}
\nabla\times\mathbf{A}_{\mathrm{N\,or\,S}} &= \nabla\times\left(M(\pm 1-\cos\theta)\nabla\phi\right)\\
&= M\left(\nabla(\pm 1-\cos\theta)\right)\times\nabla\phi\\
&= M\,\frac{\sin\theta\,\mathbf{n}_\theta}{r}\times\frac{\mathbf{n}_\phi}{r\sin\theta}\\
&= M\,\frac{\mathbf{n}_r}{r^2}\,.
\end{aligned}\tag{19}$$

The vector potentials (17) may be analytically continued to the entire 3D space (except the monopole point $r=0$) itself, but such continuations are singular. The $\mathbf{A}_N(r,\theta,\phi)$ has a so-called "Dirac string" of singularities along the negative z semi-axis ($\theta=\pi$), while the $\mathbf{A}_S(r,\theta,\phi)$ has a similar Dirac string of singularities along the positive z semi-axis ($\theta=0$). To make a non-singular picture of the monopole field, Dirac used both vector potentials $\mathbf{A}_N$ and $\mathbf{A}_S$ but restricted each potential to the region of space where it is not singular. The two regions overlap, and in the overlap we may use either $\mathbf{A}_N$ or $\mathbf{A}_S$, whichever we like.

In quantum mechanics of a charged particle, a gauge transform of the vector potential should be accompanied by a phase transform of the wave function according to

$$\mathbf{A}'(\mathbf{x}) \;=\; \mathbf{A}(\mathbf{x})\;+\;\nabla\Lambda(\mathbf{x})\quad\text{while}\quad \Psi'(\mathbf{x})\;=\;\Psi(\mathbf{x})\times\exp\left(\frac{iQ}{\hbar c}\,\Lambda(\mathbf{x})\right)\quad\text{for the same }\Lambda(\mathbf{x}).\tag{20}$$

I have explained this rule in my previous extra lecture (*cf.* my notes on classical and quantum motion of a charged particle), but this time I am using Gauss units rather than MKSA, hence the $Q/\hbar c$ factor in the exponent instead of $Q/\hbar$. Thanks to this rule, the two different gauge-equivalent vector potentials $\mathbf{A}_N(\mathbf{x})$ and $\mathbf{A}_S(\mathbf{x})$ should come with two different wave functions $\Psi_N(\mathbf{x})$ and $\Psi_S(\mathbf{x})$ in the corresponding regions of space, and in the overlap between the two regions, the $\Psi_N(\mathbf{x})$ and the $\Psi_S(\mathbf{x})$ should be related by the appropriate phase transform (20). Specifically,

$$\Lambda(r,\theta,\phi) \;=\; 2M\phi\,,\tag{21}$$

$$\mathbf{A}_N(r,\theta,\phi) \;=\; \mathbf{A}_S(r,\theta,\phi)\;+\;\nabla\Lambda(r,\theta,\phi),\tag{22}$$

$$\Psi_N(r,\theta,\phi) \;=\; \Psi_S(r,\theta,\phi)\times\exp\left(i\,\frac{Q}{\hbar c}\,\Lambda(r,\theta,\phi)\right).\tag{23}$$

Note: the gauge-transform parameter Λ in eq. (21) is multivalued since the longitude coordinate ϕ changes by 2π as we go around the equator. However, such multivalued Λ's are OK as long as both the EM potentials and the wave functions they relate are single-valued, which means that

$$\text{both} \quad \nabla\Lambda \quad \text{and} \quad \exp\left(i\,\frac{Q}{\hbar c}\,\Lambda\right) \quad \text{must be single-valued functions of } \mathbf{x}. \tag{24}$$

For the case at hand, $\nabla\phi$ and hence $\nabla\Lambda$ are single valued, but we need to check the phase

$$\exp\left(i\,\frac{Q}{\hbar c}\,\Lambda\right) \;=\; \exp\left(i\,\frac{2QM}{\hbar c}\,\phi\right). \tag{25}$$

In general, the exponential $\exp(i\nu\phi)$ for a constant ν is a single-valued function of the angle ϕ if and only if ν is an integer, so the gauge transform (21)–(23) is allowed in quantum mechanics if and only if

$$\frac{2QM}{\hbar c} \quad \text{is an integer.} \tag{26}$$

Physically, this means that a Dirac monopole of magnetic charge M may coexist with a quantum particle of electric charge Q only when the two charges obey the *Dirac quantization condition*

$$M \times Q \;=\; \tfrac{1}{2}\hbar c \times \text{an integer.} \tag{27}$$

Or in MKSA units,

$$M \times Q \;=\; \frac{2\pi\hbar}{\mu_0} \times \text{an integer.} \tag{28}$$

Suggested Reading: J. J. Sakurai, *Modern Quantum Mechanics*, §2.6.

Implications for QFT and Electric–Magnetic Duality

In quantum field theory, for every existing *species* of a charged particle there are countless virtual particles of that species everywhere. Therefore, if as much as a single magnetic monopole exist anywhere in the Universe, then the electric charges of all particle species must be quantized,

$$Q \;=\; \frac{\hbar c}{2M}\times\text{an integer.} \tag{29}$$

Historically, Dirac discovered the magnetic monopole while trying to explain the rather small *value* of the electric charge quantum e — in Gauss units,

$$e^2 \;\approx\; \frac{\hbar c}{137}\,. \tag{30}$$

The monopole gives us an excellent reason for the charge quantization in the first place, but alas it does not explain the value (30) of the quantum, and Dirac was quite disappointed.

BTW, in Gauss units, the electric and the magnetic charges have the same dimensionality. But in light of eqs. (27) and (30), they are quantized in rather different units, e for the electric charges and

$$\frac{\hbar c}{2e} \;\approx\; \frac{137}{2}\,e \tag{31}$$

for the magnetic charges. Of course, as far as the Quantum ElectroDynamics is concerned, the monopoles do not have to exist at all. But if they do exist, their charges must be quantized in units of (31). Also, the very existence of a single monopole would explain the electric charge quantization.

Today, we have other explanations of the electric charge quantization; in particular the Grand Unification of strong, weak and electromagnetic interactions at extremely high energies produces quantized electrical charges. Curiously, the same Grand Unified Theories also predict that there **are** magnetic monopoles with charges (31). More recently, several attempts to unify all the fundamental interactions within the context of the String Theory also gave rise to magnetic monopoles, with charges quantized in units of $N\hbar c/2e$, where N is an integer such as 3 or 5. It was later found that in the same theories, there were superheavy particles with fractional electric charges e/N, so the monopoles in fact had the smallest non-zero charges allowed by the Dirac condition (27)! Nowadays, most theoretical physicists believe that any fundamental theory that provides for exact quantization of the electric charge should also provide for the existence of magnetic monopoles, but this conjecture has not been proved (yet).

In quantum field theories including EM fields coupled to both electrically and magnetically charged particles, the classical electric-magnetic symmetries

$$\begin{aligned} \begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} &= \begin{pmatrix} +\cos\theta & -\sin\theta \\ +\sin\theta & +\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \\ \begin{pmatrix} Q' \\ M' \end{pmatrix} &= \begin{pmatrix} +\cos\theta & -\sin\theta \\ +\sin\theta & +\cos\theta \end{pmatrix} \begin{pmatrix} Q \\ M \end{pmatrix} \\ &\text{for the same } \theta, \end{aligned} \tag{32}$$

are generally incompatible with the Dirac's charge quantization condition (27) and hence

$$Q = m_{\text{el}} \times e, \quad M = n_{\text{mag}} \times \frac{\hbar c}{2e}\,. \tag{33}$$

The best we can do is the electric-magnetic duality relating theories with different values of the electric charge quantum e:

$$\begin{aligned} \mathbf{E}' = \mathbf{B}, &\qquad \mathbf{B}' = -\mathbf{E}, \\ n'_{\text{el}} = m_{\text{mag}}, &\qquad n'_{\text{mag}} = -n_{\text{el}}\,, \\ e' &= \frac{\hbar c}{2e}, \end{aligned} \tag{34}$$

or in terms of the dimensionless QED coupling $\alpha = \frac{e^2}{\hbar c}$,

$$\alpha' = \frac{1}{4\alpha} \tag{35}$$

The real-life quantum electrodynamics has rather weak coupling $\alpha \approx 1/137$, so we may use perturbation theory expanding various amplitudes in powers of α. Alas, the QED-like theories with stronger couplings $\alpha \sim 1$ cannot use such perturbation theory, so they are much harder to understand and work with. Fortunately, in the really strong coupling limit $\alpha \gg 1$, we may use the electric-magnetic duality (34) to map the original theory to another theory with a weak coupling $\alpha' = \frac{1}{4\alpha} \ll 1$, and then use a perturbative expansion in powers of the dual coupling α'. This way, we can understand the original theory's behavior in both the weak-coupling and the strong-coupling regimes, and then try to interpolate between the two extremes to the $\alpha \sim 1$ regime.

Angular Momentum of Dyons

A *dyon* is an elementary particle — or a particle-like bound state of several elementary particles — which has both electric and magnetic charges. The net angular momentum of the dyon includes the spins of all the constituent particles, the orbital angular momentum of their relative motion, but there is also a contribution from the momentum density

$$\frac{1}{c}\mathbf{S} \;=\; \frac{1}{4\pi c}\,\mathbf{E}\times\mathbf{B} \tag{36}$$

of the EM fields generated by the dyon. To see how this works, let's consider a simple system of a single static magnetic monopole of charge M located at some point $\mathbf{x}_m$ and a single static electric charge Q located at some other point $\mathbf{x}_q \neq \mathbf{x}_m$. For this system: (1) The net linear momentum of the EM fields happens to vanish,

$$\mathbf{P}_{\mathrm{EM}} \;=\; \frac{1}{4\pi c}\iiint \mathbf{E}\times\mathbf{B}\,d^3\mathbf{x} \;=\; 0. \tag{37}$$

Consequently, the net angular momentum of the EM fields

$$\mathbf{L}_{\mathrm{EM}} \;=\; \frac{1}{4\pi c}\iiint \mathbf{x}\times(\mathbf{E}\times\mathbf{B})\,d^3\mathbf{x} \tag{38}$$

does not depend on our choice of the coordinate origin (acting as the pivot point). However, (2) this EM angular momentum does not vanish. Instead, it has magnitude MQ/c regardless of the distance between the electric and the magnetic charges, while it's direction points from the electric charge towards the monopole. (J. J. Thomson, 1904.)

In 1936, M. N Saha used this result to derive what's now called Dirac's quantization rule

$$M \times Q \;=\; \frac{\hbar c}{2} \times \text{an integer} \tag{39}$$

slightly before Dirac, albeit in a much less rigorous fashion. Basically, he argues that *semi-classically*, any kind of angular momentum of a quantum system should have magnitude $\frac{1}{2}\hbar \times$ an integer. Imposing this requirement on the Thomson's EM angular momentum $L_{\text{EM}} = |MQ|/c$ then immediately leads to the charge quantization rule (39). Soon after Saha, Dirac re-derived this rule in a much more rigorous fashion, and that's why it's called the Dirac's quantization rule rather than the Saha's rule.

Proofs: Let's start by verifying that the net linear momentum is zero. In eq. (37),

$$\mathbf{E} \times \mathbf{B} \;=\; -(\nabla\Phi) \times \mathbf{B} \;=\; -\nabla \times (\Phi\mathbf{B}) \;+\; \Phi(\nabla \times \mathbf{B}), \tag{40}$$

where the monopole's magnetic field

$$\mathbf{B}(\mathbf{x}) \;=\; M\frac{\mathbf{x} - \mathbf{x}_m}{|\mathbf{x} - \mathbf{x}_m|^3} \;=\; -\nabla\left(\frac{M}{|\mathbf{x} - \mathbf{x}_m|}\right) \tag{41}$$

has zero curl $\nabla \times \mathbf{B} = 0$, even at the monopole's location $\mathbf{x} = \mathbf{x}_m$. Hence, eq. (40) becomes

$$\mathbf{E} \times \mathbf{B} \;=\; -\nabla \times (\Phi\mathbf{B}) \tag{42}$$

and therefore

$$P^i_{\text{EM}} \;=\; -\frac{1}{4\pi c}\iiint \epsilon^{ijk}\nabla^j(\Phi B^k)\, d^3\mathbf{x} \;=\; -\frac{1}{4\pi c}\iint \epsilon^{ijk}\Phi B^k\, d^2\text{area}^j, \tag{43}$$

or in vector notations

$$\mathbf{P}_{\text{EM}} \;=\; +\frac{1}{4\pi c}\iint d^2\mathbf{area} \times \mathbf{B}\Phi. \tag{44}$$

The surface integral here is over the surface of the integration volume in eq. (37), and since that volume was over the whole 3D space, its surface can be thought as as $R \to \infty$ limit of the sphere of radius R. In the $R \to \infty$ limit, the electrostatic potential of the $\mathbf{E}$ field behaves as $1/R$, the magnetic field of the monopole as $1/R^2$, the area as R^2, so the whole integral behaves as $1/R$ — if not smaller — and vanishes for $R \to \infty$. Consequently, the net momentum of the EM field indeed vanishes.

Now consider the net angular momentum of the EM field. Since the net linear momentum happen to vanish, we are free to put the coordinate origin wherever we like, so let's put it at the monopole's location. Consequently, in our coordinate system

$$\mathbf{B}(\mathbf{x}) \;=\; \frac{M\mathbf{n}}{r^2}\,, \tag{45}$$

hence

$$\begin{aligned} \mathbf{x}\times(\mathbf{E}\times\mathbf{B}) \;&=\; r\mathbf{n}\times\left(\mathbf{E}\times\frac{M\mathbf{n}}{r^2}\right) \\ &=\; \frac{M}{r}\big(\mathbf{n}\times(\mathbf{E}\times\mathbf{n})\big) \\ &=\; \frac{M}{r}\big(\mathbf{E}\,-\,(\mathbf{E}\cdot\mathbf{n})\mathbf{n}\big). \end{aligned} \tag{46}$$

In index notations, this formula becomes

$$\big[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})\big]^i \;=\; ME^j\,\frac{\delta^{ij}-n^in^j}{r} \;=\; ME^j\nabla^j n^i, \tag{47}$$

hence eq. (38) for the angular momentum yields

$$\begin{aligned} L^i_{\rm EM} \;&=\; \frac{1}{4\pi c}\iiint\big[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})\big]^i\,d^3\mathbf{x} \\ &=\; \frac{1}{4\pi c}\iiint ME^j\nabla^j n^i\,d^3\mathbf{x} \\ &=\; \frac{M}{4\pi c}\iint n^iE^j\,d^2{\rm area}^j \;-\; \frac{M}{4\pi c}\iiint(\nabla^jE^j)n^i\,d^3\mathbf{x}\,. \end{aligned} \tag{48}$$

Again, the surface integral on the bottom line here is over a sphere of radius $R\to\infty$. For R much larger that the distance between the electric charge and the monopole, we may approximate

$$\mathbf{E}(\mathbf{x}) \;=\; \frac{Q(\mathbf{x}-\mathbf{x}_q)}{|\mathbf{x}-\mathbf{x}_q|^3} \;\approx\; \frac{Q\mathbf{x}}{|\mathbf{x}|^3} \;=\; \frac{Q\mathbf{n}}{r^2} \tag{49}$$

hence

$$E^j\,d^2{\rm area}^j \;=\; \mathbf{E}\cdot d^2\mathbf{area} \;=\; \frac{Q}{R^2}\,R^2d^2\Omega \;=\; Q\,d^2\Omega \tag{50}$$

and therefore

$$\iint n^i E^j \, d^2\text{area}^j \;\approx\; \iint n^i Q \, d^2\Omega \;=\; 0. \tag{51}$$

This takes care of the surface integral in eq. (48) and leaves us with just the volume integral

$$L^i_{\text{EM}} \;=\; -\frac{M}{4\pi c} \iiint (\nabla^j E^j) n^i \, d^3\mathbf{x}, \tag{52}$$

or in vector notations

$$\mathbf{L}_{\text{EM}} \;=\; -\frac{M}{4\pi c} \iiint (\nabla \cdot \mathbf{E}) \mathbf{n} \, d^3\mathbf{x}. \tag{53}$$

Furthermore, for $\mathbf{E}(\mathbf{x})$ being the Coulomb field of a point electric charge Q located at $\mathbf{x}_1$, we have

$$\nabla \cdot \mathbf{E}(\mathbf{x}) \;=\; 4\pi Q \delta^{(3)}(\mathbf{x} - \mathbf{x}_q), \tag{54}$$

hence

$$\mathbf{L}_{\text{EM}} \;=\; -\frac{M}{4\pi c} \iiint 4\pi Q \delta^{(3)}(\mathbf{x} - \mathbf{x}_q) \mathbf{n}(\mathbf{x}) \, d^3\mathbf{x} \;=\; -\frac{MQ}{c} \, \mathbf{n}(\mathbf{x}_q). \tag{55}$$

In this formula, the $\mathbf{n}(\mathbf{x}_q)$ is the direction of the radius vector of the electric charge in the coordinate system whose origin is at the magnetic monopole. In a more general coordinate system, $\mathbf{n}(\mathbf{x}_q)$ becomes the unit vector pointing from the monopole to the electric charge, thus

$$\mathbf{n}(\mathbf{x}_q) \;\to\; \frac{\mathbf{x}_q - \mathbf{x}_m}{|\mathbf{x}_q - \mathbf{x}_m|} \tag{56}$$

and therefore

$$\mathbf{L}_{\text{net}} \;=\; \frac{MQ}{c} \, \frac{\mathbf{x}_m - \mathbf{x}_q}{|\mathbf{x}_q - \mathbf{x}_m|} \, . \tag{57}$$

Quod erat demonstrandum.

Although the angular momentum (57) stems from the EM fields, it combines with the particles' orbital and spinor angular momenta into a single conserved quantity. For a classical example (hence no spins), consider an electrically charged particle orbiting a monopole or a dyon surrounded by a spherically symmetric cloud of electric charges. As far as the orbiting charged particle is concerned, there is some general spherically symmetric electrostatic potential $\Phi(r)$ and the magnetic field of a static monopole placed at the center of this potential, so the net force on the particle is

$$\mathbf{F} \;=\; -Q\frac{d\Phi}{dr}\,\mathbf{n} \;+\; \frac{Q}{c}\,\mathbf{v}\times\left(\mathbf{B}=\frac{M\mathbf{n}}{r^2}\right). \tag{58}$$

For this particle, the conserved angular momentum combines the orbital angular momentum $\mathbf{x}\times m\mathbf{v}$ with the EM angular momentum (57), thus

$$\mathbf{L} \;=\; \mathbf{x}\times m\mathbf{v} \;-\; \frac{MQ}{c}\,\mathbf{n} \tag{59}$$

It is easy to see that the two individual terms here are not conserved by themselves, but their combination (59) is conserved. Indeed,

$$\frac{d}{dt}\left(\frac{MQ}{c}\,\mathbf{n}\right) \;=\; \frac{MQ}{c}\,\frac{\mathbf{v}-(\mathbf{n}\cdot\mathbf{v})\mathbf{n}}{r} \;=\; \frac{MQ}{rc}\,\mathbf{n}\times(\mathbf{v}\times\mathbf{n}) \;\neq\; 0, \tag{60}$$

and

$$\begin{aligned}
\frac{d}{dt}\left(\mathbf{x}\times m\mathbf{v}\right) \;&=\; \mathbf{v}\times m\mathbf{v} \;+\; \mathbf{x}\times\mathbf{F} \\
&=\; 0 \;+\; \mathbf{x}\times\mathbf{F}_{\text{rad}} \;+\; \mathbf{x}\times\mathbf{F}_{\text{mag}} \;=\; 0 \;+\; 0 \;+\; \mathbf{x}\times\mathbf{F}_{\text{mag}} \\
&=\; (r\mathbf{n})\times\left(\frac{Q}{c}\mathbf{v}\times\frac{M\mathbf{n}}{r^2}\right) \\
&=\; \frac{QM}{rc}\,\mathbf{n}\times(\mathbf{v}\times\mathbf{n}) \;\neq\; 0,
\end{aligned} \tag{61}$$

but

$$\frac{d}{dt}\mathbf{L} \;=\; \frac{QM}{rc}\,\mathbf{n}\times(\mathbf{v}\times\mathbf{n}) \;-\; \frac{QM}{rc}\,\mathbf{n}\times(\mathbf{v}\times\mathbf{n}) \;=\; 0. \tag{62}$$

In quantum mechanics, we have a similar conservation law for the combined orbital plus EM angular momentum (plus spin, if any), but let me leave the proof as an optional exercise for the interested students.