1. Consider a thin spherical shell of radius R with a uniform surface charge density σ. The sphere rotates about an axis through its center with angular velocity $\vec{\omega}$.

   (a) Write down the current density $\mathbf{J}(\mathbf{x})$ due to charges moving with the rotating sphere.

   (b) To help calculating the magnetic field due to this current density in the next part (c), show that

$$\iint\limits_{\text{sphere}} d^2\text{area}(\mathbf{y})\,\frac{\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} = \frac{4\pi R}{3} * \bigl(\min(r_x,R)\bigr)^3 * \frac{\mathbf{n}_x}{r_x^2} = \begin{cases} \dfrac{4\pi}{3} R r_x * \mathbf{n}_x & \text{inside the sphere,} \\ \dfrac{4\pi}{3}\dfrac{R^4}{r_x^2} * \mathbf{n}_x & \text{outside the sphere.} \end{cases} \tag{1}$$

   (c) Now use eq. (1) to calculate the vector potential $\mathbf{A}(\mathbf{x})$ and hence the magnetic field $\mathbf{B}(\mathbf{x})$ for all $\mathbf{x}$ — both inside and outside the sphere.

   (d) Calculate the magnetic dipole moment of the rotating sphere.

   Now consider two such rotating charged spheres, one inside the other. The two spheres are concentric, but they rotate around different, non-parallel axes, — thus $\vec{\omega}_1 \nparallel \vec{\omega}_2$, — although both axes go through the common center of the two spheres.

   (e) Calculate the torque between the two spheres.

2. Consider the magnetic field inside a tightly wound solenoid of finite length L and finite radius R.

   (a) Argue that in the cylindrical coordinates (z,s,ϕ), the magnetic field's components $B_z(z,s)$ and $B_s(z,s)$ are analytic functions of z and s but do not depend on ϕ, while the B_ϕ component vanishes, $B_\phi = 0$.

   (b) Now let's expand the analytic functions $B_z(z,s)$ and $B_s(z,s)$ into power series in s (with z-dependent coefficients). Re-express these series into Cartesian coordinates for the

$$B_x(x,y,z) \;=\; \frac{x}{s}B_s(s,z), \quad B_y(x,y,z) \;=\; \frac{y}{s}B_s(s,z), \quad B_z(x,y,z) \;=\; B_z(s,z), \quad (2)$$

and argue that the $B_{x,y,z}$ are analytic functions of the (x,y,z) only if the power series in s for the $B_z(z,s)$ includes only the even powers of s while the series for the $B_s(z,s)$ includes only the odd powers, thus

$$B_z(z,s,\phi) \;=\; \sum_{n=0}^{\infty} \alpha_n(z) \times s^{2n} \;=\; \alpha_0(z) \;+\; \alpha_1(z)\, s^2 \;+\; \alpha_2(z)\, s^4 \;+\; \cdots, \qquad (3)$$

$$B_s(z,s,\phi) \;=\; \sum_{n=0}^{\infty} \beta_n(z) \times s^{2n+1} \;=\; \beta_0(z)\, s \;+\; \beta_1(z)\, s^3 \;+\; \beta_2(z)\, s^5 \;+\; \cdots, \quad (4)$$

for some analytic functions $\alpha_n(z)$ and $\beta_n(z)$.

(c) Next, write the magnetostatic equations $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$ (inside the solenoid) in cylindrical coordinates, apply them to the power series (3) and (4), and derive recursive relations between the functions $\alpha_n(z)$, $\beta_n(z)$ and their derivatives. Specifically, show that

$$\frac{d}{dz}\,\beta_n(z) \;=\; +2(n+1)\alpha_{n+1}(z), \qquad \frac{d}{dz}\,\alpha_n(z) \;=\; -2(n+1)\beta_n(z), \qquad (5)$$

and therefore

$$\alpha_n(z) \;=\; \frac{(-1)^n}{2^{2n}\,(n!)^2}\left(\frac{d}{dz}\right)^{2n}\alpha_0(z), \qquad \beta_n(z) \;=\; \frac{(-1)^{n+1}}{2^{2n+1}\,(n+1)!\,n!}\left(\frac{d}{dz}\right)^{2n+1}\alpha_0(z). \qquad (6)$$

In light of parts (a–c), given the magnetic field $B_z(z,0) = \alpha_0(z)$ on the axis of the solenoid as a function of z, we may calculate the field away from the axis as power series

$$B_z(z,s) \;=\; \alpha_0(z) \;-\; \frac{s^2}{4}\,\alpha_0''(z) \;+\; \frac{s^4}{64}\,\alpha_0''''(z) \;+\; \cdots, \qquad (7)$$

$$B_s(z,s) \;=\; -\frac{s}{2}\,\alpha_0'(z) \;+\; \frac{s^3}{16}\,\alpha_0'''(z) \;-\; \frac{s^5}{384}\,\alpha_0'''''(x) \;+\; \cdots. \qquad (8)$$

So let us calculate the field $B_z(z,0)$ on the solenoid's axis as a function of z.

(d) Approximate the winding of the solenoid as a cylindrical current sheet of density $K = IN/L$ and use the Biot–Savart–Laplace formula to show that along the solenoid's axis

$$B_z(z,0) = \frac{\mu_0 IN}{L} \times \frac{1}{2}\left(\frac{(L/2)+z}{\sqrt{((L/2)+z)^2+R^2}} + \frac{(L/2)-z}{\sqrt{((L/2)-z)^2+R^2}}\right). \tag{9}$$

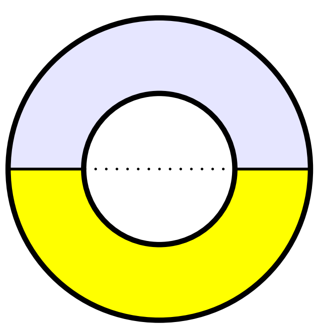
(e) Now consider a solenoid that's much longer than its radius and focus on the central region of $|z| \sim R \ll L$. Estimate the magnitudes (up to $O(1)$ factors) of the derivatives $\partial^n B_z/\partial z^n$ of the on-axis field in this region and hence the magnitudes of various terms in series (7) and (8).

(f) Calculate the leading terms in the series (7) and (8) for the central part of a long solenoid and show that

for $|z| \sim R \ll L$ and any $s < R$,

$$\begin{aligned} B_z(z,s) &= \frac{\mu_0 IN}{L}\left(1 - \frac{2R^2}{L^2} + \frac{6R^2(R^2-4z^2+2s^2)}{L^4} + O(R^6/L^6)\right), \\ B_s(z,s) &= \frac{\mu_0 IN}{L}\left(0 + \frac{24R^2 zs}{L^4} + O(R^6/L^6)\right). \end{aligned} \tag{10}$$

3. Finally, an easy problem about electrostatic boundary conditions in dielectrics. Two concentric metal spheres of respective radii a and b act as capacitor plates. Half of the space between the spheres (say, the lower hemisphere) is filled with a solid dielectric of dielectric constant ϵ while the other half is vacuum:



Note: the boundary between the dielectric and the vacuum lies in the same plane as the common center of the two spheres.

The voltage between the metal spheres is V.

(a) Show that a radial **E** field between the spheres obeys the boundary conditions at the dielectric-vacuum interface.

Then find the electric tension field **E** and the electric displacement field **D** everywhere between the spheres.

(b) Find the surface charge densities on the metal spheres, and then the capacitance of this half-filled capacitor.