

1. First, a couple of reading assignments about examples of boundary problems in dielectric and magnetic materials.
 - (a) First, read the examples of dielectric boundary problems in §4.4 of the Jackson's textbook, especially the dielectric sphere example (see also [my notes on dielectrics and magnetic materials](#)).
 - (b) Second, read §5.12 of Jackson's textbook about magnetic shielding by a spherical shell of high-permeability material. If you have trouble following the boundary condition for the magnetic scalar potential $\Psi(\mathbf{x})$ — which Jackson calls $\Phi_M(\mathbf{x})$, — go back to part (a).
2. Now consider a wire loop \mathcal{L} carrying steady current I . The loop \mathcal{L} may have any size or shape, as long as it is closed. The magnetic field \mathbf{H} generated by the current in this loop obtains from the scalar potential

$$\Psi(\mathbf{x}) = \frac{I}{4\pi} \Omega(\mathbf{x}) \tag{1}$$

where $\Omega(\mathbf{x})$ is the solid angle spanned by the loop \mathcal{L} when viewed from the point \mathbf{x} .

By convention, $\Omega(\mathbf{x})$ is positive if the current in \mathcal{L} viewed from point \mathbf{x} appears to run clockwise, and negative if the current appears to run counterclockwise. To avoid a discontinuity when \mathbf{x} is surrounded by the loop, $\Omega(\mathbf{x})$ should be analytically continued while \mathbf{x} moves from one side of the loop to another. Such continuations make Ω multivalued, with different values of $\Omega(\mathbf{x})$ at the same point \mathbf{x} differing by 4π , or more generally by $4\pi \times$ an integer.

- (a) Show that

$$\Omega(\mathbf{x}) = \iint_{\mathcal{S}} \frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3} \cdot d^2 \mathbf{area}(\mathbf{y}) \tag{2}$$

where \mathcal{S} is a surface spanning the loop \mathcal{L} .

- (b) Explain how eq. (2) leads to the sign convention for the $\Omega(\mathbf{x})$, and also how different surfaces \mathcal{S} can yield values of $\Omega(\mathbf{x})$ which differ from each other by $4\pi \times$ an integer.
- (c) Show that $\mathbf{H} = -\nabla\Psi$ for the scalar potential (1) agrees with the Biot–Savart–Laplace formula for the magnetic field of the current I in the loop \mathcal{L} .

Hint: prove and use

$$\nabla_y \times \left(\frac{(\mathbf{y} - \mathbf{x}) \times \mathbf{c}}{|\mathbf{y} - \mathbf{x}|^3} \right) = (\mathbf{c} \cdot \nabla_y) \frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3} \quad (3)$$

for $\mathbf{y} \neq \mathbf{x}$ and any constant vector \mathbf{c} .

3. Next, a capacitor force problem.

- (a) Consider an infinitesimal change in the geometry of a charged capacitor which changes its capacitance C by δC . During this process, the capacitor may be disconnected, or connected to a DC power supply with a fixed voltage, or even connected to some circuitry that changes both the voltage of the capacitor and the charge it stores. Show that in any case, the *mechanical* work involved in changing the capacitor's geometry is

$$\delta W_{\text{mech}} = -\frac{V^2}{2} \times \delta C. \quad (4)$$

Note: this is mechanical work on the capacitor by the external forces; the mechanical work by the capacitor itself has opposite sign.

- (b) Inserting an extra piece of dielectric into the capacitor generally increases its capacitance. Consequently, there is a net electric force on that piece of dielectric pulling it in. Show that the magnitude of this force is

$$F = \frac{V^2}{2} \times \frac{dC}{dx}. \quad (5)$$

- (c) Finally, consider the capacitor made from two large parallel vertical metal plates at small distance d between them, immersed part way into transformer oil with dielectric constant ϵ and *mass density* ρ . Connect the plates by wires to a battery or any other DC power supply of voltage V .

Show that this makes the oil between the plates rise to the height

$$h = \frac{(\epsilon - 1)\epsilon_0 V^2}{2\rho g d^2} \quad (6)$$

relative to the oil outside the plates. ($g = 9.8 \text{ N/kg}$ is the gravitational field.)

4. Finally, an easy problem on Faraday's Induction Law. Consider a long straight vertical wire moving at constant velocity \mathbf{v} in a horizontal direction. The wire carries a constant current I , which creates a magnetic field; in the quasi-static approximation (valid for $v \ll c$), $\mathbf{B}(\mathbf{x}, t)$ obtains via the Biot–Savart–Laplace equation using the wire's location at time t . But the moving wire make this quasi-static field move, so at a fixed location \mathbf{x} the magnetic field changes with time.

Find the electric field $\mathbf{E}(\mathbf{x}, t)$ *induced* by this time-dependent magnetic field.

Hint: calculate the vector potential $\mathbf{A}(\mathbf{x}, t)$ for the magnetic field of the moving wire.