1. Consider a bunch of permanent magnets. For simplicity, assume the ferromagnetic material in the magnets is so hard that its magnetization $\mathbf{M}$ stays constant despite the variable $\mathbf{H}$ field of the other magnets and/or electromagnets.

   Let's start with one permanent magnet and one current-carrying coil. The magnet is fixed in place, but the coil may move around it. Also, the current in the coil may change with time.

   (a) Show that for this system, the net electric + mechanical work is *reversible*, that is

   $$W_{\text{electric}} \;+\; W_{\text{mechanic}} \;=\; \Delta U(I, \text{coil's position}). \tag{1}$$

   for some well-defined magnetic energy U — there is no irreversibly lost work due to hysteresis. Also, show that

   $$U \;=\; \frac{\mu_0}{2} \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{H}^2(\mathbf{x})\, d^3\mathbf{x} \;+\; \text{const.} \tag{2}$$

   Note: the $\mathbf{H}$ field here is the net field due to both the current in the coil and the magnetization $\mathbf{M}$ of the permanent magnet. Thus, even though the magnetization $\mathbf{M}$ does not appear in eq. (2) *directly*, it does affect the net magnetic energy of the system via its effect on the $\mathbf{H}$ field.

   Now consider a system of several permanent magnets, each having constant magnetization $\mathbf{M}$ despite the $\mathbf{H}$ fields from the other magnets. But there are no coils or other macroscopic electric currents.

   (b) Argue that the magnetic forces and torques on the magnets follow from the potential energy $U(\text{geometry})$ which has exactly the same form as in eq. (2).

   Hint: first, in a setup of part (a) move the magnet around the coil and use the relativity of motion. Second, replace the coil with another permanent magnet. Finally, generalize to several magents.

(c) Show that without conduction currents

$$\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{H}\cdot\mathbf{B}\,d^3\mathbf{x} \;=\; 0, \tag{3}$$

then use this formula to rewrite the magnetic energy (2) as

$$U \;=\; -\frac{\mu_0}{2}\sum_{i\neq j}^{\text{magnets}}\;\iiint\limits_{\text{magnet}\#i} \mathbf{M}_i\cdot\mathbf{H}[\text{magnet}\#j]\,d^3\mathbf{x} \;+\; \text{const.} \tag{4}$$

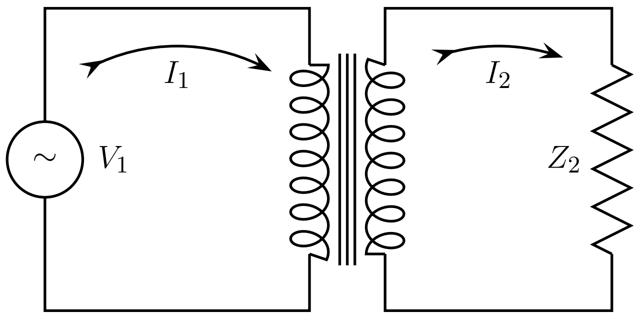
(d) To check this formula, consider a system of two small magnets separated by a much larger distance. Approximating each magnet as a pure dipole of magnetic moment $\mathbf{m}_1$ or $\mathbf{m}_2$, show that for this system

$$U \,+\, \text{const} \;=\; -\frac{\mu_0}{2}\big(\mathbf{M}_1\cdot\mathbf{H}_2(\mathbf{x}_1)+\mathbf{M}_2\cdot\mathbf{H}_1(\mathbf{x}_2)\big) \;=\; -\mathbf{M}_1\cdot\mathbf{B}_2(\mathbf{x}_1) \;=\; -\mathbf{M}_2\cdot\mathbf{B}_1(\mathbf{x}_2). \tag{5}$$

Then use eq. (5) to argue that the forces and the torques on the magnets stemming from the magnetic energy (4) = (5) agree with the usual formulae for the forces and the torques on magnetic dipoles,

$$\mathbf{F} \;=\; \nabla(\mathbf{m}\cdot\mathbf{B}), \quad \vec{\tau} \;=\; \mathbf{m}\times\mathbf{B}. \tag{6}$$

2. A transformer is made of 2 coils on a common ferromagnetic core. The coils have respective self-inductances L_1 and L_2 and mutual inductance $M_{12} = M_{21} = k\sqrt{L_1 L_2}$. The primary coil is plugged into an AC power source of voltage V_1 and frequency ω, while the secondary coil is connected to a load of impedance Z_2:



For simplicity, consider an ideal transformer: perfectly linear ferromagnetic core with no hysteresis, no eddy currents in the core, no ohmic losses in the wiring of the coils, and perfect *magnetic coupling* of the two coils, $k = 1$.

(a) Write down linear equations for the complex amplitudes of the currents in the two coils and the voltages on them. Then solve the equations and show that

$$\frac{V_2}{V_1} = n, \quad \frac{I_2}{I_1} = \frac{1}{n} \times \frac{j\omega L_2}{j\omega L_2 + Z_2} \tag{7}$$

for n being the *stepping ratio*

$$n = \sqrt{\frac{L_2}{L_1}}. \tag{8}$$

In particular, show that even for an ideal transformer, the simple ratios

$$\frac{V_2}{V_1} = n, \quad \frac{I_2}{I_1} = \frac{1}{n} \tag{9}$$

obtain only for $|Z_2| \ll \omega L_2$.

(b) Now consider a somewhat less ideal transformer with a coupling coefficient k just a little bit smaller than 1, so that $1 - k^2 \ll 1$. Again, calculate the transformer ratios V_2/V_1 and I_2/I_1 and show that they approximate the simple ratios (9) for the load impedance Z_2 in the range

$$\omega L_2 \;\gg\; |Z_2| \;\gg\; (1 - k^2) \times \omega L_2\,, \tag{10}$$

but outside of this range we need more complicated formulae.

(c) Finally, for a transformer made of two coils of respectively N_1 and N_2 turns wound around a common toroidal ferromagnetic core, check that $n \approx N_2/N_1$. Also, explain what causes $k < 1$ and argue that in the limit of very high permeability μ of the ferromagnetic core $k \to 1$.