1. A thin toroidal coil of N turns has mean radius R, cross-section $a \ll R^2$, and carries steady current I. A point charge Q is placed at the center of the toroid. For simplicity, assume that electric field of this charge penetrates inside the coil without any distortion by the coil's wires.

   Initially, the whole system — the change, the coil, the battery and the wires providing the current — is at rest.

   (a) Calculate the net momentum of the electromagnetic fields in the system.

   (b) Pick realistic values of the input parameters and calculate the resulting momentum. What kind of a mechanical system would have a similar momentum? A speeding bullet? A crawling ant? A proton in the LHC accelerator? Something else?

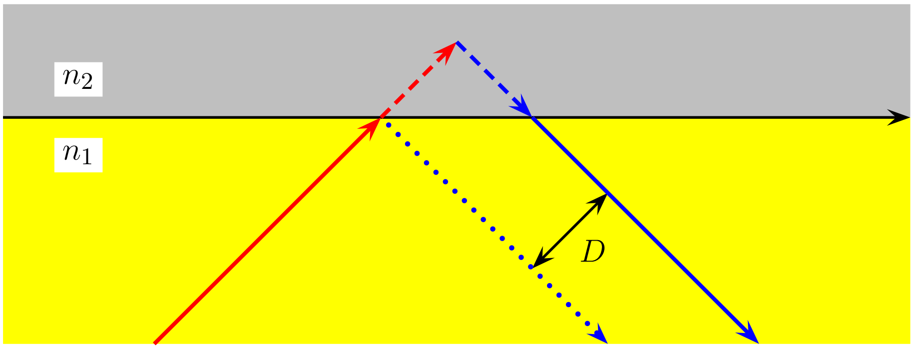
   (c) Let's turn off the current in the coil. The transient electric field induced by the dropping magnetic field imparts an impulse on the change. Show that the net impulse given to the charge in this process is precisely the former momentum of the EM fields.

2. Next, a reading assignment plus a simple exercise.

   (a) Read §7.3 of the Jackson's textbook about reflection and refraction of the EM waves. Unlike my notes on the same subject, the textbook analysis allows for magnetic media on both sides of the boundary.

   (b) To check your understanding, rewrite the textbook formulae (7.39), (7.41), and (7.42) in terms of the wave impedance ratio Z'/Z of the two media.

3. Finally, consider the Goos–Hänchen effect: In a total internal reflection, the reflected ray is displaced sideways relative to the incoming ray — as if it's reflected not from the boundary itself but from a small distance behind it.



(1)

The key to the Goos–Hänchen effect is the complex reflection coefficient

$$r(\alpha) \;=\; \exp(i\phi(\alpha)), \tag{2}$$

its magnitude in a total internal reflection is 1, but the phase depends on the incidence angle α.

To see how this works, let the incident wave have a finite but large width $a \gg (1/k_0)$ in the direction $\perp$ to the wave within the plane of incidence,

$$\mathbf{E}_1(\mathbf{x},t) \;=\; \mathcal{E}_0\,\mathbf{e}_1\,\exp\bigl(ik_0\mathbf{n}_1\cdot\mathbf{x}-i\omega t\bigr)\exp\left(-\frac{(\mathbf{m}_1\cdot\mathbf{x})^2}{2a^2}\right) \tag{3}$$

where $\vec{\mathcal{E}_0}$ is the overall amplitude, $\mathbf{e}_1$ is polarization vector of the incident wave,

$$\mathbf{n}_1 \;=\; (\sin\alpha, 0, \cos\alpha) \tag{4}$$

is the unit vector in the direction of the incident beam, and

$$\mathbf{m}_1 \;=\; (\cos\alpha, 0, -\sin\alpha) \tag{5}$$

is the unit vector in the direction across the beam: $\mathbf{m}_1 \perp \mathbf{n}_1$ within the incidence plane.

(a) Fourier transform the incident wave from the $\mathbf{x}$ space to the $\mathbf{k}$ space, show that

$$\tilde{\mathbf{E}}_1(\mathbf{k},t) = (2\pi)^{5/2} a\,\mathcal{E}_0\,\mathbf{e}_1\,e^{-i\omega t}\,\delta(k_y)\delta(\mathbf{n}_1\cdot\mathbf{k}-k_0)\exp\bigl(-\tfrac{1}{2}a^2(\mathbf{m}_1\cdot\mathbf{k})^2\bigr) \tag{6}$$

and therefore, the Fourier-transformed reflected wave is

$$\begin{aligned}\tilde{\mathbf{E}}_3(\mathbf{x},t) &= (2\pi)^{5/2} a\,\mathcal{E}_0\,\mathbf{e}_3\,e^{-i\omega t}\,\delta(k_y)\delta(\mathbf{n}_3\cdot\mathbf{k}-k_0)\exp\bigl(-\tfrac{1}{2}a^2(\mathbf{m}_3\cdot\mathbf{k})^2\bigr)\\ &\quad *\exp\bigl(i\phi(\mathrm{dir}(\mathbf{k}))\bigr)\end{aligned} \tag{7}$$

for the unit vectors

$$\mathbf{n}_3 = (\sin\alpha, 0, -\cos\alpha) \quad\text{and}\quad \mathbf{m}_3 = (\cos\alpha, 0, +\sin\alpha) \tag{8}$$

pointing along and across the reflected beam.

(b) Now Fourier transform the reflected wave (7) back to the coordinate space and show that

$$\begin{aligned}\mathbf{E}_3(\mathbf{x},t) &= \mathcal{E}_0\,\mathbf{e}_3\,\exp\bigl(ik_0\mathbf{m}_3\cdot\mathbf{x} - i\omega t\bigr)\int\frac{a\,dk_\perp}{\sqrt{2\pi}}\exp\begin{pmatrix} ik_\perp(\mathbf{m}_3\cdot\mathbf{x}) - \tfrac{1}{2}a^2k_\perp^2 \\ +\,i\phi(\mathrm{dir}(k_0\mathbf{n}_3+k_\perp\mathbf{m}_3))\end{pmatrix}\\ &\approx \mathcal{E}_0\,\mathbf{e}_3\,\exp\bigl(ik_0\mathbf{m}_3\cdot\mathbf{x} - i\omega t\bigr)\exp\left(-\frac{(\mathbf{m}_3\cdot\mathbf{x}-D)^2}{2a^2}\right)\end{aligned} \tag{9}$$

for the displacement

$$D = -\frac{\partial\phi(\mathrm{dir}(k_0\mathbf{n}_3+k_\perp\mathbf{m}_3))}{\partial k_\perp} = -\frac{1}{k_0}\,\frac{\partial\phi}{\partial\alpha}\,. \tag{10}$$

At the end of my notes on refraction and reflection of the EM waves, I calculate the reflection coefficients $r_\parallel$ and $r_\perp$ — including their phases $\phi_\parallel$ and $\phi_\perp$ — for the regime of the total internal reflection. Please note two different formulae for the two linear polarizations of the EM wave, in-plane and normal-to-the-plane of incidence.

(c) Plug those phases into eq. (10) for the sideways displacement of the reflected wave and show that

$$D_\perp = \frac{2}{k} \frac{\sin\alpha}{\sqrt{\sin^2\alpha - (n_2/n_1)^2}}, \tag{11}$$

$$D_\parallel = D_\perp \times \frac{1}{(1 + (n_1/n_2)^2)\sin^2\alpha \; - \; 1}. \tag{12}$$