1. In conducting materials, the EM waves attenuate with distance. For a specific example, consider a uniform material with dielectric constant ϵ and conductivity σ; assume the frequency is low enough that ϵ and σ are real. Also assume negligible magnetism, $\mu = 1$.

   The attenuating plane wave propagating in $\mathbf{z}$ direction has general form

   $$\mathbf{E}(x, y, z, t) \;=\; \vec{\mathcal{E}} \exp(ik_r z - \kappa z - i\omega t), \quad \mathbf{H}(x, y, z, t) \;=\; \vec{\mathcal{H}} \exp(ik_r z - \kappa z - i\omega t). \quad (1)$$

   (a) Write down formulae for k_r and κ as functions of ω. Also, relate the electric amplitude $\vec{\mathcal{E}}$ and the magnetic amplitude $\vec{\mathcal{H}}$ to each other.

   Now consider a boundary between a conducting material and the vacuum. Suppose an EM wave comes from the vacuum side and hits the boundary head-on.

   (b) Calculate the reflectivity $R = |r|^2$ of the boundary.

   (c) Show that for a good conductor

   $$R \;\approx\; 1 \;-\; \frac{4\,\mathrm{Re}(n)}{|n^2|} \;\approx\; 1 \;-\; \frac{4\pi\delta}{\lambda_0} \quad (2)$$

   where $n(\omega)$ is the complex refraction coefficient of the conductor, $\delta(\omega)$ is the skin depth of that conductor, while λ_0 is the wavelength of the EM wave in the vacuum.

   (d) As an example, find the reflectivity of sea water ($\sigma \approx 5\mho/\mathrm{m}$) at an FM radio frequency $\omega = 2\pi \times 100$ MHz.

2. Next, consider charge density perturbations $\rho(\mathbf{x}, t)$ in a metal.

   (a) Fourier transform time-dependence of macroscopic EM fields and charge/current densities to frequency dependence. Use

   $$\mathbf{J}_{\mathrm{net}}(\mathbf{x}, \omega) \;=\; \mathbf{J}_{\mathrm{cond}}(\mathbf{x}, \omega) \;-\; i\omega\mathbf{D}(\mathbf{x}, \omega) \;=\; \sigma_c(\omega)\mathbf{E}(\mathbf{x}, \omega) \quad (3)$$

   where

   $$\sigma_c(\omega) \;=\; \sigma(\omega) \;-\; i\omega\epsilon(\omega)\epsilon_0 \quad (4)$$

   is the effective complex conductivity of the metal, as well as the Fourier-transformed

Maxwell equations to show that the transformed charge density $\rho(\mathbf{x}, \omega)$ obeys

$$\sigma_c(\omega)\rho(\omega, \mathbf{x}) = 0. \tag{5}$$

Hint: $\nabla \cdot \mathbf{J}_{\text{net}} = 0$.

Drude–Lorentz formula tell us that in metals

$$\sigma_c(\omega) = \frac{ne^2 f_0}{m_e^*} \frac{1}{\gamma_0 - i\omega} - i\omega\epsilon_b\epsilon_0 \approx \epsilon_b\epsilon_0 \left(\frac{\omega_p^2}{\gamma_0 - i\omega} - i\omega \right) \tag{6}$$

where $\gamma_0 = (1/\tau)$ is the rate at which the conduction electrons lose their average velocity vector to collisions with ions, and ω_p is the plasma frequency of the metal.

(b) Show that for the Drude-Lorentz compex conductivity (6), eq. (5)becomes

$$(\omega_p^2 - i\gamma_0\omega - \omega^2)\rho(\mathbf{x}, \omega = 0) = 0 \tag{7}$$

and hence the time-dependent charge density $\rho(\mathbf{x}, t)$ obeys the differential equation

$$\left(\frac{\partial^2}{\partial t^2} + \gamma^0 \frac{\partial}{\partial t} + \omega_p^2 \right) \rho(\mathbf{x}, t) = 0. \tag{8}$$

(c) Finally, solve eq. (8) for the charge density oscillations in a metal with $\omega_p \gg \gamma_0$.

3. Now consider a 1D wave propagating through a linear and homogeneous but dispersive media with refraction index $n(\omega)$, *i.e.*, the phase velocity of a wave $v(\omega) = c/n(\omega)$. To allow for absorption, $n(\omega)$ may be complex rather than real.

(a) Show that the most general solution of the dispersive wave equation is

$$\psi(x, t) = \int\limits_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left(A(\omega) \times \exp(+i\omega n(\omega)x/c) + B(\omega) \times \exp(-i\omega n(\omega)x/c) \right) \tag{9}$$

for some arbitrary complex functions $A(\omega)$ and $B(\omega)$.

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(b) Show that a real wave $\psi(x,t)$ requires $n(-\omega) = n^*(+\omega)$ as well as $A(-\omega) = A^*(+\omega)$ and $B(-\omega) = B^*(+\omega)$.

(c) Suppose at $x = 0$ we observe ψ and its x derivative as functions of time. Show that in terms of these data

$$
\begin{aligned}
A(\omega) &= \int\limits_{-\infty}^{+\infty} dt\, e^{+i\omega t} \left[\frac{1}{2}\psi(0,t) - \frac{ic}{2\omega n(\omega)} \frac{\partial\psi}{\partial x}(0,t) \right], \\
B(\omega) &= \int\limits_{-\infty}^{+\infty} dt\, e^{+i\omega t} \left[\frac{1}{2}\psi(0,t) + \frac{ic}{2\omega n(\omega)} \frac{\partial\psi}{\partial x}(0,t) \right].
\end{aligned}
\tag{10}
$$

4. Finally, show that in the regime of normal dispersion — *i.e.*, at frequencies not too close to any of the resonances — the group velocity of the EM wave is always less than c. For simplicity, assume negligible magnetism $\mu \approx 1$ and use the low-density approximation to the dielectric constant,

$$
\epsilon(\omega) \approx 1 + \frac{ne^2}{\epsilon_0 m_e} \sum_i^{\text{resonances}} \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}.
\tag{11}
$$

Hint: show that $v_{\text{group}} < v_{\text{phase}}$ and $v_{\text{group}} \times v_{\text{phase}} < c^2$.

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