⋆ This problem set has 5 problems, or rather 4 problems and one reading assignment. Problems 1,2 and 3 concern emission of photons by atoms and nuclei, while problems 4 and 5 are about radiation by long antennas with $L \gtrsim \lambda$.

1. A quantum hydrogen atom initially in the excited 2p state drops to the ground 1s state while emitting a photon. Calculate the matrix element of the electric dipole operator between these two states and hence the transition rate (in the electric dipole approximation).

   For the sake of definiteness, let the initial 2p state be $|n=2,\ell=1,m=0\rangle$ with the wave function

$$\Psi_{2p}(\mathbf{x}) \;=\; \frac{1}{\sqrt{32\pi a^5}}\, r e^{r/2a}\,\cos\theta \tag{1}$$

   while the final 1s state $|n=1,\ell=0,m=0\rangle$ has wavefunction

$$\Psi_{1s}(\mathbf{x}) \;=\; \frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}, \tag{2}$$

   where

$$a \;=\; \frac{4\pi\epsilon_0\,\hbar^2}{e^2 m_e} \;=\; \frac{\hbar}{\alpha m_e c} \;\approx\; 0.53\ \text{Å} \tag{3}$$

   is the Bohr radius. For simplicity, ignore the electron's spin.

2. In a hydrogen-like atom or ion, the closest quantum analogues of circular electron orbits are the states $|n,\ell,m\rangle$ with $m=\ell=n-1$. For $n \gg 1$, the wave function of such a state is strongly peaked in a thin torus around the classical circular orbit. (This is FYI, you do not need to prove this.)

   Show that the only allowed transition from such a state to a lower-energy state is to a similar state $|n',\ell',m'\rangle$ with $m'=\ell'=n'-1$ for $n'=n-1$. Consequently, if an atom is initially in such a state with large n, then it de-excites down to the ground state through a cascade of transitions

$$\begin{aligned} |n, n-1, n-1\rangle \to |n-1, n-2, n-2\rangle \to |n-2, n-3, n-3\rangle \to \\ \to \cdots \to |3,2,2\rangle \to |2,1,1\rangle \to |1,0,0\rangle\,. \end{aligned} \tag{4}$$

This cascade is the quantum analogy of the classical circular orbit spiraling down to the nucleus.

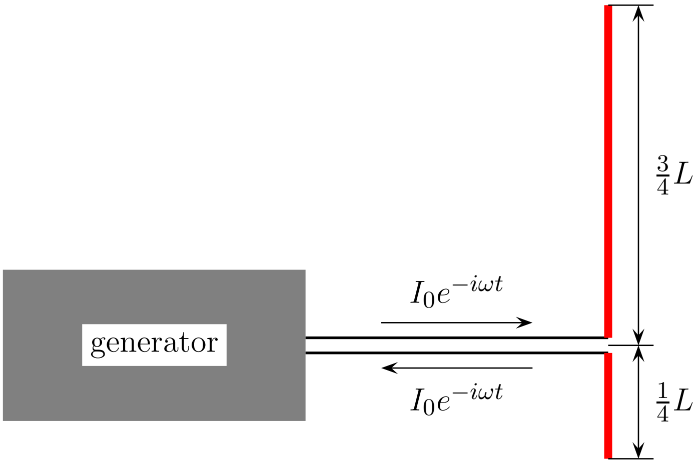
3. A few metastable nuclear isomers have extraordinarily long lifetimes. I have discussed the technetium $\mathrm{Tc}^{99\mathrm{m}}$, $\mathrm{Tc}^{97\mathrm{m}}$, and $\mathrm{Tc}^{95\mathrm{m}}$ isomers in class, but there is also cobalt $\mathrm{Co}^{58\mathrm{m}1}$ (half-life of 9 hours), hafnium $\mathrm{Hf}_{72}^{178\mathrm{m}2}$ (half-life of 31 years), holmium $\mathrm{Ho}_{67}^{166\mathrm{m}1}$ (half-life of 1200 years), and and most remarkably tantalum $\mathrm{Ta}_{73}^{180\mathrm{m}}$ — its half-life is estimated to be so longer than 10^{15} years but nobody have actually seen it decay! Most of these metastable nuclei decays by emission of γ rays or by internal conversion, — the nucleus emits a γ-ray photon but the atomic electrons immediately absorb it and use its energy to kick an electron out. The exception is holmium $\mathrm{Ho}_{67}^{166\mathrm{m}1}$ whose γ-decay rate is so slow that it's preempted by the β-decay to erbium Er^{166}.

   The extraordinarily slow γ-decay rate by all these metastable nuclear states are due to high-order multipoles involved in the transitions. Your task is to find which multipole order is responsible for the γ decays (or internal conversions) of cobalt $\mathrm{Co}^{58\mathrm{m}1}$, hafnium $\mathrm{Hf}_{72}^{178\mathrm{m}2}$, and tantalum $\mathrm{Ta}_{73}^{180\mathrm{m}}$.

   Look up Wikipedia articles *"isotopes of Cobalt", "isotopes of Hafnium",* and *"isotopes of Tantalum"* to find out the angular momenta and the parities of the 3 isomers in question as well as the angular momenta and the parities of the states they decay to. (If you Google up a better source for this information, let me know.) Once you have these data, use the selection rules to find the lowest multipole order — electric or magnetic — allowed for each transition and hence the power of the small (kR) factor suppressing the transition rate.

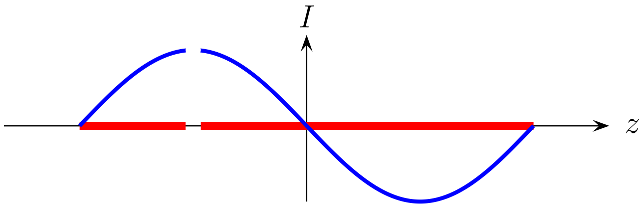
4. Next, a reading assignment for the students who have missed the make-up class on November 15: my notes on long antennas.

5. Consider a linear antenna that's precisely one wavelength long $L = \lambda$. The antenna is fed at a point at distance $L/4$ from one end rather than at in the middle,



For simplicity, approximate the current in the antenna by a sine wave with nodes at both ends, thus

$$I(z) \;=\; -I_0 \sin \frac{2\pi z}{L = \lambda} \qquad (5)$$



Note: this sine wave is different from the current waves in the center-fed antennas, so the radiation pattern of this antenna is quite different from the $L = \lambda$ center-fed antenna discussed in class.

(a) Calculate the $\mathbf{f}(\mathbf{n})$ for this antenna without using the multipole expansion.

(b) Plot the angular dependence of the power (per solid angle) emitted by the antenna in question in the direction $\mathbf{n}$ as a function of the angle θ between that direction and the antenna's axis.

(c) Calculate the net power emitted by the antenna and hence the antenna's radiation resistance. Note: the integral here requires special functions or numeric integration. Don't try to do it by hand but use Mathematica or equivalent software.

Although the antenna in question is too long to trust the multipole expansion, let's use it anyway and see how far off the mark we would get by using just the leading multipoles. In terms of the multipole expansion,

$$\mathbf{f}(\mathbf{n}) = \sum_{m=0}^{\infty} \mathbf{f}_m(\mathbf{n}), \tag{6}$$

$$\mathbf{f}_m(\mathbf{n}) = \frac{(-ik)^m}{4\pi m!} \iiint\limits_{\text{antenna}} d^3\mathbf{y}\, \mathbf{J}(\mathbf{y})\, (\mathbf{n}\cdot\mathbf{y})^m. \tag{7}$$

(d) Use symmetries of the antenna in question to argue that it has zero magnetic multipole moments for all ℓ, while the electric multipole moments vanish for all odd ℓ. Thus, the only multipole moments for this antenna are the electric quadrupole, electric 16-pole, electric 64-pole, *etc.*.

To avoid the messy indexologies of the higher multipole moments, it is easier to directly calculate the $\mathbf{f}_m(\mathbf{n})$ for the antenna in question. In light of part (d), the $\mathbf{f}_m$ should vanish for all even $m = 0, 2, 4, 6, \ldots$.

(e) Verify this, then calculate the three leading non-zero terms $\mathbf{f}_m(\mathbf{n})$ for the odd $m = 1, 3, 5$.

(f) Use successive approximations

$$\begin{aligned} \mathbf{f}_\alpha(\mathbf{n}) &= \mathbf{f}_1(\mathbf{n}), \\ \mathbf{f}_\beta(\mathbf{n}) &= \mathbf{f}_1(\mathbf{n}) + \mathbf{f}_3(\mathbf{n}), \\ \mathbf{f}_\gamma(\mathbf{n}) &= \mathbf{f}_1(\mathbf{n}) + \mathbf{f}_3(\mathbf{n}) + \mathbf{f}_5(\mathbf{n}), \end{aligned} \tag{8}$$

to calculate the $dP/d\Omega$ and the net power emitted by the antenna. Compare the angular power distributions $(dP/d\Omega)_\alpha$, $(dP/d\Omega)_\beta$, and $(dP/d\Omega)_\gamma$ you get from these approximations to the $(dP/d\Omega)$ from part (b), and plot them all on the same graph. Also, compare the net powers P_α, P_β, and P_γ from the α, β, γ approximations to the exact net power from part (c).