1. Let's start with scattering of a plane EM wave from a perfectly conducting sphere of small radius $a \ll$ wavelength λ.

   (a) Because of skin effect, a perfect conductor acts as a perfect diamagnetic to an oscillating magnetic field. Use this fact to show that the incident EM wave induces an oscillating magnetic dipole moment in the sphere with amplitude

   $$\mathbf{m} \;=\; -2\pi a^3 \mathbf{H}_{\rm inc}\,. \tag{1}$$

   (b) Besides the magnetic dipole, the wave also induces an oscillating electric dipole moment

   $$\mathbf{p} \;=\; +4\pi a^3 \epsilon_0 \mathbf{E}_{\rm inc}\,. \tag{2}$$

   Verify this formula, then show that the electric and the magnetic dipole moments are related to each other as

   $$\frac{\mathbf{m}}{c} \;=\; -\frac{1}{2}\,\mathbf{n}_0 \times \mathbf{p} \tag{3}$$

   (c) Calculate $\mathbf{f}(\mathbf{n})$ due to combined electric and magnetic dipoles and hence the EM fields $\mathbf{E}_{\rm sc}$ and $\mathbf{H}_{\rm sc}$ of the scattered wave in the far zone.

   (d) Derive the polarized partial cross-section for scattering from the conducting sphere. Show that for general polarizations of the incident and the scattered waves

   $$\frac{d\sigma(\mathbf{n}_0, \mathbf{e}_0 \to \mathbf{n}, \mathbf{E})}{d\Omega} \;=\; k^4 a^6 \times \left|\mathbf{e}^* \cdot \mathbf{e}_0 \;-\; \tfrac{1}{2}(\mathbf{n} \times \mathbf{e}^*) \cdot (\mathbf{n}_0 \times \mathbf{e}_0)\right|^2. \tag{4}$$

   In particular, for the linear polarizations $\perp$ and $\|$ to the scattering plane,

   $$\begin{aligned}
   \frac{d\sigma(\perp\to\perp)}{d\Omega} &= k^4 a^6 \times \left(1 - \tfrac{1}{2}\cos\theta\right)^2,\\
   \frac{d\sigma(\perp\to\|)}{d\Omega} &= 0,\\
   \frac{d\sigma(\|\to\perp)}{d\Omega} &= 0,\\
   \frac{d\sigma(\|\to\|)}{d\Omega} &= k^4 a^6 \times \left(\tfrac{1}{2} - \cos\theta\right)^2.
   \end{aligned} \tag{5}$$

   (e) Calculate the un-polarized partial cross-section as a function of scattering angle θ. Note that unlike in the dielectric sphere example explained in class, the scattering off a conducting sphere does not have a forward-backward symmetry $\theta \to \pi - \theta$.

Also, calculate the polarization degree $\Pi(\theta)$ of the scattered EM wave for the unpolarized incident wave.

(f) Finally, calculate the net scattering cross-section and the forward-backward asymmetry

$$A = \frac{\sigma(\theta < 90^\circ) - \sigma(\theta > 90^\circ)}{\sigma(\theta < 90^\circ) + \sigma(\theta > 90^\circ)}. \tag{6}$$

2. Next, consider a spherical vacuum-filled microwave resonator cavity of radius R.

(a) Argue that the resonant modes of such a cavity are the $\mathrm{TE}_{\ell,m}$ and the $\mathrm{TM}_{\ell,m}$ waves described in my notes on spherical waves, and the EM fields $\mathbf{E}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ of these modes should be exactly as in eqs. (152) through (156) on pages 24–25 of my notes, except that the $g_\ell(kr)$ radial functions in those equations should be replaced with the *regular* spherical Bessel functions $j_\ell(kr)$.

Note: 'argue' does not mean 're-derive from scratch'!

(b) Suppose the surface of the spherical cavity is perfectly conducting. Apply the boundary conditions at that surface to the TE and the TM waves and show that they resonate at frequencies for which

$$j_\ell(x) = 0 \text{ @ } x = kR \quad \text{for a TE}_\ell \text{ wave,} \tag{7.a}$$

$$y j_\ell'(y) + j_\ell(y) = 0 \text{ @ } y = kR \quad \text{for a TM}_\ell \text{ wave.} \tag{7.b}$$

In other words, the resonant frequencies are

$$\omega_n(\mathrm{TE}_\ell) = \frac{c}{R} \times x_{\ell,n}, \qquad \omega_n(\mathrm{TM}_\ell) = \frac{c}{R} \times y_{\ell,n}, \tag{8}$$

where $x_{\ell,n}$ is the n^{th} positive zero of $j_\ell(x)$ while $y_{\ell,n}$ is the n^{th} positive zero of

$$F_\ell(y) = y j_\ell'(y) + j_\ell(y) = \frac{d}{dy}\big(y j_\ell(y)\big). \tag{9}$$

(c) Use Mathematica to find the 4 lowest frequencies numerically (in units of c/R). Also, state which modes these frequencies belong to.

(d) Now suppose the outer wall of the spherical cavity has a small surface resistivity R_s. Calculate the quality factor Q of the spherical resonator for all the modes and show that the TE modes have

$$Q = \frac{Z_0}{2R_s} \times (x_{\ell,n} = \omega R/c), \tag{10}$$

while the TM modes have

$$Q = \frac{Z_0}{2R_s} \times \left(y_{\ell,n} - \frac{\ell(\ell+1)}{y_{\ell,n}} \right) = \frac{Z_0}{2R_s} \times \left((\omega R/c) - \frac{\ell(\ell+1)}{(\omega R/c)} \right). \tag{11}$$

Math help: spherical Bessel functions obey all kinds of rather obscure identities. In particular, here are a couple of integral identities you need for this problem:

(i) For any $X = x_{\ell,n}$ such that $j_\ell(X) = 0$,

$$\int_0^X dx\, x^2 \left(j_\ell(x)\right)^2 = \frac{X^3}{2} \times \left(j'_\ell(X)\right)^2. \tag{12}$$

(ii) For any $Y = y_{\ell,n}$ such that $Y j'_\ell(Y) + j_\ell(Y) = 0$,

$$\int_0^Y dx\, x^2 \left(j_\ell(x)\right)^2 = \frac{Y}{2}\left(Y^2 - \ell(\ell+1)\right) \times \left(j_\ell(Y)\right)^2. \tag{13}$$

3. Finally, a reading assignment: §10.3–4 of the Jacksons textbook about partial wave analysis of EM waves, and also §10.11 about the optical theorem for the EM waves.