BOUNDARY PROBLEMS IN ELECTROSTATICS

The basic differential equations of electrostatics are rather simple: The *static* electric field $\mathbf{E}(\mathbf{x})$ — where $\mathbf{x}$ is the radius vector (x, y, z) — has zero curl,

$$\nabla \times \mathbf{E}(\mathbf{x}) \;=\; 0 \quad \forall \mathbf{x}, \tag{1}$$

while its divergence obeys the Gauss Law

$$\nabla \cdot \mathbf{E}(\mathbf{x}) \;=\; \frac{1}{\epsilon_0}\rho(\mathbf{x}). \tag{2}$$

Thanks to the zero-curl equation, the electric field is a gradient of a scalar field, namely (minus) the electric potential,

$$\mathbf{E}(\mathbf{x}) \;=\; -\nabla\Phi(\mathbf{x}). \tag{3}$$

In terms of this potential, the zero-curl equation is automatic, while the Gauss Law becomes the Poisson equation

$$\triangle\Phi(\mathbf{x}) \;=\; \frac{-1}{\epsilon_0}\rho(\mathbf{x}), \tag{4}$$

where $\triangle$ is the *Laplacian* — the Laplace operator —

$$\triangle \;=\; \nabla^2 \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,. \tag{5}$$

Naively, the Poisson equation (4) for a general electric charge distribution $\rho(\mathbf{x})$ has a simple general solution — the Coulomb potential

$$\Phi(\mathbf{x}) \;=\; \iiint \frac{\rho(\mathbf{y})}{4\pi\epsilon_0\,|\mathbf{x}-\mathbf{y}|}\,d^3\mathbf{y}\,. \tag{6}$$

Alas, this is true only when we know the charge distribution $\rho(\mathbf{y})$ throughout the entire 3D space: Given this information, we may indeed use the Coulomb integral (6) to obtain the potential $\Phi(\mathbf{x})$, also throughout the entire 3D space.

Much more commonly, we know the electric charges and their density $\rho(\mathbf{y})$ only inside some limited volume $\mathcal{V}$, but not outside that volume or even on its surface $\mathcal{S}$. Naturally, with this limited knowledge we cannot possibly find the potential $\Phi(\mathbf{x})$ outside the volume $\mathcal{V}$, but even to find the potential inside $\mathcal{V}$ we need additional information. Specifically, we need *boundary conditions* for the potential $\Phi(\mathbf{x})$ for $\mathbf{x} \in \mathcal{S}$, on the surface of the volume $\mathcal{V}$ in question.

There are many different kinds of boundary conditions for different kinds of problems. Here are some examples:

1. Dirichlet and Dirichlet-like boundary conditions:

   - The Dirichlet condition,

   $$\Phi(\mathbf{x}) \;=\; 0 \quad \text{at all } \mathbf{x} \in \mathcal{S}. \tag{7}$$

   Physically, this condition obtains when the boundary is a grounded conducting surface.

   - When the boundary surface is conducting but not grounded, we have a modified Dirichlet condition

   $$@\mathbf{x} \in \mathcal{S}: \quad \Phi(\mathbf{x}) \;=\; \text{const} \;\neq\; 0. \tag{8}$$

   - More generally, we may happen to know the potential along the surface but not inside the volume in question. In this case, the boundary condition becomes

   $$@\mathbf{x} \in \mathcal{S}: \quad \Phi(\mathbf{x}) \;=\; \text{given } \Phi_b(\mathbf{x}). \tag{9}$$

   We shall see examples of such boundary conditions in the next couple of lectures.

2. Neumann or Neumann-like boundary conditions:

   ◦ The Neumann boundary condition, zero normal derivative of the potential at the boundary,

$$@\mathbf{x}\in\mathcal{S}:\quad \frac{\partial\Phi}{\partial x_n} \;=\; 0 \tag{10}$$

where x_n is the local coordinate $\perp$ to the boundary $\mathcal{S}$. Or in vector notations,

$$@\mathbf{x}\in\mathcal{S}:\quad \mathbf{n}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) \;=\; 0 \tag{11}$$

where $\mathbf{n}(\mathbf{x})$ is the unit vector $\perp\mathcal{S}$ at the point $\mathbf{x}$.

- ○ More generally, we may happen to know the normal component of the electric field at the boundary. In this case, we get a generalized Neumann boundary condition

$$@\mathbf{x}\in\mathcal{S}:\quad \mathbf{n}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) \;=\; \text{given } E_b^n(\mathbf{x}). \tag{12}$$

3. For volumes $\mathcal{V}$ which extend to infinity in *some* directions, we need asymptotic conditions on the potential's behavior for $\mathbf{x}\to\infty$ (within $\mathcal{V}$).

   ∗ Most commonly, we want the potential to asymptote to zero at the infinity,

$$\Phi(\mathbf{x}) \;\to\; 0 \quad\text{for}\quad \mathbf{x}\to 0 \text{ (within } \mathcal{V}). \tag{13}$$

   ∗ But sometimes, we want the electric field to asymptote to a given external field $\mathbf{E}_e(\mathbf{x})$, thus

$$\mathbf{E}(\mathbf{x}) \;\to\; \text{given } \mathbf{E}_e(\mathbf{x}) \quad\text{for}\quad \mathbf{x}\to 0 \text{ (within } \mathcal{V}). \tag{14}$$

4. In many coordinate systems — such as polar coordinates in 2D and spherical or cylindrical coordinates in 3D — there are pseudo-boundary conditions stemming from the coordinate singularities or periodicity conditions. Mathematically, such pseudo-boundary conditions act very similarly to the real boundary conditions.

   ⊙ In particular, the potential must be a periodic function of the angular coordinate ϕ in spherical coordinates:

$$\Phi(r,\theta,\phi+2\pi) \;=\; \Phi(r,\theta,\phi) \tag{15}$$

   and likewise for the cylindrical coordinates or polar coordinates.

⊙ Unless there is an actual point charge at the origin of a spherical coordinate system, the potential should have a finite limit for $\mathbf{x} \to 0$, same in any direction:

$$\text{finite } \lim_{r\to 0} \Phi(r,\theta,\phi), \quad \text{same } \forall\theta,\phi. \tag{16}$$

Likewise, unless there is a line charge along the axis of the cylindrical coordinate system, we should have

$$\lim_{\rho\to 0} \Phi(\rho,\phi,z) \;=\; \text{finite } \Phi_0(z), \quad \text{same } \forall\phi. \tag{17}$$

⊙ Similarly, in the spherical coordinates, the potential $\Phi(r,\theta,\phi)$ should not have any singularities at the spherical poles $\theta = 0$ or $\theta = \pi$ unless there are actual point or line charges at those locations.

5. Finally, the surface $\mathcal{S}$ of the volume $\mathcal{V}$ in question may have several distinct parts, with different kinds of boundary conditions for different parts. We shall see examples of such mixed boundary conditions in our next lecture.

There is no general method for solving the electrostatic boundary problems! Instead, there is a collection of methods, each method suitable for a limited type of problems. In this class, I shall cover 3 methods: (1) The *image charge* method (briefly); (2) the *separation of variables* method; and (3) the *Green's function* method.

Image Charge Method

The image charge method works for a few particularly simple geometries of the volume $\mathcal{V}$ in question, namely a half-space with a conducting boundary, outside a conducting sphere, or inside a conducting spherical cavity. This method is explained in most undergraduate-level textbooks as well as in Jackson §2.1–5 (pages 57–64), so let me be brief in these notes.

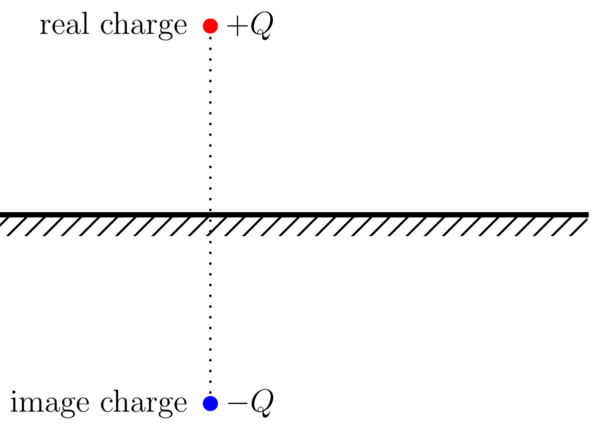
Consider a half-space above a grounded conducting plane at $z = 0$. The potential in this half-space is subject to the Dirichlet boundary condition at $z = 0$,

$$\Phi(x,y,z=0) \;=\; 0 \tag{18}$$

as well as the asymptotic condition

$$\Phi(\mathbf{x}) \ \to \ 0 \quad \text{for } |\mathbf{x}| \to \infty \text{ in any direction above the ground.} \tag{19}$$

Then the potential $\Phi(x,y,z)$ of a single point charge Q in this half space is the same as the Coulomb potential of two charges: Q itself, and an image charge $-Q$ located at the mirror image of the real Q mirror-reflected from the $z=0$ plane:



Specifically, for the real charge located at $(0,0,+a)$ — and hence the image charge at $(0,0,-a)$ — the potential above the ground plane — and only above the ground plane — is

$$\Phi(x,y,z) \ = \ \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+a)^2}}\right). \tag{20}$$

It's easy to see that this potential obeys both boundary conditions: For $\mathbf{x}=(x,y,z)\to\infty$ (in any direction) both terms in the potential (20) asymptote to zero, while for $z=0$ the two terms cancel each other thus $\Phi=0$. As to the Poisson equation, formally

$$-\epsilon_0\triangle\Phi(x,y,z) \ = \ +Q\delta(x)\delta(y)\delta(z-a) \ - \ Q\delta(x)\delta(y)\delta(z+a). \tag{21}$$

However, for (x,y,z) restricted to the upper half-space $z>0$, the second term here becomes an identical zero since $\delta(z+a)\equiv 0$ for $z>0$. Consequently, *for* (x,y,z) *restricted to the upper half-space*

$$-\epsilon_0\triangle\Phi(x,y,z) \ = \ +Q\delta(x)\delta(y)\delta(z-a) \ = \ \rho_{\text{real}}(x,y,z), \tag{22}$$

so the potential (20) indeed obeys the Poisson equation.

By linearity, for any other kind of potential distribution $\rho(x,y,z)$ in the upper half-space, the potential in that half space obtain as the Coulomb potential of combined real and image charges,

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\iiint\limits_{z'>0} d^3\mathbf{x}'\,\rho(\mathbf{x}')\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}-\frac{1}{|\mathbf{x}-\mathrm{image}(\mathbf{x}')|}\right)\\ \text{where}\quad \mathrm{image}(x',y',z') &= (+x',+y',-z').\end{aligned} \tag{23}$$

This pretty much summarizes the image charge method for a half-space. Similar methods work for the spaces outside or inside of a conducting spherical surface. Please look them up in Jackson's textbook, §2.2–5.