POLARIZED SCATTERING OF EM WAVES

In a polarized scattering problem, we take the incident wave to be 100% polarized and also detect the polarization of the scattered wave. Specifically, let the incident wave be a plane wave in direction $\mathbf{n}_0$ polarized in direction $\mathbf{e}_0 \perp \mathbf{n}_0$, thus

$$\mathbf{E}_{\rm inc}(\mathbf{x},t) \;=\; E_0\mathbf{e}_0 \exp(ik\mathbf{n}_0\cdot\mathbf{x} - i\omega t). \tag{1}$$

This wave creates harmonic current $\mathbf{J}(\mathbf{x})e^{-i\omega t}$ in the scattering body, which in turn radiates the scattered wave. In the far zone of this scattered wave, the electric field is

$$\mathbf{E}_{\rm sc}(\mathbf{x},t) \;=\; -ikZ_0E_0\,\frac{e^{ikr-i\omega t}}{r}\,\big(\mathbf{n}_x\times(\mathbf{n}_x\times\mathbf{F})\big) \tag{2}$$

where

$$\mathbf{F}(\mathbf{n}_0,\mathbf{e}_0;\mathbf{n}_x) \;=\; \frac{\mu_0}{4\pi E_0}\int\! d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\exp(ik\mathbf{n}_x\cdot\mathbf{y}). \tag{3}$$

Projecting the scattered electric field onto a particular polarization $\mathbf{e}\perp\mathbf{n}_x$, we have

$$E_{\rm sc}[\mathbf{e}] \;=\; \mathbf{e}^*\cdot\mathbf{E}_{\rm sc} \;=\; -ikZ_0E_0\,\frac{e^{ikr-i\omega t}}{r}\,\big(\mathbf{e}^*\cdot(\mathbf{n}\times(\mathbf{n}\times\mathbf{F}))\big), \tag{4}$$

where

$$\mathbf{e}^*\cdot(\mathbf{n}\times(\mathbf{n}\times\mathbf{F})) \;=\; (\mathbf{n}\times\mathbf{F})\cdot(\mathbf{e}^*\times\mathbf{n}) \;=\; (\mathbf{n}\cdot\mathbf{e}^*)(\mathbf{F}\cdot\mathbf{n}) \;-\; \mathbf{n}^2(\mathbf{F}\cdot\mathbf{e}^*) \;=\; 0 \;-\; \mathbf{e}^*\cdot\mathbf{F}, \tag{5}$$

thus

$$E_{\rm sc}[\mathbf{e}] \;=\; +ikZ_0E_0\,\frac{e^{ikr-i\omega t}}{r}\,(\mathbf{e}^*\cdot\mathbf{F}). \tag{6}$$

Consequently, comparing the scattered power in the direction $\mathbf{n}$ and polarization $\mathbf{e}$

$$\frac{dP(\mathbf{n},\mathbf{e})}{d\Omega} \;=\; r^2\times\frac{|E_{\rm sc}(\mathbf{n},\mathbf{e})|^2}{2Z_0} \tag{7}$$

to the incident power flux $S_{\rm inc} = |E_0|^2/2Z_0$, we find the polarized scattering cross-section to be

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\mathbf{n}_0, \mathbf{e}_0 \to \mathbf{n}, \mathbf{e}) &= \frac{1}{S_{\text{inc}}}\frac{dP}{d\Omega} = \frac{r^2 ||E_{\text{sc}}(\mathbf{n}, \mathbf{e})|^2}{|E_0|^2} \\ &= k^2 Z_0^2 \left|\mathbf{e}^* \cdot \mathbf{F}(\mathbf{n}_0, \mathbf{e}_0; \mathbf{n})\right|^2.\end{aligned} \tag{8}$$

For example, consider scattering off a small dielectric sphere of radius $a \ll \lambda$. The incident wave's electric field is approximately uniform over such a small sphere, so it induces in the sphere the electric dipole moment

$$\mathbf{p} \approx \alpha \mathbf{E}(\text{center}) = \alpha E_0 \mathbf{e}_0 e^{-i\omega t} \tag{9}$$

where α is the sphere's polarizability,

$$\alpha = 4\pi a^3 \times \frac{\epsilon - 1}{\epsilon + 2}\,\epsilon_0\,. \tag{10}$$

Neglecting the higher electric and magnetic multipole moments compared to this dipole moment, we have

$$E_0 \mathbf{F} = \mathbf{f} \approx \mathbf{f}_{\text{el.dipole}} = -\frac{i\omega}{4\pi}\mathbf{p}_{\text{amplitude}} = -\frac{i\omega}{4\pi}\alpha E_0 \mathbf{e}_0\,, \tag{11}$$

hence

$$\mathbf{F} = -i\frac{\omega\alpha}{4\pi}\,\mathbf{e}_0\,, \tag{12}$$

and therefore

$$\frac{d\sigma}{d\Omega}(\mathbf{n}_0, \mathbf{e}_0 \to \mathbf{n}, \mathbf{e}) = \left(\frac{k Z_0 \omega\alpha}{4\pi}\right)^2 \left|\mathbf{e}^* \cdot \mathbf{e}_0\right| \tag{13}$$

Note that this polarized scattering cross-section depends on the polarization vectors of the incident and scattering waves but seems to be independent of their directions $\mathbf{n}_0$ and $\mathbf{n}$; but there implicit dependence on the two wave's directions through the constraints $\mathbf{e}_0 \perp \mathbf{n}_0$ and $\mathbf{e} \perp \mathbf{n}$.

As to the overall coefficient of the scattering cross-section, for the dielectric sphere

$$\frac{kZ_0\omega\alpha}{4\pi} = k^2\times cZ_0\times\left(\frac{\alpha}{4\pi} = a^3\frac{\epsilon-1}{\epsilon+2}\epsilon_0\right) = k^2a^3\,\frac{\epsilon-1}{\epsilon+2}\times\bigl(cZ_0\epsilon_0 = 1\bigr) = k^2a^3\,\frac{\epsilon-1}{\epsilon+2}\,, \tag{14}$$

thus polarized scattering cross-section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{polarized}} = k^4a^6\left(\frac{\epsilon-1}{\epsilon+2}\right)^2\times\left|\mathbf{e}^*\cdot\mathbf{e}_0\right|^2. \tag{15}$$

Now let's work out the polarized cross-sections in the specific basis of planar polarizations: one polarization tangent to the scattering plane and the other normal to that plane, where the scattering plane is the plane spanning the directions $\mathbf{n}_0$ and $\mathbf{n}$ of both the incident and the scattered waves. Without loss of generality, let the incident wave run in z direction while the scattered wave run in the (zx) plane at angle θ to the incident wave. Then in this coordinate system

$$\begin{aligned}
\mathbf{n}_0 &= (0,0,1), & \mathbf{n} &= (\sin\theta, 0, \cos\theta),\\
\mathbf{e}_0(\|) &= (1,0,0), & \mathbf{e}(\|) &= (\cos\theta, 0, -\sin\theta),\\
\mathbf{e}_0(\perp) &= (0,1,0), & \mathbf{e}(\perp) &= (0,1,0),
\end{aligned} \tag{16}$$

and therefore

$$\left|\mathbf{e}^*\cdot\mathbf{e}_0\right|^2 = \begin{cases} \cos^2\theta & \text{for } \|\to\|,\\ 0 & \text{for } \|\to\perp,\\ 0 & \text{for } \perp\to\|,\\ 1 & \text{for } \perp\to\perp. \end{cases} \tag{17}$$

Consequently, the polarized partial scattering cross-sections (15) come out to be

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)(\|\to\perp) &= \left(\frac{d\sigma}{d\Omega}\right)(\perp\to\|) = 0,\\
\left(\frac{d\sigma}{d\Omega}\right)(\|\to\|) &= k^4a^6\left(\frac{\epsilon-1}{\epsilon+2}\right)^2\times\cos^2\theta,\\
\text{and}\quad\left(\frac{d\sigma}{d\Omega}\right)(\perp\to\perp) &= k^4a^6\left(\frac{\epsilon-1}{\epsilon+2}\right)^2 \qquad \langle\!\langle\,\text{isotropic!}\,\rangle\!\rangle.
\end{aligned} \tag{18}$$

What about the un-polarized cross-sections? Suppose the detector of the scattered wave measure only its total intensity and is blind to the wave's polarization. At the same time, the incident wave is un-polarized, meaning: 50% of its power belong to one polarization and 50% to the other polarization. In this case, we measure the un-polarized partial cross-section which obtains by summing the polarized cross-sections over the scattered wave's polarizations and averaging over the incident wave's polarizations,

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} &= \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)(\|\!\to \text{any}) + \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)(\perp\!\to \text{any}) \\ &= \frac{1}{2}\left(\left(\frac{d\sigma}{d\Omega}\right)(\|\!\to\!\|) + \left(\frac{d\sigma}{d\Omega}\right)(\|\!\to\!\perp) + \left(\frac{d\sigma}{d\Omega}\right)(\perp\!\to\!\|) + \left(\frac{d\sigma}{d\Omega}\right)(\perp\!\to\!\perp)\right) \\ &= \frac{1}{2}\times k^4a^6\left(\frac{\epsilon-1}{\epsilon+2}\right)^2\times\left(\cos^2\theta + 0 + 0 + 1\right) \\ &= k^4a^6\left(\frac{\epsilon-1}{\epsilon+2}\right)^2\times\frac{1+\cos^2\theta}{2}.\end{aligned} \tag{19}$$

And the total scattering cross-section obtains by integrating this partial cross-section over the 4π directions of the scattered waves,

$$\sigma_{\text{tot}} = \int d^2\Omega\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpolarized}} = k^4a^6\left(\frac{\epsilon-1}{\epsilon+2}\right)^2\times\frac{8\pi}{3}. \tag{20}$$

Finally, suppose the incident EM wave is unpolarized, but the detector of the scattered wave is sensitive to its polarization. In this case, the detector will show the scattered wave to be partially polarized, because one polarization of the incident wave scatters stronger than the other. In terms of planar polarizations $\|$ and $\perp$ to the plane of scattering, the *degree of polarization* for the scattered wave (in a particular direction) is

$$\Pi(\theta) = \frac{dP_\perp - dP_\|}{dP_\perp + dP_\|} = \frac{d\sigma(\perp) - d\sigma(\|)}{d\sigma(\perp) + d\sigma(\|)}, \tag{21}$$

which for the small dielectric ball evaluates to

$$\Pi(\theta) = \frac{1-\cos^2\theta}{1+\cos^2\theta}. \tag{22}$$

Note the direction dependence of this degree of polarization: the wave scattered backward or forward is unpolarized ($\Pi = 0$) while the wave scattered at 90° angle is 100% polarized $\perp$ to the plane of scattering.