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Electrostatic Energy

2 point charges

$$U_{pot} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}} \qquad R_{12} = |\vec{x}_1 - \vec{x}_2|$$

N point charges

$$U = \sum_{j=2}^{N} Q_j \times \sum_{i=1}^{j-1} \frac{Q_i}{4\pi\varepsilon_0 R_{ij}}$$

$$= \frac{1}{2} \sum_{i \neq j} \frac{Q_i Q_j}{4\pi\varepsilon_0 R_{ij}}$$

compensates for double counting each (i,j) pair

$$U = \frac{1}{2} \sum_j Q_j \times \sum_{i \neq j} \frac{Q_i}{4\pi\varepsilon_0 R_{ij}} = \frac{1}{2} \sum_j Q_j \times \Phi'(\vec{x}_j)$$

$\Phi'(\vec{x}_j)$ is potential due to all <u>other</u> charges $Q_{i \neq j}$

U of assembling a continuous charge $\rho(\vec{x})$

$$U = \frac{1}{2} \iiint d^3\vec{x}\, \rho(\vec{x})\, \Phi'(\vec{x})$$

or $\frac{1}{2} \iint d^2\text{area}\, \sigma(x)\, \Phi'(\vec{x})$

for a surface charge.

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For volume & surface charges
(but not line or point charges)

$$\phi'(\vec{x}) = \phi(\vec{x}).$$

$$\delta\phi(\vec{x}) = \phi(\vec{x}) - \phi'(\vec{x})$$

is ϕ due to charge ρd^3x itself

$$\delta\phi \sim \frac{\text{charge}}{\text{size}} \sim \frac{\rho\,(\text{size})^3}{\text{size}} \sim \rho\,(\text{size})^2$$

$\to 0$ for infinitesimal size.

Ditto, for ~~sur~~ surface $dQ = \sigma\, d^2\text{area}$

$$\delta\phi \sim \frac{\text{charge}}{\text{size}} \approx \frac{\sigma\,(\text{size})^2}{\text{size}} = \sigma\,(\text{size}) \to 0.$$

$$\boxed{U = \frac{1}{2}\iiint \rho(\vec{x})\,\phi(\vec{x})\,d^3x}$$

In vacuum $\rho = \epsilon_0\,\vec{\nabla}\cdot\vec{E}$

$$U = \frac{\epsilon_0}{2}\iiint (\vec{\nabla}\cdot E)\,\phi\, d^3x$$

$$= \frac{\epsilon_0}{2}\oint_{\text{surface}} \phi\,\vec{E}\,d^2\vec{\text{area}} + \frac{\epsilon_0}{2}\iiint \vec{E}\cdot(\underbrace{-\vec{\nabla}\phi}_{=\vec{E}})\,d^3\vec{x}$$

$$U = \frac{\epsilon_0}{2}\iiint_{\text{vol}} \vec{E}^2 d^3x + \frac{\epsilon_0}{2}\oint_{\text{surface}} \phi\,\vec{E}\,d^2\vec{\text{area}}$$

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volume → whole space.

volume = large ball, rad $R \to \infty$

$$\phi \sim \frac{Q_{net}}{4\pi\epsilon_0 R}$$

$$\bar{E}\cdot\hat{n} = \frac{Q_{net}}{4\pi\epsilon_0 R^2}$$

$$\text{Area} = 4\pi R^2$$

$$\oiint_{\text{surface}} \phi \vec{E}\cdot d\vec{\text{area}} = \frac{Q_{net}^2}{4\pi\epsilon_0^2} \times \frac{1}{R} \xrightarrow[R\to\infty]{} 0$$

$$\boxed{U_{net} = \frac{\epsilon_0}{2} \iiint_{\substack{\text{whole}\\ \text{space}}} \bar{E}^2 d^3x}$$

$U_{net} > 0$, always.

compare to pot. energy of 2 charges

$$U_{pot} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}}$$: may have either sign

U_{pot} may turn negative because it has different $U=0$ baseline

there $U_{net} = \frac{\epsilon_0}{2}\iiint \bar{E}^2 d^3\bar{x}$.

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$U_{potential}$ for point charges

~~is the ext~~ accts for the motion of the charges; its the work of bringing them in from ∞

but it does not account for assembling the individual charges.

OOH, U_{tot} starts with all charges dispersed to ∞ in all directions.

→ U_{tot} includes the work of assembling each individual charge.

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2 compact charges Q_1, Q_2

$$\Rightarrow \vec{E} = \vec{E}_1 + \vec{E}_2$$

$$U_{tot} = \frac{\epsilon_0}{2}\iiint(\vec{E}_1+\vec{E}_2)^2 d^3x$$

$$= \frac{\epsilon_0}{2}\iiint \vec{E}_1^2 d^3x + \frac{\epsilon_0}{2}\iiint \vec{E}_2^2 d^3x$$

$$+ \epsilon_0 \iiint \vec{E}_1\cdot\vec{E}_2\, d^3x$$

$$= U_1 + U_2 + U_{int}.$$

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U_1 : Energy of Assembling Q_1

U_2 : " — " — " — " Q_2

U_{int} : Energy of their relative motion

potential

For 2 point charges, $U_{int} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}}$

$$U_{int} = \epsilon_0 \iiint \vec{E}_1 \cdot (\vec{E}_2 = -\vec{\nabla}\Phi_2)\, d^3x$$

$$\text{by parts} = + \iiint \Phi_2 (\epsilon_0 \vec{\nabla}\cdot\vec{E}_1)\, d^3x$$

$$= + \iiint \Phi_2(x)\, \rho_1(x)\, d^3x$$

For point Q_1 $= Q_1 \Phi_2(\vec{x}_1) = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}}$

Electric energy in a Dielectric.

Baseline : All dielectric in place.

But no free charges

: work for moving the free

charges in from ∞.

charges move by themselves

8/6)

Move a ~~dist~~ free charge Q thru $\delta\vec{x}$

work $Q_{free} \vec{E}(\vec{x}) \cdot \delta \vec{x}$

↳ E from all ~~sep~~ sources free & bound.

Dotto, change $\rho_{free}(\vec{x})$ by $\delta\rho_{free}(\vec{x})$

work $\delta w = \iiint d^3\vec{x}\, \delta\rho_{free}(\vec{x})\, \Phi(\vec{x})$

from all sources

Assume a linear dielectric

(or a system of several linear dielectrics)

$\Phi(\vec{x}) = \iiint d^3y\, G(\vec{x},\vec{y})\, \rho_f(\vec{y})$

for some complicated $G(\vec{x},\vec{y})$

G does not depend on ρ_{fre}

$\delta w = \iiint d^3\vec{x} \iiint d^3\vec{y}\; G(\vec{x},\vec{y})\, \delta\rho_f(\vec{x})\, \rho_f(\vec{y})$

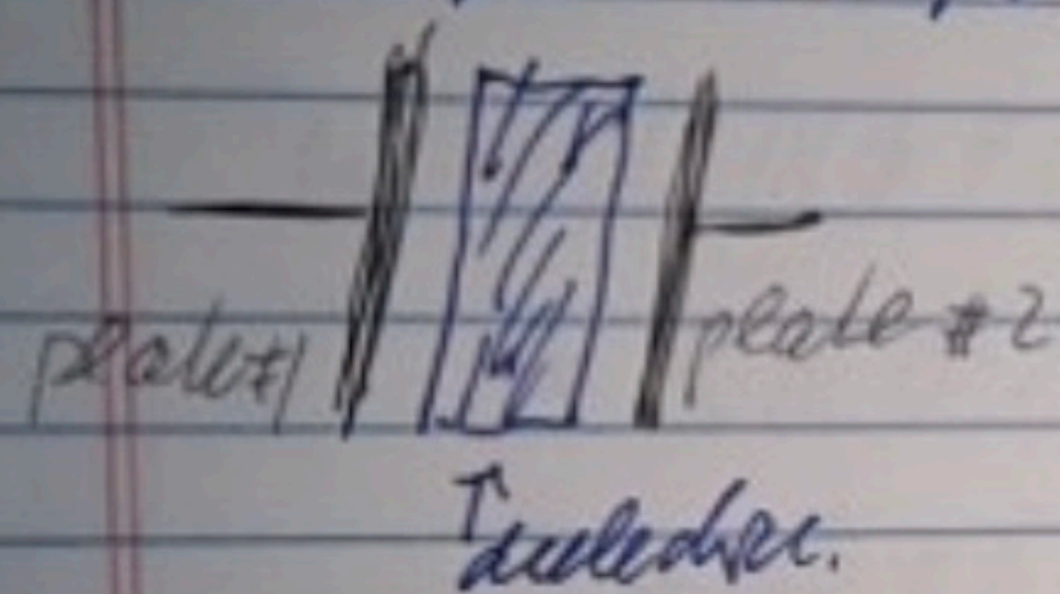
δu

$\iiint d^3y\; G(\vec{x},\vec{y})\, \rho_f(\vec{x})\, \rho_f(\vec{y})$

8/7 $\boxed{U = \frac{1}{2}\iiint d^3x\, \rho_f(\vec{x})\, \phi(\vec{x})}$

Only for linear dielectrics

Example: A capacitor



$Q_1 = +Q$
$Q_2 = -Q$.

Free charges are only on the plates

Also, each plate has uniform ϕ.

but $\phi_1 \neq \phi_2$

$\phi_1 - \phi_2 = V =$ voltage.

$$U = \frac{1}{2}\iiint dQ\, \phi = \frac{1}{2}(Q_1\phi_1 + Q_2\phi_2)$$

$$= \frac{1}{2}(Q\phi_1 - Q\phi_2) = \frac{1}{2}QV.$$

$Q = C \times V \quad C =$ capacitance.

$$\boxed{\frac{1}{2}QV = \frac{C}{2}V^2 = \frac{1}{2C}Q^2}$$

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General Linear Dielectric

$$U = \frac{1}{2} \iiint \Phi \, \rho_{free} \, d^3x$$

$$\rho_{free} = \vec{\nabla} \cdot \vec{D}$$

$$U = \frac{1}{2} \iiint \Phi \, \vec{\nabla} \cdot \vec{D} \, d^3x = \text{by parts}$$

$$= \frac{1}{2} \iiint \left(\vec{D} \cdot \left(-\nabla \Phi = \vec{E} \right) \right) d^3x$$

$$+ \underbrace{\frac{1}{2} \oiint \Phi \, \vec{D} \cdot d^2\text{area}}_{\to \infty \text{ for } \iiint \text{ over whole space}}$$

$$\boxed{U = \frac{1}{2} \iiint_{\text{whole space}} \vec{E} \cdot \vec{D} \, d^3x}$$

$$\boxed{U = \frac{\epsilon_0}{2} \iiint_{\text{whole space}} \epsilon(\vec{x}) \, \vec{E}^2 \, d^3x}$$

... vacuum $\epsilon = 1$

8/9 Assume non-linear dielectric

$$\delta W = \iiint d^3x \, \delta\rho_f(\vec{x}) \phi(\vec{x})$$

$$= \iiint_{\substack{\text{whole} \\ \text{space}}} d^3x \, \delta\vec{D}(\vec{x}) \cdot \vec{E}(\vec{x}).$$

If $D(\vec{x})$ depends only on $\vec{E}(\vec{x})$
& not on E elsewhere or
long history of $\vec{E}$.

Then $\vec{E} \cdot \delta\vec{D}(\vec{E}) = \delta f(\vec{E})$
for some $f(\vec{E})$.

$$\Rightarrow U = \iiint d^3\vec{x} \, f(\vec{E})$$

Linear case $\vec{D} = \epsilon\epsilon_0\vec{E}$

$$\vec{E} \cdot \delta\vec{D} = \tfrac{1}{2}\epsilon\epsilon_0 \, \delta(\vec{E}^2)$$

$$= \tfrac{1}{2}\delta(\vec{E}\cdot\vec{D}) \rightarrow f(\vec{E})$$

$$\iiint d^3x \, \tfrac{1}{2}\vec{D}\cdot\vec{E}$$

...e: different $f(\vec{E}) \neq \frac{\vec{D}\cdot\vec{E}}{2}$

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If Hysteresis, $\vec{D}$ depends on history of $\vec{E}$

Then over a <u>closed cycle</u>

$$\oint \vec{E} \cdot \delta \vec{D} \neq 0.$$

$\Rightarrow$ net work (closed cycle) $\neq 0$

$\rightarrow$ energy loss.