Separation of Variables in Polar and Spherical Coordinates

Polar Coordinates

Consider an infinitely long cylindrical cavity of radius R. There may be some charges outside the cavity and/or on its surface, but there are no charges inside the cavity, so the potential obeys $\triangle V(s,\phi,z)=0$ for $s<R$. Suppose we are given the potential $V_b(\phi,z)$ on the cavity's surface, and we need to find the potential everywhere inside the cavity. For simplicity, suppose the boundary potential depends only on the angular coordinate ϕ and not on the z coordinate along the cylinder, $V_b(\phi,z)=V_b(\phi\text{ only})$, so the potential inside the cylinder should also be independent on z, $V(s,\phi,z)=V(s,\phi\text{ only})$.

Mathematically, we have a two-dimensional problem: Find $V(s,\phi)$ such that:
[1] $\nabla^2 V(s,\phi)\equiv 0$ for $s\le R$.
[2] V is periodic in ϕ, $V(s,\phi+2\pi)=V(s,\phi)$.
[3] V is well-behaved at $s=0$ (the axis).
[4] At $s=R$ (the surface), $V(s,\phi)=\text{given }V_b(\phi)$.

In the separation-of-variables method, we start by looking at solutions to conditions [1,2,3] (but not [4]) in the form

$$V(s,\phi)\;=\;f(s)\times g(\phi). \tag{1}$$

Let's start with the 2D Laplacian, which in polar coordinates (s,ϕ) acts as

$$\triangle V(s.\phi)\;=\;\frac{\partial^2 V}{\partial s^2}\;+\;\frac{1}{s}\times\frac{\partial V}{\partial s}\;+\;\frac{1}{s^2}\times\frac{\partial^2 V}{\partial\phi^2}\,. \tag{2}$$

For the potential $V(s,\phi)$ of the form (1), this Laplacian becomes

$$\triangle V\;=\;f''(s)\times g(\phi)\;+\;\frac{f'(s)}{s}\times g(\phi)\;+\;\frac{f(s)}{s^2}\times g''(\phi), \tag{3}$$

hence

$$\frac{s^2}{V}\times\triangle V\;=\;\frac{s^2 f''(s)}{f(s)}\;+\;\frac{sf'(s)}{f(s)}\;+\;\frac{g''(\phi)}{g(\phi)}\,. \tag{4}$$

Note that the first two terms here depend only on s while the third term depends only on ϕ. OOH, $\triangle V$ and hence the whole combination must vanish for all s and all ϕ, which is possible only if

$$\frac{s^2 f''(s)}{f(s)} + \frac{s f'(s)}{f(s)} = \text{constant } C \quad \text{and} \quad \frac{g''(\phi)}{g(\phi)} = -C. \tag{5}$$

Next, consider the g equation $g''(\phi) + Cg(\phi) = 0$ for a constant C. In general, the solutions to this equation are

$$\text{for } C = +m^2 \geq 0, \quad g(\phi) = A\cos(m\phi) + B\sin(m\phi), \tag{6}$$

$$\text{for } C = -\mu^2 \leq 0, \quad g(\phi) = A\cosh(\mu\phi) + B\sinh(\mu\phi). \tag{7}$$

However, we want not just any solution but a *periodic solution* $g(\phi+2\pi) = g(\phi)$, which requires trigonometric rather than hyperbolic sine and cosine, hence $C = +m^2 > 0$. Moreover, *a period compatible with* 2π *requires integer* $m = 0, 1, 2, 3, 4, \ldots$. Thus,

$$C = +m^2 \text{ for } m = 0, 1, 2, 3, \ldots \quad \text{and} \quad g(\phi) = A\cos(m\phi) + B\sin(m\phi). \tag{8}$$

Now consider the f equation for $C = +m^2$,

$$s^2 \times f''(s) + s \times f'(s) - m^2 \times f(s) = 0. \tag{9}$$

This equation is linear in f and *homogeneous* in s, so let's look for solutions of the form $f(s) = s^\alpha$ for some power α. Indeed, plugging such f into the equation yields

$$0 = s^2 \times \alpha(\alpha-1)s^{\alpha-2} + s \times \alpha s^{\alpha-1} - m^2 \times s^\alpha = s^\alpha \times \left(\alpha^2 - m^2\right), \tag{10}$$

which is satisfied whenever

$$\alpha^2 - m^2 = 0 \quad \Longrightarrow \quad \alpha = \pm m. \tag{11}$$

For $m \neq 0$ there are two distinct roots, hence two independent solutions to eq. (9), so the general solution looks like

$$f(s) = D \times s^{+m} + E \times s^{-m} \tag{12}$$

for some constants D and E. For $m = 0$ the roots (11) coincide so we only get one solution, while the other solution involves the logarithm $\ln(s)$, thus in general

$$f(s) \;=\; D \;+\; E\times\ln(s). \tag{13}$$

In any case, we want more than a general solution to the equation (9), we want the solution which obeys condition [3], namely no singularity at the cylinder's axis $s = 0$. This condition rules out negative powers of s for $m \neq 0$ or the logarithm for $m = 0$, which leaves us with

$$f(s) \;=\; \text{const}\times s^{+m} \;=\; \text{const}'\times\left(\frac{s}{R}\right)^m. \tag{14}$$

Altogether, we have an infinite series of solutions to conditions [1,2,3], namely

$$\begin{aligned} V_0(s,\phi) \;&=\; A_0 \;=\; \text{const for } m=0, \text{ and}\\ V_m(s,\phi) \;&=\; A\cos(m\phi)\times(s/R)^m \;+\; B\sin(m\phi)\times(s/R)^m \quad \text{for integer } m=1,2,3,\ldots. \end{aligned} \tag{15}$$

Consequently, a general solution to [1,2,3] is

$$V(s,\phi) \;=\; A_0 \;+\; \sum_{m=1}^{\infty}\Bigl(A_m\cos(m\phi) \;+\; B_m\sin(m\phi)\Bigr)\times\left(\frac{s}{R}\right)^m \tag{16}$$

for some constant coefficients A_m and B_m. Or in terms of complex exponentials $e^{\pm im\phi}$ with complex coefficients,

$$\begin{aligned} V(s,\phi) \;&=\; A_0 \;+\; \sum_{m=1}^{\infty}\Bigl(\tfrac{1}{2}(A_m+iB_m)e^{+im\phi} \;+\; \tfrac{1}{2}(A_m-iB_m)e^{-im\phi}\Bigr)\times\left(\frac{s}{R}\right)^m\\ &=\; \sum_{m=-\infty}^{+\infty} C_m\times e^{im\phi}\times\left(\frac{s}{R}\right)^{|m|}, \end{aligned} \tag{17}$$

$$\text{where } C_0 \;=\; A_0, \quad C_{+m} \;=\; \tfrac{1}{2}(A_m+iB_m), \quad C_{-m} \;=\; \tfrac{1}{2}(A_m-iB_m) \;=\; C^*_{+m}. \tag{18}$$

Finally, the coefficients C_m follows from the boundary condition [4] on the surface of the cylinder:

$$@s=R, \quad V(R,\phi) \;=\; \sum_{m=-\infty}^{+\infty} C_m\times e^{im\phi} \;=\; \text{given } V_b(\phi), \tag{19}$$

so the C_m obtain from expanding the periodic $V_b(\phi)$ into the Fourier series. Hence, the reverse

Fourier transform gives

$$C_m = \frac{1}{2\pi}\int\limits_0^{2\pi} V_b(\phi)\times e^{-im\phi}\,d\phi. \tag{20}$$

Or if you prefer the expansion (16) into real sine and cosine waves,

$$\begin{aligned} B_m &= \frac{2}{2\pi}\int\limits_0^{2\pi} V_b(\phi)\sin(m\phi)\,d\phi, \\ A_m &= \frac{2}{2\pi}\int\limits_0^{2\pi} V_b(\phi)\cos(m\phi)\,d\phi, \\ \text{except } A_0 &= \frac{1}{2\pi}\int\limits_0^{2\pi} V_b(\phi)\,d\phi. \end{aligned} \tag{21}$$

As a specific example, suppose the cylinder's surface is split in two halves with potentials $\pm V_0$, for example

$$V_b(\phi) = \begin{cases} +V_0 & \text{for } 0<\phi<\pi, \\ -V_0 & \text{for } \pi<\phi<2\pi. \end{cases} \tag{22}$$

By antisymmetry $V_b(2\pi-\phi) = -V_b(\phi)$, the Fourier transform of this potential has no cosine waves but only sine waves, thus all $A_m = 0$ while

$$\begin{aligned} B_m &= \frac{V_0}{\pi}\int\limits_0^{\pi}\sin(m\phi)\,d\phi \;-\; \frac{V_0}{\pi}\int\limits_{\pi}^{2\pi}\sin(m\phi)\,d\phi \\ &= \frac{V_0}{m\pi}\big[\cos(0) - 2\cos(m\pi) + \cos(2m\pi)\big] = \frac{V_0}{m\pi}\times\begin{cases} 4 & \text{for odd } m, \\ 0 & \text{for even } m. \end{cases} \end{aligned} \tag{23}$$

Consequently, the potential inside the cylinder is given by the series

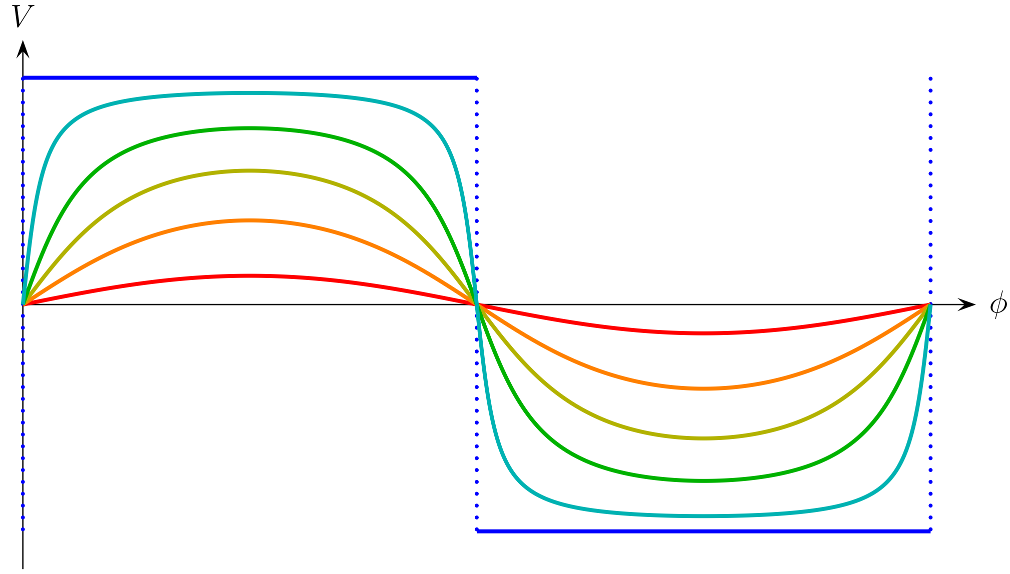
$$V(s,\phi) = \frac{4V_0}{2\pi}\sum_{m=1,3,5,\ldots}^{\text{odd } m}\frac{\sin(m\phi)}{m}\times\left(\frac{s}{R}\right)^m, \tag{24}$$

which can be analytically summer up to

$$V(r,s) = \frac{4V_0}{2\pi}\times\arctan\left(\frac{2Rs}{R^2-s^2}\times\sin\phi\right). \tag{25}$$

To illustrate this potential graphically, let me plot it as a function of ϕ for $s = 0.1R$, $0.3R$,

$s = 0.5R$, $s = 0.7R$, $s = 0.9R$, and $s = R$:



(26)

Note: the closer we are to the axis, the smaller is the amplitude of the $V(\phi)$ curve, and the curve looks more and more like the sine wave. Mathematically, this happens because the larger–m terms in the series (24) carry larger powers of (s/R), so for small s/R ratios they become small compared to the leading $m = 1$ term. Consequently, close to the axis where $(s/R) \ll 1$ we may approximate the whole series by its leading term $(s/R)\sin(\phi)$.

Outside the Cylinder

Now consider a slightly different problem: instead of a cylindrical cavity, we have a charged cylinder surrounded by empty space. We don't know the charges inside the cylinder or on its surface, all we know is the boundary potential $V_b(\phi)$ which happens to be independent of z coordinate, and we need to find out the potential $V(s, \phi)$ outside the cylinder (which we presume to be also z independent).

Proceeding similarly to the previous example, we start by looking for $V(s, \phi) = f(s) \times g(\phi)$ which obeys the Laplace equation and is periodic in ϕ. This leads us to

$$C = +m^2 \text{ for } m = 0, 1, 2, 3, \ldots \quad \text{and} \quad g(\phi) = A\cos(m\phi) + B\sin(m\phi). \tag{8}$$

and hence

$$f(s) = \begin{cases} D + E \times \ln(s) & \text{for } m = 0, \\ D \times s^{+|m|} + E \times s^{-|m|} & \text{for } m \neq 0. \end{cases} \tag{27}$$

However, this time we are concerned with the asymptotic behavior for $s \to \infty$ rather than the axis of the cylinder at $s = 0$. Specifically, we want the potential to go to zero — or at least to stay finite — for $s \to \infty$, and this rules out the positive powers of s as well as $\ln(s)$. Consequently, outside of the cylinder

$$f(s) \;=\; \text{const} \times s^{-|m|} \tag{28}$$

instead of $f(s) \propto s^{+|m|}$ inside the cylinder.

Combining the s and ϕ dependence, we find

$$V(s,\phi) \;=\; A_0 \;+\; \sum_{m=1}^{\infty} \Big(A_m \cos(m\phi) \;+\; B_m \sin(m\phi) \Big) \times \left(\frac{R}{s}\right)^m \tag{29}$$

for some constants A_m and B_m, or in terms of complex exponentials $e^{\pm im\phi}$,

$$V(s,\phi) \;=\; \sum_{m=-\infty}^{+\infty} C_m \times e^{im\phi} \times \left(\frac{R}{s}\right)^{|m|}. \tag{30}$$

Finally, the complex coefficients $C_m = C^*_{-m}$ here — or if you prefer, the real coefficients A_m and B_m, — obtain from expanding the boundary potential into the Fourier series, precisely as in eqs. (20) or (21).

Spherical Coordinates

Now consider a 3D problem: Find the potential $V(r,\theta,\phi)$ inside a spherical cavity — or outside a sphere — when we are given the potential $V_b(\theta,\phi)$ on the spherical surface. For simplicity, let's focus on potentials with axial symmetry:

$$V_b(\theta,\phi) \;=\; V_b(\theta\text{ only}) \quad\Longrightarrow\quad V(r,\theta,\phi) \;=\; V(r,\theta\text{ only}). \tag{31}$$

Mathematically, we seek the potential which:

[1] Obeys the 3D Laplace equation.

[2] Is single-valued, non-singular, and smooth as a function of θ.

[3] Is well behaved at the center $r \to 0$ if we work inside the sphere, or asymptotes to zero for $r \to \infty$ if we work outside the sphere.

[4] Has given boundary values at the sphere's surface, $V(r = R, \theta) = V_b(\theta)$.

Using the separation of variables method, we first seek to satisfy the conditions [1,2,3] for a potential of the form

$$V(r,\theta) \;=\; f(r)\times g(\theta), \tag{32}$$

find an infinite series of solutions, then look for a linear combination which satisfies the condition [4].

Let's start with the Laplace equation in the spherical coordinates:

$$\begin{aligned}\triangle V(r,\theta,\phi) \;=&\; \frac{\partial^2 V}{\partial r^2} \;+\; \frac{2}{r}\times\frac{\partial V}{\partial r}\\ &+\; \frac{1}{r^2}\times\frac{\partial^2 V}{\partial\theta^2} \;+\; \frac{1}{r^2\,\tan\theta}\times\frac{\partial V}{\partial\theta}\\ &+\; \frac{1}{r^2\,\sin^2\theta}\times\frac{\partial^2 V}{\partial\phi^2}\,.\end{aligned} \tag{33}$$

For the potential of the form (32), the Laplacian becomes

$$\triangle V \;=\; \left(f''(r)\,+\,\frac{2f'(r)}{r}\right)\times g(\theta) \;+\; \frac{f(r)}{r^2}\times\left(g''(\theta)\,+\,\frac{g'(\theta)}{\tan\theta}\right), \tag{34}$$

hence

$$\frac{r^2}{V}\times\triangle V \;=\; \left(\frac{r^2 f''(r)}{f(r)}\,+\,\frac{2rf(r)'}{f(r)}\right) + \left(\frac{g''(\theta)}{g(\theta)}\,+\,\frac{g'(\theta)}{g(\theta)\,\tan\theta}\right), \tag{35}$$

where the two terms inside the first () depend only on radius r while the two terms inside the second () depend only on the latitude θ. Consequently, the Laplace equation $\triangle V\equiv 0$ for all r,θ requires

$$r^2\times\frac{f''(r)}{f(r)} \;+\; 2r\times\frac{f'(r)}{f(r)} \;=\; +C, \tag{36}$$

$$\frac{g''(\theta)}{g(\theta)} \;+\; \frac{1}{\tan\theta}\times\frac{g'(\theta)}{g(\theta)} \;=\; -C, \tag{37}$$

$$\text{for the same constant } C. \tag{38}$$

Next, consider the g equation (37), or equivalently

$$g''(\theta) \;+\; \frac{g'(\theta)}{\tan\theta} \;+\; C\times g(\theta) \;=\; 0. \tag{39}$$

Let's change the independent variable here from θ to $x=\cos\theta$, thus

$$g(\theta) \;=\; P(\cos\theta) \tag{40}$$

for some function $P(x)$. Consequently, by the chain rule for derivatives,

$$\frac{dg}{d\theta} \;=\; -\sin\theta\times\left.\frac{dP}{dx}\right|_{x=\cos\theta} \tag{41}$$

and hence

$$\frac{d^2g}{d\theta^2} \;=\; -\cos\theta\times\left.\frac{dP}{dx}\right|_{x=\cos\theta} \;+\; \sin^2\theta\times\left.\frac{d^2P}{dx^2}\right|_{x=\cos\theta}, \tag{42}$$

so plugging these derivatives into eq. (39) we arrive at

$$\begin{aligned} 0 \;&=\; -\cos\theta\times\frac{dP}{dx} \;+\; \sin^2\theta\times\frac{d^2P}{dx^2} \;+\; \frac{-\sin\theta}{\tan\theta}\times\frac{dP}{dx} \;+\; C\times P \\ &=\; (1-\cos^2\theta)\times\frac{d^2P}{dx^2} \;-\; (\cos\theta+\cos\theta)\times\frac{dP}{dx} \;+\; C\times P. \end{aligned} \tag{43}$$

In terms of $x=\cos\theta$, this is the *Legendre equation* for the $P(x)$,

$$(1-x^2)\times P''(x) \;-\; 2x\times P'(x) \;+\; C\times P(x) \;=\; 0. \tag{44}$$

Without explaining how to solve this equation, let me briefly summarize its solutions. For generic C, all non-zero solutions to this equation have logarithmic singularities at $x=+1$ (which corresponds to $\theta=0$) and/or at $x=-1$ (which corresponds to $\theta=\pi$). The non-singular solutions obtain only for

$$C = \ell(\ell+1), \quad \text{integer } \ell = 0,1,2,3,\ldots, \tag{45}$$

in which case the good solution is the *Legendre polynomial* of degree ℓ,

$$P_\ell(x) = \frac{1}{2^\ell\,\ell!}\,\frac{d^\ell}{dx^\ell}\,(x^2-1)^\ell. \tag{46}$$

The overall coefficient here is chosen such that at $x=+1$ all these polynomials become $P_\ell(1) = 1$, while for $x=-1$ $P_\ell(-1) = (-1)^\ell$. Here are a few explicit Legendre polynomials for small ℓ:

$$\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= x,\\
P_2(x) &= \tfrac{3}{2}x^2 - \tfrac{1}{2},\\
P_3(x) &= \tfrac{5}{2}x^3 - \tfrac{3}{2}x,\\
P_4(x) &= \tfrac{35}{8}x^4 - \tfrac{15}{4}x^2 + \tfrac{3}{8},\\
P_5(x) &= \tfrac{63}{8}x^5 - \tfrac{35}{4}x^3 + \tfrac{15}{8}x,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{47}$$

The Legendre polynomial are 'orthogonal' to each other when we use $\int_{-1}^{+1} dx$ as the measure,

$$\int\limits_{-1}^{+1} dx\, P_\ell(x)\times P_{\ell'}(x) = \begin{cases} 0 & \text{for any } \ell' \neq \ell,\\ \dfrac{2}{2\ell+1} & \text{for } \ell' = \ell. \end{cases} \tag{48}$$

Consequently, any analytic function of x ranging from -1 to $+1$ may be expanded in a series of Legendre polynomials,

$$\text{any } H(x) = \sum_{\ell=0}^{\infty} H_\ell \times P_\ell(x) \quad \text{for} \quad H_\ell = \frac{2\ell+1}{2}\int\limits_{-1}^{+1} H(x)\times P_\ell(x)\,dx. \tag{49}$$

Anyhow, for $C=\ell(\ell+1)$ and $g(\theta) = P_\ell(\cos\theta)$, the f equation (36) becomes

$$r^2\times f''(r) + 2r\times f'(r) - \ell(\ell+1)\times f(r) = 0. \tag{50}$$

This equation is linear in f and *homogeneous* in r, so let's look for the solutions of the form

$f(r) = r^\alpha$ for some constant power α. Indeed plugging such an f into the equation (50) yields

$$\begin{aligned} 0 &= r^2 \times \alpha(\alpha-1)r^{\alpha-2} \;+\; 2r \times \alpha r^{\alpha-1} \;-\; \ell(\ell+1) \times r^\alpha \\ &= r^\alpha \times \big(\alpha(\alpha-1) + 2\alpha - \ell(\ell+1)\big) \end{aligned} \tag{51}$$

so the differential equation is satisfied whenever

$$\alpha(\alpha-1) \;+\; 2\alpha \;=\; \alpha(\alpha+1) \;=\; \ell(\ell+1) \quad\Longrightarrow\quad \alpha \;=\; \ell \;\textbf{or}\; \alpha \;=\; -\ell \;-\; 1. \tag{52}$$

Thus, he general solution to eq. (36) has form

$$f(r) \;=\; A \times r^\ell \;+\; \frac{B}{r^{\ell+1}}\,. \tag{53}$$

The specific solution we need depends on whether we are looking for the potential inside the sphere or outside the sphere.

- For the inside-the-sphere solution we want the potential to be non-singular at the center, which rules out negative powers of the radius r. In terms of eq. (53) this means $B = 0$ and hence

$$f(r) \;=\; \text{const} \times r^\ell \;=\; \text{const}' \times \left(\frac{r}{R}\right)^\ell. \tag{54}$$

- For the outside-the-sphere solution, we want the potential to asymptote to zero for $r \to \infty$, which rules out positive powers of the radius. In terms of eq. (53) this means $A = 0$ and hence

$$f(r) \;=\; \frac{\text{const}}{r^{\ell+1}} \;=\; \text{const}' \times \left(\frac{R}{r}\right)^{\ell+1}. \tag{55}$$

Altogether, the general solution to the conditions [1,2,3] is given by the series:

Inside the sphere,

$$V(r,\theta) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \left(\frac{r}{R}\right)^\ell. \tag{56}$$

Outside the sphere,

$$V(r,\theta) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \left(\frac{R}{r}\right)^{\ell+1}. \tag{57}$$

In both cases the coefficients C_ℓ are constants, whose values are determined by the remaining condition [4], namely the boundary condition at the sphere's surface:

$$V(r=R,\theta) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta)\times 1 = \text{ given } V_b(\theta). \tag{58}$$

To solve this condition for the C_ℓ, we use the orthogonality of the Legendre polynomials and hence eq. (49): Treat the given boundary potential $V_b(\theta)$ as a function of $x=\cos\theta$, then

$$V_b(x) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(x) \quad \text{for} \quad C_\ell \;=\; \frac{2\ell+1}{2}\int_{-1}^{+1} V_b(x)\times P_\ell(x)\,dx. \tag{59}$$

Or in terms of θ rather than $x=\cos\theta$,

$$C_\ell \;=\; \frac{2\ell+1}{2}\int_{0}^{\pi} V_b(\theta)\times P_\ell(\cos\theta)\times \sin\theta\,d\theta. \tag{60}$$

Example: $V_b(\theta)=V_0\times\cos(3\theta)$.
For boundary potentials which are manifest polynomials of $\cos\theta$ — or can be brought to such form using simple trigonometry, such as in our example

$$V_b \;=\; V_0\cos(3\theta) \;=\; 4V_0\cos^3\theta \;-\; 3V_0\cos\theta, \tag{61}$$

— we do not need to evaluate the integrals (60) to find the coefficients C_ℓ. Instead, we may simply expand the polynomial $V_b(\cos\theta)$ as a finite sum — rather than an infinite series — of Legendre polynomials using their explicit forms (47). Indeed, power by power in $x=\cos\theta$ we have

$$\begin{aligned} x \;=\; P_1(x),\quad x^2 \;&=\; \tfrac{1}{3}\big(2P_2(x)+P_0(x)\big),\quad x^3 \;=\; \tfrac{1}{5}\big(2P_3(x)+3P_1(x)\big),\\ x^4 \;&=\; \tfrac{1}{35}\big(8P_4(x)+20P_2(x)+7P_0(x)\big),\quad \ldots \end{aligned} \tag{62}$$

Consequently, for our example we have

$$V_b(\cos\theta) \;=\; 4V_0\times\frac{2P_3(\cos\theta)+3P_1(\cos\theta)}{5} - 3V_0\times P_1(\cos\theta) \;=\; \tfrac{8}{5}V_0\times P_3(\cos\theta) - \tfrac{3}{5}V_0\times P_1(\cos\theta), \tag{63}$$

hence

$$C_1 \;=\; -\tfrac{3}{5}V_0, \quad C_3 \;=\; +\tfrac{8}{5}V_0, \quad \text{all other } C_\ell \;=\; 0. \tag{64}$$

Therefore, inside the sphere the potential is

$$V(r,\theta) \;=\; -\tfrac{3}{5}V_0 \times P_1(\cos\theta) \times \left(\frac{r}{R}\right) \;+\; \tfrac{8}{5}V_0 \times P_3(\cos\theta) \times \left(\frac{r}{R}\right)^3, \tag{65}$$

while outside the sphere

$$V(r,\theta) \;=\; -\tfrac{3}{5}V_0 \times P_1(\cos\theta) \times \left(\frac{R}{r}\right)^2 \;+\; \tfrac{8}{5}V_0 \times P_3(\cos\theta) \times \left(\frac{R}{r}\right)^4. \tag{66}$$

Charges on the spherical surface

Consider a thin spherical shell with some surface charge density $\sigma(\theta,\phi)$ — and no other charges inside or outside the shell. For simplicity, assume axial symmetry, thus $\sigma(\theta$ only). Let's find out the potential both inside and outside the spherical shell due to this charge density.

Surface charge densities make for discontinuous electric fields, but the potential V is continuous across the charged surface. Thus, while in the present situation we do not know the boundary potential $V_b(\theta)$ on the spherical surface, we do know its the same potential both immediately inside and immediately outside the surface. Consequently, the potential $V(r,\theta)$ inside and outside the sphere is given by the equations (56) and (57) for the same coefficients C_ℓ, whatever they are. In other words,

$$\forall\, r,\theta: \quad V(r,\theta) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \begin{cases} \left(\dfrac{r}{R}\right)^\ell & \text{for } r < R, \\ \left(\dfrac{R}{r}\right)^{\ell+1} & \text{for } r > R. \end{cases} \tag{67}$$

Next, consider the radial component of the electric field:

$$E_r \;=\; -\frac{\partial V(r,\theta)}{\partial r} \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \begin{cases} -\ell\,\dfrac{r^{\ell-1}}{R^\ell} & \text{for } r < R, \\ +(\ell+1)\,\dfrac{R^{\ell+1}}{r^{\ell+2}} & \text{for } r > R. \end{cases} \tag{68}$$

Unlike the potential, this radial electric field is discontinuous across the sphere. Indeed, near the sphere

$$E_r(r \approx R) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \begin{cases} \dfrac{-\ell}{R} & \text{just inside the sphere,} \\[2ex] \dfrac{+(\ell+1)}{R} & \text{just outside the sphere,} \end{cases} \tag{69}$$

with discontinuity

$$\mathrm{disc}(E_r) \;=\; E_r(r = R+\epsilon) \;-\; E_r(r = R-\epsilon) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \frac{2\ell+1}{R}\,. \tag{70}$$

Physically, this discontinuity is caused by the surface charge density on the sphere,

$$\mathrm{disc}(E_r) \;=\; \frac{\sigma}{\epsilon_0}\,. \tag{71}$$

Consequently, the charge density as a function of θ is related to the coefficients C_ℓ of the potential (67) according to

$$\sigma(\theta) \;=\; \epsilon_0 \;\mathrm{disc}(E_r(\theta)) \;=\; \frac{\epsilon_0}{R} \times \sum_{\ell=0}^{\infty} (2\ell+1) \times C_\ell \times P_\ell(\cos\theta). \tag{72}$$

We may also reverse this relation according to eq. (49) to get the coefficients C_ℓ from the $\sigma(\theta)$,

$$C_\ell \;=\; \frac{R}{2\epsilon_0} \times \int\limits_0^\pi \sigma(\theta) \times P_\ell(\cos\theta) \times \sin\theta\, d\theta. \tag{73}$$

For example, suppose the sphere is neutral on the whole, but has a quadrupole charge density

$$\sigma(\theta) \;=\; \hat{\sigma} \times \frac{3\cos^2\theta - 1}{2} \;=\; \hat{\sigma} \times P_2(\cos\theta). \tag{74}$$

Comparing this angular dependence with eq. (72), we immediately see that the only non-zero coefficient C_ℓ is the C_2, specifically

$$C_2 \;=\; \frac{R\hat{\sigma}}{5\epsilon_0}\,. \tag{75}$$

Consequently, inside the sphere the potential is

$$V(r,\theta) \;=\; \frac{\hat{\sigma}}{5\epsilon_0} \times \frac{r^2}{R} \times P_2(\cos\theta), \tag{76}$$

while outside the sphere

$$V(r,\theta) \;=\; \frac{\hat{\sigma}}{5\epsilon_0} \times \frac{R^4}{r^3} \times P_2(\cos\theta). \tag{77}$$

Metal Sphere in External Electric Field

Now consider another example: a metal sphere in uniform external electric field. That is, far away from the sphere the electric field asymptotes to the uniform $\mathbf{E} = E\hat{\mathbf{z}}$, hence

$$\text{for } r \to \infty, \quad V \;\to\; -Ez \;=\; -Er \times \cos\theta \;=\; -Er \times P_1(\cos\theta). \tag{78}$$

The sphere itself is neutral, so without loss of generality we may assume it has zero potential.

Let's find the potential outside the sphere for these boundary conditions. Since we no longer have $V \to 0$ at infinity, the radial function $f_\ell(r)$ could be a general combination of two solutions,

$$f_\ell(r) \;=\; A_\ell \times r^\ell \;+\; \frac{B_\ell}{r^{\ell+1}} \tag{79}$$

with $A_\ell \neq 0$. On the other hand, asking for $V = 0$ all over the sphere requires $f_\ell(r = R) = 0$ and hence

$$B_\ell = \; -R^{2\ell+1} \times A_\ell\,. \tag{80}$$

Consequently, the general form of the potential outside the sphere looks like

$$V(r,\theta) \;=\; \sum_{\ell=0}^{\infty} A_\ell \times P_\ell(\cos\theta) \times \left(r^\ell \;-\; \frac{R^{2\ell+1}}{r^{\ell+1}}\right) \tag{81}$$

for some coefficients A_ℓ.

To find these coefficients, we compare the asymptotic behavior of the potential (81) for large r,

$$V \longrightarrow \sum_{\ell=0}^{\infty} A_\ell \times P_\ell(\cos\theta) \times r^\ell \tag{82}$$

to the desired asymptotics (78). This comparison immediately tells us that

$$A_1 = -E, \quad \text{all other } A_\ell = 0, \tag{83}$$

hence

$$V(r,\theta) = -E\left(r - \frac{R^3}{r^2}\right) \times \cos\theta, \tag{84}$$

or in Cartesian coordinates

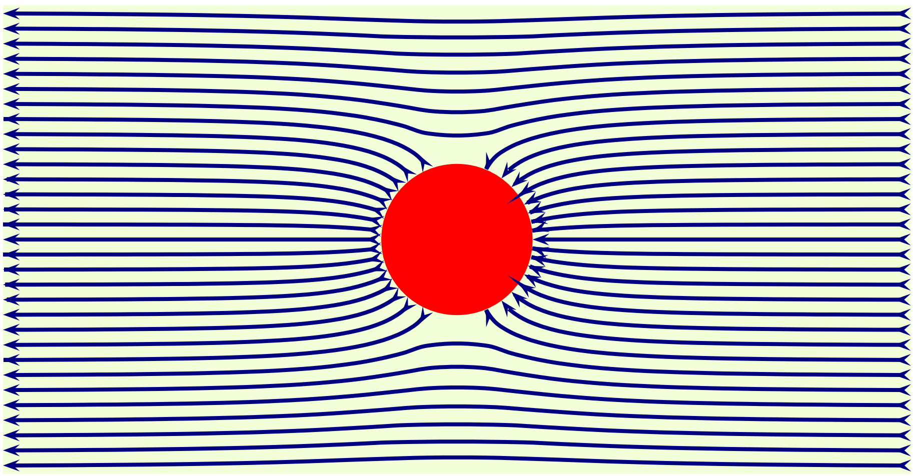
$$V(x,y,z) = -Ez + ER^3 \times \frac{z}{(x^2+y^2+z^2)^{3/2}} \tag{85}$$

The first term here is due to the external electric field, while the second term is due to induced charges on the sphere's surface.

Taking the gradient of the potential (85), we obtain the net electric field,

$$\mathbf{E}(x,y,z) = E\hat{\mathbf{z}} + ER^3\left(\frac{3z}{r^4}\,\hat{\mathbf{r}} - \frac{1}{r^3}\,\hat{\mathbf{z}}\right) = E\hat{\mathbf{z}} + \frac{ER^3}{r^3}\left(2\frac{z}{r}\,\hat{\mathbf{z}} - \frac{x}{r}\,\hat{\mathbf{x}} - \frac{y}{r}\,\hat{\mathbf{y}}\right). \tag{86}$$

Here is the picture of the field lines for this electric field:



Spherical Harmonics

Finally, consider a more general 3D problem with a spherical boundary, but with a given boundary potential $V_b(\theta, \phi)$ (or a given boundary charge $\sigma(\theta, \phi)$) which is not axially symmetric but depends on both angular coordinates θ and ϕ. In this case, instead of the Legendre polynomials $P_\ell(\cos\theta)$ we should use the *spherical harmonics* $Y_{\ell,m}(\theta, \phi)$. You will study these spherical harmonics in some detail in the Quantum Mechanics class in the context of angular momentum quantization, hydrogen atom wavefunctions, *etc.*, *etc.* For the moment, let me skip the details and simply summarize a few key properties of the spherical harmonics.

- The spherical harmonics are solutions to the partial differential equation

$$\frac{\partial^2 Y}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial Y}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2} = -\ell(\ell+1)Y \tag{87}$$

  subject to the conditions of single-valuedness and no singularities anywhere on the sphere. In terms of the θ and ϕ coordinates this means periodicity in ϕ and no singularities at the poles $\theta = 0$ and $\theta = \pi$.

- The solutions exist only for integer $\ell = 0, 1, 2, 3, \ldots$. For each such ℓ, there are $2\ell + 1$ independent solutions $Y_{\ell,m}(\theta, \phi)$ labeled by another integer m running from $-\ell$ to $+\ell$.

- The $Y_{\ell,m}$ have form $Y_{\ell,m}(\theta, \phi) = (\text{const}) \times P_{\ell(m)}(\cos\theta) \times \exp(im\phi)$ where the $P_{\ell(m)}(x)$ are called the *associate Legendre polynomials*, even though some of them are not really polynomials. Instead, $P_{\ell(m)}(\cos\theta) = (\sin\theta)^{|m|} \times$ degree $(\ell - |m|)$ polynomial of $\cos\theta$.

- For $m \neq 0$ the spherical harmonics are complex; by convention, $Y^*_{\ell,m} = (-1)^m Y_{\ell,-m}$. Also, all the harmonics with $m \neq 0$ vanish at the poles $\theta = 0$ and $\theta = \pi$.

- The only harmonics which do not vanish at the poles are the $Y_{\ell,0}$. These harmonics are independent of ϕ and are proportional to $P_\ell(\cos\theta)$, but have different normalization: $Y_{\ell,0}(\theta, \not\phi) = \sqrt{(2\ell+1)/4\pi} \times P_\ell(\cos\theta)$.

- The spherical harmonics are orthogonal to each other and normalized to 1. That is

$$\iint Y^*_{\ell,m}(\theta, \phi)\, Y_{\ell',m'}(\theta, \phi)\, d^2\Omega(\theta, \phi) = \delta_{\ell,\ell'}\delta_{m,m'}\,. \tag{88}$$

- Any smooth, single-valued function $g(\theta, \phi)$ can be decomposed into a series of spherical harmonics,

$$g(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} C_{\ell,m}Y_{\ell,m}(\theta,\phi) \quad \text{for} \quad C_{\ell,m} = \iint g(\theta,\phi)\,Y^*_{\ell,m}(\theta,\phi)\,d^2\Omega(\theta,\phi). \quad (89)$$

- Let $F(r,\theta,\phi) = r^\ell \times Y_{\ell,m}(\theta,\phi)$. Then in Cartesian coordinates, $F(x,y,z)$ is a homogeneous polynomial in x,y,z of degree ℓ. Moreover, $F(x,y,z)$ obeys the Laplace equation.

Now let's apply the spherical harmonics to the electrostatic potential problems with spherical boundaries but with ϕ-dependent boundary conditions. Mathematically, we look for a function $V(r,\theta,\phi)$ which:
[1] Obeys the Laplace equation inside or outside some sphere of radius R.
[2] Is smooth and single-valued everywhere in the volume in question; in particular, V is periodic in ϕ and has no singularities at $\theta = 0$ or $\theta = \pi$.
[3] For the inside of a spherical cavity, V is smooth at $r \to 0$; for the outside or a sphere, V asymptotes to zero for $r \to \infty$.
[4] On the spherical boundary the potential has given form, $V(R,\theta,\phi) = V_b(\theta,\phi)$.

Using the separation of variables method, we start by looking for solutions to conditions [1,2,3] (but not [4]) of the form

$$V(r,\theta,\phi) = f(r) \times g(\theta,\phi); \quad (90)$$

note incomplete separation of variables at this stage. In light of eq. (33) for the Laplace operator in spherical coordinates,

$$\frac{r^2}{V} \times \Delta V = \frac{r^2 f''}{f} + \frac{2rf'}{f} + \frac{1}{g}\left(\frac{\partial^2 g}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial g}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\phi^2}\right), \quad (91)$$

so to get a solution to the Laplace equation $\Delta V = 0$ we need

$$\frac{\partial^2 g}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial g}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\phi^2} + C \times g = 0, \quad (92)$$

$$r^2\frac{d^2 f}{dr^2} + 2r\frac{df}{dr} - C \times f = 0 \quad (93)$$

for the same constant C. By inspection, eq. (92) is the same as eq. (87), so we know that the solutions exist only for $C = \ell(\ell+1)$ for integer $\ell = 0, 1, 2, 3, \ldots$, and the solutions are the spherical harmonics $g(\theta, \phi) = Y_{\ell,m}(\theta, \phi)$ or their linear combinations. Thus,

$$V(r, \theta.\phi) \;=\; f(r) \times Y_{\ell,m}(\theta, \phi) \tag{94}$$

where the radial function $f(r)$ obeys

$$r^2 f''(r) \;+\; 2r f'(r) \;-\; \ell(\ell+1) f(r) \;=\; 0. \tag{95}$$

As we saw earlier in these notes, the solutions to this equation have form

$$f(r) \;=\; A \times r^\ell \;+\; \frac{B}{r^{\ell+1}} \tag{96}$$

for some constants A and B. For a spherical cavity, regularity of the solution at the center requires $B = 0$ while for an outside of a sphere the asymptotic condition at ∞ requires $A = 0$. However, for a space between two spherical boundaries, we may have both $A \neq 0$ and $B \neq 0$.

Altogether, the general solution to conditions [1,2,3] for the inside of a spherical cavity has form

$$V(r, \theta, \phi) \;=\; \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} C_{\ell,m} \times \left(\frac{r}{R}\right)^{\ell} \times Y_{\ell,m}(\theta, \phi), \tag{97}$$

while the general solution for the outside of a sphere looks like

$$V(r, \theta, \phi) \;=\; \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} C_{\ell,m} \times \left(\frac{R}{r}\right)^{\ell+1} \times Y_{\ell,m}(\theta, \phi). \tag{98}$$

In both cases, the constant coefficients $C_{\ell,m}$ follow from the boundary condition [4] at the spherical surface:

$$V(R, \theta, \phi) \;=\; \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} C_{\ell,m} \times Y_{\ell,m}(\theta, \phi). \tag{99}$$

Since the spherical harmonics form a complete orthonormal basis for the functions of the spherical angles (θ, ϕ), we may use eq. (89) to obtain the coefficients $C_{\ell,m}$ for any given boundary potential $V_b(\theta, \phi)$ on the spherical surface, namely

$$C_{\ell,m} \;=\; \iint V_b(\theta, \phi) \times Y^*_{\ell,m}(\theta, \phi)\, d^2\Omega(\theta, \phi). \tag{100}$$