Problem **2**(a):

The boundary problem of the form

$$\begin{aligned} \Phi(x,y,z) &= 0 \quad \text{for } x=0 \text{ or } x=L \text{ or } y=0 \text{ or } y=L \text{ or } z=0, && \text{(S.1)} \\ \Phi(x,y,z) &= \text{given } \Phi_b(x,y) \quad \text{for } z=L, && \text{(S.2)} \end{aligned}$$

is best solved using separation of variables in Cartesian coordinates: First, we look for solutions of the Laplace equation in the form

$$\Phi(x,y,z) = A(x)B(y)C(z) \tag{S.3}$$

which obey the homogeneous boundary conditions (S.1), and then we look for a linear combination of such solutions which also obeys the in-homogeneous boundary condition (S.2). As explained in class, for the potential in the form (S.3), the Laplace equation becomes

$$\begin{aligned} &A''(x) = aA(x), \quad B''(y) = bB(y), \quad C''(z) = c\,C(z), \\ &\quad \text{for some constants } a,b,c \text{ such that } a+b+c = 0, \end{aligned} \tag{S.4}$$

while the homogeneous boundary conditions (S.1) translate to

$$A(0) = A(L) = 0, \quad B(0) = B(L) = 0, \quad C(0) = 0. \tag{S.5}$$

Solving the equations (S.4) with boundary conditions (S.5), we obtain

$$\begin{aligned} A(x) &= \sin\frac{m\pi x}{L} \quad \text{for an integer } m = 1,2,3,\ldots, \\ B(y) &= \sin\frac{n\pi y}{L} \quad \text{for an integer } n = 1,2,3,\ldots, \\ C(z) &= \sinh(\kappa_{m,n} z) \quad \text{for } \kappa_{m,n}^2 = (m\pi/L)^2 + (n\pi/L)^2. \end{aligned} \tag{S.6}$$

Consequently, a most general solution of the Laplace equation plus the homogeneous boundary conditions (S.1) has form

$$\Phi(x,y,z) \;=\; \sum_{m,n} \alpha_{m,n}\,\sin\frac{m\pi x}{L}\,\sin\frac{n\pi y}{L}\,\sinh(\kappa_{m,n}z) \tag{S.7}$$

for some real coefficients $\alpha_{m,n}$. Adding the in-homogeneous boundary conditions (S.2) fixes the values of these coefficients to whatever it takes to get

$$\sum_{m,n} \alpha_{m,n}\,\sin\frac{m\pi x}{L}\,\sin\frac{n\pi y}{L}\,\sinh(\kappa_{m,n}L) \;=\; \text{given }\Phi_b(x,y). \tag{S.8}$$

For the problem at hand, the boundary potential (1) at the cube's lid amounts to

$$\Phi_b(x,y) \;=\; V_0 \sin\frac{3\pi x}{L}\,\sin\frac{4\pi y}{L}, \tag{S.9}$$

so we may immediately determine

$$\alpha_{3,4}\times\sinh(\kappa_{3,4}L) \;=\; V_0, \quad \text{all other }\alpha_{n,m} \;=\; 0, \tag{S.10}$$

and therefore

$$\Phi(x,y,z) \;=\; V_0\times\sin\frac{3\pi x}{L}\times\sin\frac{4\pi y}{L}\times\frac{\sinh(\kappa_{3,4}z)}{\sinh(\kappa_{3,4}L)}\,. \tag{S.11}$$

Finally,

$$\kappa_{3,4}^2 \;=\; (3\pi/L)^2 \;+\; (4\pi/L)^2 \;=\; (\pi/L)^2\times(3^2+4^2=5^2) \quad\Longrightarrow\quad \kappa_{3,4} \;=\; 5\pi/L, \tag{S.12}$$

and consequently

$$\Phi(x,y,z) \;=\; V_0\times\sin\frac{3\pi x}{L}\times\sin\frac{4\pi y}{L}\times\frac{\sinh(5\pi z/L)}{\sinh(5\pi)}\,. \tag{S.13}$$

Problem **2**(b):

The best coordinate system for the hollow cylindrical pipe in question are the cylindrical coordinates (r, ϕ, z). In these coordinates, we have homogeneous boundary (and boundary-like) conditions

$$\begin{aligned} \Phi(r,\phi,z) &= 0 \quad \text{for } r = R \text{ and for } z \to +\infty, \\ \text{periodic } \Phi(r,\phi,z) &= \Phi(r,\phi+2\pi,z), \\ \text{smooth } \Phi(r,\phi,z) & \quad \text{for } r \to 0, \end{aligned} \tag{S.14}$$

as well as the inhomogeneous boundary condition

$$\Phi(r,\phi,z) = \text{given } \Phi_b(r,\phi) \text{ for } z = 0, \tag{S.15}$$

in our case

$$\Phi_b(r,\phi) = V_0 \times J_1(kr) \times \cos\phi \tag{S.16}$$

where J_1 is the Bessel function J_n for $n = 1$.

As explained in the textbook, we start by looking at the solutions of the Laplace equation and the homogeneous boundary conditions (S.14) in the form

$$\Phi(r,\phi,z) = A(r) \times B(\phi) \times C(z), \tag{S.17}$$

and then we look for a linear combination of such solutions which also obeys the inhomogeneous boundary condition (S.15). For the potential of the form (S.17), the Laplace equation becomes

$$\frac{\nabla^2\Phi}{\Phi} = \frac{A''(r)}{A(r)} + \frac{A'(r)}{rA(r)} + \frac{1}{r^2}\frac{B''(\phi)}{B(\phi)} + \frac{C''(z)}{C(z)} = 0, \tag{S.18}$$

which calls for

$$A''(r) + \frac{1}{r}A'(r) - \frac{n^2}{r^2}A(r) = k^2 A(r), \tag{S.19}$$

$$B''(\phi) = -n^2 B(\phi), \tag{S.20}$$

$$C''(z) \;=\; -k^2 C(z), \tag{S.21}$$

for some constants n^2 and k^2, while the homogeneous boundary conditions (S.14) become

$$\begin{aligned} A(r=R) \;&=\; 0, \quad \text{smooth } A(r) \text{ for } r \to 0, \\ B(\phi) \;&=\; B(\phi + 2\pi), \\ C(z) \;&\to\; 0 \text{ for } z \to +\infty. \end{aligned} \tag{S.22}$$

As explained in the textbook, solving these equations and boundary conditions gives us

$$\begin{aligned} B(\phi) \;&=\; \alpha\cos(n\phi) \;+\; \beta\sin(n\phi) \\ &\quad\quad \text{for an integer } n, \\ A(r) \;&=\; J_n(kr) \quad \langle\!\langle \text{ Bessel function } \#n \,\rangle\!\rangle \\ &\quad\quad \text{for } k \text{ such that } J_n(kR) \;=\; 0, \\ C(z) \;&=\; \exp(-kz). \end{aligned} \tag{S.23}$$

Consequently, the general solution of the Laplace equations plus homogeneous boundary conditions has form

$$\Phi(r,\phi,z) \;=\; \sum_{n,m} \bigl(\alpha_{n,m}\cos(n\phi) \;+\; \beta_{n,m}\sin(n\phi)\bigr) \times J_n(k_m(n)\times r) \times \exp(-k_m(n)\times z) \tag{S.24}$$

for some real coefficients $\alpha_{n,m}$ and $\beta_{n,m}$, where

$$k_m(n) \;=\; \frac{1}{R} \times \bigl(m^{\text{th}} \text{ zero of the Bessel function } J_n\bigr). \tag{S.25}$$

The values of the coefficients $\alpha_{n,m}$ and $\beta_{n,m}$ follow from the inhomogeneous boundary condition at $z = 0$, namely whatever it takes to get

$$\sum_{n,m} \bigl(\alpha_{n,m}\cos(n\phi) \;+\; \beta_{n,m}\sin(n\phi)\bigr) \times J_n(k_m(n)\times r) \;=\; \text{given } \Phi_b(r,\phi). \tag{S.26}$$

For the problem at hand, we are given

$$\Phi_b(r,\phi) \;=\; V_0 \times J_1(kr) \times \cos\phi \tag{S.27}$$

where k is one of the $k_m(1)$ (since we are told that $J_1(kR) = 0$), say $k = k_{m_0}(1)$. Therefore, we immediately identify

$$\alpha_{1,m_0} = V_0\,, \quad \text{all other } \alpha_{n,m} = 0, \quad \text{all } \beta_{n,m} = 0, \tag{S.28}$$

so the solution inside the pipe is

$$\Phi(r,\phi,z) = V_0 \times J_1(k_{m_0}(1)\times r)\times\cos\phi\times\exp(-k_{m_0}(1)\times z). \tag{S.29}$$

Problem **2**(c):

To keep our notations simple, let's use the spherical coordinates with the origin at the sphere's center and the "north pole" $\theta=0$ pointing in the direction of the asymptotic electric field $\mathbf{E}_0$. In these spherical coordinates, we have a Dirichlet boundary condition $\Phi(r,\theta,\phi)=0$ for $r=R$, and the asymptotic condition

$$\Phi(r,\theta,\phi) \xrightarrow[r\to\infty]{} -E_0 z = -E_0 r\cos\theta. \tag{S.30}$$

To solve this problem, we first use the axial symmetry to look for a ϕ-independent $\Phi(r,\theta)$, then separate the variables to look for solutions in the form

$$\Phi(r,\theta) = A(r)\times B(\theta), \tag{S.31}$$

and ultimately look for a linear combinations of such solutions which has the right asymptotic behavior (S.30).

As explained in class, the Laplace equation for potential of the form (S.31) becomes

$$A''(r) + \frac{2}{r}A'(r) - \frac{\ell(\ell+1)}{r^2}A(r) = 0, \tag{S.32}$$

$$B''(\theta) + \frac{1}{\tan\theta}B'(\theta) + \ell(\ell+1)B(\theta) = 0. \tag{S.33}$$

where in lieu of the boundary conditions $B(\theta)$ must be non-singular at both $\theta=0$ and at $\theta=\pi$. The solutions of these conditions exist only for integer $\ell=0,1,2,3,\ldots$, and have form

$$B(\theta) \;=\; P_\ell(\cos\theta) \tag{S.34}$$

where $P_\ell(x)$ is the Legendre polynomial of degree ℓ. As to eq. (S.32) for the $A(r)$, the general solution is

$$A(r) \;=\; \alpha\times r^\ell \;+\; \frac{\beta}{r^{\ell+1}} \tag{S.35}$$

for some coefficients α and β. However, the Dirichlet boundary condition on the conducting sphere's surface — which translates to $A(r=R)=0$, — requires

$$\alpha\times R^\ell \;+\; \frac{\beta}{R^{\ell+1}} \;=\; 0, \tag{S.36}$$

hence $\beta=-R^{2\ell+1}\times\alpha$ and therefore

$$A(r) \;=\; \alpha\times\left(r^\ell \;-\; \frac{R^{2\ell+1}}{r^{\ell+1}}\right). \tag{S.37}$$

Altogether, the general axially-symmetric potential outside the conducting sphere has form

$$\Phi(r,\theta) \;=\; \sum_{\ell=0}^{\infty}\alpha_\ell\times\left(r^\ell \;-\; \frac{R^{2\ell+1}}{r^{\ell+1}}\right)\times P_\ell(\cos\theta) \tag{S.38}$$

for some coefficients α_ℓ. To find these coefficients, we look at the asymptotic behavior of the potential (S.38) at long distances from the sphere, $r\gg R$:

$$\Phi(r,\theta) \;\xrightarrow[r\to\infty]{}\; \sum_{\ell=0}^{\infty}\alpha_\ell\times r^\ell\times P_\ell(\cos\theta). \tag{S.39}$$

Matching this asymptotic behavior to the uniform electric field (S.30), — which we may rewrite as

$$\Phi(r,\theta) \xrightarrow[r\to\infty]{} -E_0\times r\times P_1(\cos\theta) \tag{S.40}$$

since $P_1(x)=x$, — we immediately see that we want

$$\alpha_1 \;=\; -E_0\,, \quad \text{all other } \alpha_\ell \;=\; 0. \tag{S.41}$$

Consequently, the potential (S.38) at finite distances of the sphere is

$$\Phi(r,\theta) \;=\; -E_0\left(r\,-\,\frac{R^3}{r^2}\right)\times\cos\theta. \tag{S.42}$$

Finally, the electric field $\mathbf{E}=-\nabla\Phi$ outside the sphere is best obtained in vector or index notations:

$$\Phi \;=\; -(\mathbf{E}_0\cdot\mathbf{n})\left(r\,-\,\frac{R^3}{r^2}\right), \tag{S.43}$$

$$\begin{aligned}
\mathbf{E} \;&=\; -\nabla\Phi \;=\; \big(\nabla(\mathbf{n}\cdot\mathbf{E}_0)\big)\left(r\,-\,\frac{R^3}{r^2}\right)\;+\;(\mathbf{n}\cdot\mathbf{E}_0)\,\nabla\left(r\,-\,\frac{R^3}{r^2}\right)\\
&=\; \frac{\mathbf{E}_0\,-\,(\mathbf{E}_0\cdot\mathbf{n})\mathbf{n}}{r}\left(r\,-\,\frac{R^3}{r^2}\right)\;+\;(\mathbf{n}\cdot\mathbf{E}_0)\left(1\,+\,\frac{2R^3}{r^3}\right)\mathbf{n}\\
&=\; \mathbf{E}_0\;+\;\frac{R^3}{r^3}\big(3(\mathbf{E}_0\cdot\mathbf{n})\mathbf{n}\,-\,\mathbf{E}_0\big).
\end{aligned} \tag{S.44}$$

Problem **3**(intro):

The separation of variables method in spherical coordinates was explained in class, but let me briefly repeat it here. For a region of space bounded by a sphere or 2 concentric spheres and containing no electric charges, we look for solutions of the Laplace equation in the form

$$\Phi(r,\theta,\phi) \;=\; A(r)\times B(\theta,\phi). \tag{S.45}$$

For these kinds of potentials, the Laplace equation becomes

$$A''(r)\;+\;\frac{2}{r}A'(r)\;-\;\frac{\ell(\ell+1)}{r^2}A(r)\;=\;0, \tag{S.46}$$

$$\mathbf{L}^2 B(\theta,\phi) \;-\; \ell(\ell+1)B(\theta,\phi) \;=\; 0, \tag{S.47}$$

and the $B(\theta,\phi)$ factor is subject to boundary-like conditions

$$B(\theta,\phi+2\pi) \;=\; B(\theta,\phi) \quad \text{and} \quad B \text{ is smooth for } \theta\to 0 \text{ and } \theta\to\pi. \tag{S.48}$$

As you (should have) learned in a Quantum Mechanics class, eq. (S.47) subject to conditions (S.48) has solutions only for integer $\ell = 0,1,2,\ldots$, and the solutions are the spherical harmonics $Y_{\ell,m}(\theta,\phi)$ for $m = -\ell, 1-\ell, \ldots, \ell-1, \ell$. As to the eq. (S.46) for the $A(r)$ factor, the general solution is

$$A(r) \;=\; \alpha\times r^\ell \;+\; \frac{\beta}{r^{\ell+1}} \tag{S.49}$$

for some coefficients α and β.

Consequently, without specifying the boundary conditions at the spherical boundaries, the most general solution of the Laplace equation can be written as

$$\Phi(r,\theta,\phi) \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\alpha_{\ell,m}r^\ell \;+\; \frac{\beta_{\ell,m}}{r^{\ell+1}}\right)\times Y_{\ell,m}(\theta,\phi) \tag{S.50}$$

for some coefficients $\alpha_{\ell,m}$ and $\beta_{\ell,m}$.

Problem **3**(a):

Now, let's apply eq. (S.50) to the problem at hand. In our case, the volume in question is everywhere outside a sphere of radius R — which serves as the inner boundary, — and we are given the potential $\Phi_b(\theta,\phi)$ which we should have at that boundary. There is no outer boundary — the volume in question extends to infinity in all directions, — but there is an implicit asymptotic condition of $\Phi(\mathbf{x})\to 0$ for $r\to\infty$. Therefore, for any direction (θ,ϕ), the expansion of the potential in powers of r should contain no positive powers of r but only negative powers, which means that in the series (S.50) we must have

$$\alpha_{\ell,m} \;=\; 0 \quad \text{for all } \ell, m. \tag{S.51}$$

Thus, the series (S.50) becomes

$$\Phi(r,\theta,\phi) \;=\; \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{\beta_{\ell,m}}{r^{\ell+1}} \times Y_{\ell,m}(\theta,\phi) \tag{S.52}$$

where the coefficients $\beta_{\ell,m}$ obtain from the boundary condition at the sphere of radius R. Specifically,

$$\begin{aligned} \text{for } r = R, \quad \Phi(R,\theta,\phi) \;&=\; \sum_{\ell,m} \frac{\beta_{\ell,m}}{R^{\ell+1}} \times Y_{\ell,m}(\theta,\phi) \\ \text{should be } \;&=\; \text{given } \Phi_b(\theta,\phi). \end{aligned} \tag{S.53}$$

To solve this equation, we use the orthogonality of the spherical harmonics:

$$\iint d^2\Omega(\theta,\phi)\, Y^*_{\ell,m}(\theta,\phi)\, Y_{\ell',m'}(\theta,\phi) \;=\; \delta_{\ell,\ell'}\delta_{m,m'}, \tag{S.54}$$

hence given eq. (S.53), we must have

$$\begin{aligned} \iint d^2\Omega(\theta,\phi)\, Y^*_{\ell,m}(\theta,\phi) \times \Phi_b(\theta,\phi) \;&=\; \sum_{\ell',m'} \frac{\beta_{\ell',m'}}{R^{\ell'+1}} \times \iint d^2\Omega(\theta,\phi)\, Y^*_{\ell,m}(\theta,\phi)\, Y_{\ell',m'}(\theta,\phi) \\ &=\; \sum_{\ell',m'} \frac{\beta_{\ell',m'}}{R^{\ell'+1}} \times \delta_{\ell,\ell'}\delta_{m,m'} \\ &=\; \frac{\beta_{\ell,m}}{R^{\ell+1}}\,. \end{aligned} \tag{S.55}$$

Consequently,

$$\begin{aligned} \beta_{\ell,m} \;&=\; R^{\ell+1} \times \iint d^2\Omega(\theta,\phi)\, Y^*_{\ell,m}(\theta,\phi) \times \Phi_b(\theta,\phi) \\ &=\; R^{\ell-1} \times \iint\limits_{\text{sphere}} d^2\mathbf{y}\, Y^*_{\ell,m}(\mathbf{n}_\mathbf{y}) \times \Phi_b(\mathbf{y}), \end{aligned} \tag{S.56}$$

where the second line follows from

$$d^2\mathbf{y} \;\equiv\; d^2\text{Area}(\mathbf{y}) \;=\; R^2 \times d^2\Omega(\theta,\phi) \quad \text{for } \mathbf{y} \text{ spanning the sphere of radius } R. \tag{S.57}$$

Finally, let's plug the coefficients (S.56) into eq. (S.52) for the potential outside the sphere:

$$\begin{aligned}\Phi(r,\theta,\phi) &= \sum_{\ell,m} \frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}} \times \left(\beta_{\ell,m} = R^{\ell-1} \iint\limits_{\text{sphere}} d^2\mathbf{y}\, Y^*_{\ell,m}(\mathbf{n_y}) \times \Phi_b(\mathbf{y})\right) \\ &= \iint\limits_{\text{sphere}} d^2\mathbf{y}\,\Phi_b(\mathbf{y}) \times \sum_{\ell,m} \frac{R^{\ell-1}}{r^{\ell+1}} \times Y_{\ell,m}(\theta,\phi)\, Y^*_{\ell,m}(\mathbf{n}_y),\end{aligned} \tag{S.58}$$

or in other words,

$$\Phi(\mathbf{x}) = \iint\limits_{\text{sphere}} d^2\mathbf{y}\,\Phi_b(\mathbf{y}) \times F(\mathbf{x},\mathbf{y}) \tag{3}$$

for

$$F(\mathbf{x},\mathbf{y}) \stackrel{\text{def}}{=} \sum_{\ell=0}^{\infty} \frac{R^{\ell-1}}{|\mathbf{x}|^{\ell+1}} \times \sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\mathbf{n_x})\, Y^*_{\ell,m}(\mathbf{n_y}). \tag{4}$$

Quod erat demonstrandum.

Problem **3**(b):

First, let's complete eq. (8) using eq. (7):

$$\begin{aligned}\sum_{\ell=0}^{\infty}(2\ell+1)t^{\ell} \times P_{\ell}(c) &= \left(2t\frac{\partial}{\partial t}+1\right)\sum_{\ell=0}^{\infty} t^{\ell} \times P_{\ell}(c) \\ &\quad\langle\!\langle \text{ plugging in eq. (7) } \rangle\!\rangle \\ &= \left(2t\frac{\partial}{\partial t}+1\right)\frac{1}{\sqrt{1-2ct+t^2}} \\ &= 2t \times \frac{c-t}{(1-2ct+t^2)^{3/2}} + \frac{1}{\sqrt{1-2ct+t^2}} \\ &= \frac{2t(c-t) + (1-2ct+t^2)}{(1-2ct+t^2)^{3/2}} \\ &= \frac{1-t^2}{(1-2ct+t^2)^{3/2}}.\end{aligned} \tag{8*}$$

Next, let's use eq. (6) to sum over m in eq. (4):

$$\begin{aligned}
F(\mathbf{x},\mathbf{y}) &= \sum_{\ell=0}^{\infty} \frac{R^{\ell-1}}{r^{\ell+1}} \times \sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\mathbf{n_x})\, Y^*_{\ell,m}(\mathbf{n_y}) \\
&\qquad \langle\!\langle \text{ where } r = |\mathbf{x}| \,\rangle\!\rangle \\
&= \sum_{\ell=0}^{\infty} \frac{R^{\ell-1}}{r^{\ell+1}} \times \frac{2\ell+1}{4\pi} P_\ell(\mathbf{n_x}\cdot\mathbf{n_y}) \\
&= \frac{1}{4\pi Rr} \sum_{\ell=0}^{\infty} (2\ell+1)(R/r)^\ell \times P_\ell(\mathbf{n_x}\cdot\mathbf{n_y}).
\end{aligned} \tag{S.59}$$

Physically, $c = (\mathbf{n_x}\cdot\mathbf{n_y})$ is the cosine of the angle between the directions of the vectors $\mathbf{x}$ and $\mathbf{y}$, so we always have $|c| \leq 1$. At the same time $\mathbf{x}$ is outside the sphere of radius R while $\mathbf{y}$ is on the surface of that sphere, hence

$$r = |\mathbf{x}| > R = |\mathbf{y}| \quad \Longrightarrow \quad t = \frac{R}{r} < 1. \tag{S.60}$$

Consequently, eq. (7) — and hence also eq. (8*) — are valid for $c = (\mathbf{n_x}\cdot\mathbf{n_y})$ and $t = (R/r)$, so applying eq. (8*) to the series on the bottom line of eq. (S.59), we get

$$\begin{aligned}
F(\mathbf{x},\mathbf{y}) &= \frac{1}{4\pi Rr} \sum_{\ell=0}^{\infty} (2\ell+1)(t = R/r)^\ell \times P_\ell(c = \mathbf{n_x}\cdot\mathbf{n_y}) \\
&= \frac{1}{4\pi Rr} \times \frac{1-(t=R/r)^2}{[1 \,-\, 2(t=R/r)(c=\mathbf{n_x}\cdot\mathbf{n_y}) \,+\, (t=R/r)^2]^{3/2}} \\
&= \frac{1}{4\pi Rr} \times \frac{r(r^2-R^2)}{[r^2 \,-\, 2rR(\mathbf{n_x}\cdot\mathbf{n_y}) \,+\, R^2]^{3/2}} \\
&= \frac{r^2-R^2}{4\pi R} \times \frac{1}{[\mathbf{x}^2 \,-\, 2\mathbf{x}\cdot\mathbf{y} \,+\, \mathbf{y}^2]^{3/2}} \\
&= \frac{\mathbf{x}^2-R^2}{4\pi R} \times \frac{1}{|\mathbf{x}-\mathbf{y}|^3}\,.
\end{aligned} \tag{5}$$

Quod erat demonstrandum.