Problem **1**(a):

The Green's function $G_D(\mathbf{x},\mathbf{y})$ for the space outside the sphere with Dirichlet boundary condition $\Phi = 0$ on the sphere's surface is the answer to the following physical problem: Let the sphere of radius $R$ be a grounded conductor — hence the Dirichlet boundary condition $\Phi = 0$, — and put a point charge $q$ at some location $\mathbf{y}$ outside the sphere, then the potential outside the sphere is

$$\Phi(\mathbf{x}) \;=\; \frac{q}{\epsilon_0}\,G_D(\mathbf{x},\mathbf{y}). \tag{S.1}$$

The image charge method has another way to solve exactly the same problem: the potential $\Phi(\mathbf{x})$ (for $\mathbf{x}$ outside the sphere) is the combined Coulomb potential of two point charges, the real charge $q$ at $\mathbf{y}$ outside the sphere, and the image charge $Q_{\rm im} = -(R/|\mathbf{y}|)Q$ inside the sphere at $\mathbf{y}_{\rm im} = (R^2/|\mathbf{y}|)\mathbf{n}_y = (R^2/|\mathbf{y}|^2)\mathbf{y}$. Altogether,

$$\begin{aligned}
\Phi(\mathbf{x}) \;&=\; \frac{Q}{4\pi\epsilon_0|\mathbf{x}-\mathbf{y}|} \;+\; \frac{Q_{\rm im}}{4\pi\epsilon_0|\mathbf{x}-\mathbf{y}_{\rm im}|}\\
&=\; \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|} \;-\; \frac{(R/|\mathbf{y}|)}{|\mathbf{x}\;-\;\frac{R^2}{|\mathbf{y}|^2}\mathbf{y}|}\right)\\
&=\; \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2-2xyc}} \;-\; \frac{R}{\sqrt{x^2y^2+R^4-2R^2xyc}}\right)
\end{aligned} \tag{S.2}$$

where $x = |\mathbf{x}|$, $y = |\mathbf{y}|$, and $c = \mathbf{n}_x\cdot\mathbf{n}_y$. In terms of eq. (S.1), this means that the Dirichlet Green's function for the sphere is

$$G_D(\mathbf{x},\mathbf{y}) \;=\; \frac{1}{4\pi\sqrt{x^2+y^2-2xyc}} \;-\; \frac{R}{4\pi\sqrt{x^2y^2+R^4-2R^2xyc}}\,, \tag{S.3}$$

exactly as in eq. (2). *Quod erat demonstrandum.*

For completeness sake — although I do not expect the students to do this, — let me formally verify that (2) is indeed the Green's function of the Laplace equations for the

Dirichlet boundary conditions. First, let's take the Laplacian WRT $\mathbf{x}$ of eq. (2):

$$\begin{aligned} G_D(\mathbf{x},\mathbf{y}) &= \frac{1}{4\pi\sqrt{x^2+y^2-2xyc}} - \frac{R}{4\pi\sqrt{x^2y^2+R^4-2R^2xyc}} \\ &= \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} - \frac{R/|\mathbf{y}|}{4\pi|\mathbf{x}-\mathbf{y}_{\rm im}|}\,, \\ -\nabla_x^2 G(\mathbf{x},\mathbf{y}) &= -\nabla_x^2\left(\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} - \frac{R/|\mathbf{y}|}{4\pi|\mathbf{x}-\mathbf{y}_{\rm im}|}\right) \\ &= \delta^{(3)}(\mathbf{x}-\mathbf{y}) - (R/|\mathbf{y}|)\,\delta^{(3)}(\mathbf{x}-\mathbf{y}_{\rm im}). \end{aligned} \tag{S.4}$$

In the second term here, $\mathbf{y}_{\rm im} = (R^2/|\mathbf{y}|^2)\mathbf{y}$, so if the real charge is at $\mathbf{y}$ outside the sphere then its image is at $\mathbf{y}_{\rm im}$ inside the sphere. Consequently, when we restrict both arguments $\mathbf{x}$ and $\mathbf{y}$ of the Green's function to the outside the conducting sphere, then $\mathbf{x} \neq \mathbf{y}_{\rm im}$ so the second delta-function in eq. (S.4) vanishes for all valid $\mathbf{x}$ and $\mathbf{y}$. Thus,

$$\text{for both } \mathbf{x} \text{ and } \mathbf{y} \text{ outside the sphere,}\quad -\nabla_x^2 G(\mathbf{x},\mathbf{y}) = \delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{S.5}$$

which confirms that $G(\mathbf{x},\mathbf{y})$ in eq. (2) is a valid Green's function.

Finally, let's confirm that this $G_D$ obeys the Dirichlet boundary condition at the sphere's surface, that is, $G_D(\mathbf{x},\mathbf{y}) = 0$ for $\mathbf{x} \in$ surface. In terms of $x = |\mathbf{x}|$, $y = |\mathbf{y}|$, and $c = \mathbf{n}_x \cdot \mathbf{n}_y$, this means $G(x,y,c) = 0$ for $x = R$, and indeed,

$$\begin{aligned} G(\mathbf{x},\mathbf{y}) &= \frac{1}{4\pi\sqrt{x^2+y^2-2cxy}} - \frac{R}{4\pi\sqrt{x^2y^2+R^4-2cR^2xy}} \\ &\underset{x=R}{\longrightarrow} \frac{1}{4\pi\sqrt{R^2+y^2-2cRy}} - \frac{R}{4\pi\sqrt{R^2y^2+R^4-2cR^3y}} \\ &= 0. \end{aligned} \tag{S.6}$$

Problem **1**(b):

Note: In the Gauss theorem and related formulae, the normal direction $\mathbf{n}$ to the boundary should point from the inside of the volume in question to the outside. Since the volume in question is the *outside* of the sphere, the normal direction $\mathbf{n}$ points from the outside of the sphere to its inside, which makes it opposite to the radial direction. Thus,

$$\mathbf{n}(\mathbf{x})\cdot\nabla_x G(\mathbf{x},\mathbf{y}) \;=\; -\frac{\partial}{\partial|\mathbf{x}|}\,G(\mathbf{x},\mathbf{y}), \tag{S.7}$$

or in the context of $G(x,y,c)$ as in eq. (2),

$$\mathbf{n}(\mathbf{x})\cdot\nabla_x G(\mathbf{x},\mathbf{y}) \;=\; -\frac{\partial G(x,y,c)}{\partial x}\,. \tag{S.8}$$

Specifically,

$$\begin{aligned}
-\frac{\partial G(x,y,c)}{\partial x} &= \frac{x-cy}{4\pi[x^2+y^2-2cxy]^{3/2}} - \frac{R(xy^2-cyR^2)}{4\pi[x^2y^2+R^4-2cxyR^2]^{3/2}}\\
&\underset{x=R}{\longrightarrow} \frac{R-cy}{4\pi[R^2+y^2-2cRy]^{3/2}} - \frac{R^2(y^2-cyR)}{4\pi[R^2y^2+R^4-2cR^3y]^{3/2}}\\
&= \frac{R-cy}{4\pi[R^2+y^2-2cRy]^{3/2}} - \frac{R^2(y^2-cyR)}{4\pi R^3\,[y^2+R^2-2cRy]^{3/2}}\\
&= \frac{1}{4\pi[R^2+y^2-2cRy]^{3/2}}\left((R-cy)-\frac{y^2-cyR}{R}=\frac{R^2-y^2}{R}\right)\\
&= \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|^3}\times\frac{R^2-\mathbf{y}^2}{R}\,.
\end{aligned} \tag{S.9}$$

<u>Problem **1**(c):</u>

Eq. (1) follows from eq. (S.9) and the Green theorem explained in class: Given the Green's function $G_D(\mathbf{x},\mathbf{y})$ for some volume with Dirichlet boundary condition $\Phi=0$ on its surface, the potential $\Phi(\mathbf{y})$ throughout the volume in questions for any given charge density $\rho(\mathbf{x})$ inside the volume and any given potential $\Phi_b(\mathbf{x})$ in its surface obtains as

$$\Phi(\mathbf{y}) \;=\; \frac{1}{\epsilon_0}\iiint\limits_{\mathcal{V}} d^4\mathbf{x}\,\rho(\mathbf{x})\times G_D(\mathbf{x},\mathbf{y}) \;-\; \iint\limits_{\text{boundary}} d^2\text{Area}(\mathbf{x})\,\Phi(\mathbf{x})\,\big(\mathbf{n}(\mathbf{x})\cdot\nabla_x G_D(\mathbf{x},\mathbf{y})\big). \tag{S.10}$$

For the problem at hand, there are no charges inside the volume in question (outside the sphere), and the normal derivative of the Green's function (with Dirichlet BC) is given in eq. (S.9). Thus,

$$\Phi(\mathbf{y}) \;=\; +\frac{y^2-R^2}{4\pi R}\iint\limits_{\text{sphere}}\frac{d^2\mathbf{x}\,\Phi(\mathbf{x})}{[R^2+y^2-2cRy]^{3/2}} \;=\; +\frac{|\mathbf{y}|^2-R^2}{4\pi R}\iint\limits_{\text{sphere}}\frac{d^2\mathbf{x}\,\Phi(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^3}\,. \tag{S.11}$$

in perfect agreement with eq. (1). *Quod erat demonstrandum.*

Problem **2**, preface to (a–c):

The potentials (3), (4), and (8) in parts (a–c) all have form

$$\Phi(\mathbf{x}) \;=\; \frac{A(\mathbf{x})B(\mathbf{x})}{4\pi\epsilon} \tag{S.12}$$

where $A(\mathbf{x})$ is a polynomial in $(x,y,z)$ and $B(\mathbf{x})$ is negative power of $r$. Ignoring possible subtleties at the origin, we may evaluate the the Laplacian at all other points using the product rule

$$4\pi\epsilon_0\,\nabla^2\Phi \;=\; \nabla^2(AB) \;=\; A(\nabla^2 B) \;+\; 2(\nabla A)\cdot(\nabla B) \;+\; (\nabla^2 A)B. \tag{S.13}$$

We shall see in a moment that for all the potentials (3), (4), and (8), this Laplacian vanishes if and only if the corresponding multipole moment tensor has zero trace.

Problem **2**(a):

For the potential (3), we take $A(\mathbf{x}) = Q_{ij}x_ix_j$ and $B(\mathbf{x}) \;=\; 1/r^5$. Consequently

$$\nabla_k B \;=\; -\frac{5n_k}{r^6} \;=\; -\frac{5x_k}{r^7}\,, \tag{S.14}$$

$$\nabla^2 B \;=\; \left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)\frac{1}{r^5} \;=\; +\frac{5\times 6}{r^7} \;-\; \frac{2\times 5}{r^7} \;=\; +\frac{20}{r^7}\,, \tag{S.15}$$

while

$$\nabla_k(A=Q_{ij}x_ix_j) \;=\; Q_{ij}(\delta_{ik}x_j+\delta_{jk}x_i) \;=\; Q_{kj}x_j \;+\; Q_{ik}x_i \;=\; 2Q_{kj}x_j \tag{S.16}$$

where the last equality stems from the symmetry $Q_{ij} = Q_{ji}$, and hence

$$\nabla^2(A = Q_{ij}x_ix_k) \;=\; \nabla_k\bigl(2Q_{kj}x_j\bigr) \;=\; 2Q_{kj}\delta_{kj} \;=\; 2\,\mathrm{tr}(Q). \tag{S.17}$$

Plugging all these formulae into the Laplacian-of-the-product rule (S.13), we arrive at

$$4\pi\epsilon_0\,\nabla^2\Phi \;=\; (Q_{ij}x_ix_j)\times\frac{20}{r^7} \;+\; 2\bigl(2Q_{kj}x_j\bigr)\times\frac{-5x_k}{r^7} \;+\; \mathrm{tr}(Q)\times\frac{1}{r^5}\,, \tag{S.18}$$

where the first two terms on the RHS cancel each other. Therefore, for the would-be quadrupole potential (3) we have

$$\nabla^2\Phi(\mathbf{x}) \;=\; \frac{\mathrm{tr}(\mathcal{Q})}{r^5}\,, \tag{S.19}$$

which vanishes if and only if the would-be quadrupole moment tensor $\mathcal{Q}_{ij}$ is traceless, $\mathrm{tr}(\mathcal{Q}) = \mathcal{Q}_{kk} = 0$. *Quad erat demonstrandum.*

Problem **2**(b):

For the would-be octupole potential (4), we take $A(\mathbf{x}) = \mathcal{O}_{ijk}x_ix_jx_k$ while $B(\mathbf{x}) = 1/r^7$. Consequently,

$$\nabla_\ell B \;=\; -\frac{7n_\ell}{r^8} \;=\; -\frac{7x_\ell}{r^9}\,, \tag{S.20}$$

$$\nabla^2 B \;=\; \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\frac{1}{r^7} \;=\; +\frac{7\times 8}{r^9} \;-\; \frac{2\times 7}{r^9} \;=\; +\frac{42}{r^7}\,, \tag{S.21}$$

while

$$\begin{aligned}\nabla_\ell A \;&=\; \mathcal{O}_{ijk}\nabla_\ell(x_ix_jx_k)\\ &=\; \mathcal{O}_{ijk}\bigl(\delta_{\ell i}x_jx_k \;+\; \delta_{\ell,j}x_ix_k \;+\; \delta_{\ell,k}x_ix_j\bigr)\\ &=\; \mathcal{O}_{\ell jk}x_jx_k \;+\; \mathcal{O}_{i\ell k}x_ix_k \;+\; \mathcal{O}_{ij\ell}x_ix_j \qquad \text{(S.22)}\\ &\qquad \langle\!\langle\,\text{thanks to the symmetry (5) of the } \mathcal{O}_{ijk} \text{ tensor}\,\rangle\!\rangle\\ &=\; 3\mathcal{O}_{\ell jk}x_jx_k\end{aligned}$$

and hence

$$\begin{aligned}
\nabla^2 A \;&=\; \nabla_\ell \nabla_\ell A \;=\; \nabla_\ell \bigl(3\mathcal{O}_{\ell jk} x_j x_k\bigr) \\
&=\; 3\mathcal{O}_{\ell jk}\bigl(\delta_{\ell j} x_k \,+\, \delta_{\ell k} x_i\bigr) \\
&=\; 3\mathcal{O}_{\ell\ell k} x_k \;+\; 3\mathcal{O}_{\ell j\ell} x_j \\
&=\; 6\mathcal{O}_{\ell\ell k}\, x_k \quad \langle\!\langle \text{ by symmetry } \rangle\!\rangle \\
&=\; 6\,\mathrm{tr}(\mathcal{O})_k\, x_k\,.
\end{aligned} \tag{S.23}$$

Altogether, using eq. (S.13) for the Laplacian of the product, we get

$$\begin{aligned}
4\pi\epsilon_0\, \nabla^2\Phi(\mathbf{x}) \;&=\; \bigl(\mathcal{O}_{ijk} x_i x_j x_k\bigr) \times \frac{42}{r^9} \;+\; 2\bigl(3\mathcal{O}_{\ell jk} x_j x_k\bigr) \times \frac{-7x_\ell}{r^9} \;+\; \bigl(6\,\mathrm{tr}(\mathcal{O})_k\, x_k\bigr) \times \frac{1}{r^7} \\
&\quad \langle\!\langle \text{ where the first two terms cancel each other } \rangle\!\rangle \\
&=\; \frac{6\,\mathrm{tr}(\mathcal{O})_k\, x_k}{r^7}\,.
\end{aligned} \tag{S.24}$$

Obviously, this Laplacian vanishes for all $\mathbf{x} \neq \mathbf{0}$ if and only if

$$\mathrm{tr}(\mathcal{O})_k \;\equiv\; \mathcal{O}_{iik} \;\equiv\; \delta_{ij}\mathcal{O}_{ijk} \;=\; 0 \quad \forall k = x, y, z. \tag{S.25}$$

*Quod erat demonstrandum.*

Problem **2**(c):

For $\ell > 3$, the would-be multipole potential (8) can be handled exactly as the would-be quadrupole (3) or would-be octupole (4). For general $\ell \geq 2$ we take

$$A(\mathbf{x}) \;=\; \mathcal{M}^{(\ell)}_{i_1,\ldots,i_\ell}\, x_{i_1} \cdots x_{i_\ell}\,, \quad B(\mathbf{x}) \;=\; \frac{1}{r^{2\ell+1}}\,, \tag{S.26}$$

hence

$$\nabla_j B \;=\; -(2\ell+1)\frac{n_j}{r^{2\ell+2}} \;=\; -(2\ell+1)\frac{x_j}{r^{2\ell+3}}\,, \tag{S.27}$$

$$
\begin{aligned}
\nabla^2 B &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\frac{1}{r^{2\ell+1}} \\
&= \frac{(2\ell+1)(2\ell+2)}{r^{2\ell+3}} - \frac{2(2\ell+1)}{r^{2\ell+3}} \\
&= \frac{(2\ell)(2\ell+1)}{r^{2\ell+3}}\,,
\end{aligned}
\tag{S.28}
$$

while

$$
\begin{aligned}
\nabla_j A &= \mathcal{M}^{(\ell)}_{i_1,\ldots,i_\ell}\left(\delta_{j,i_1}x_{i_2}\cdots x_{i_\ell} + \cdots + \delta_{j,i_\ell}x_{i_1}\cdots x_{i_{\ell-1}}\right) \\
&= \sum_{n=1}^{\ell}\mathcal{M}^{(\ell)}_{i_1,\ldots,i_{n-1},j,i_{n+1},\ldots i_\ell}\times x_{i_1}\cdots x_{i_{n-1}}\,x_{i_{n+1}}\cdots x_{i_\ell} \\
&\quad\langle\!\langle\text{ by the symmetry (9) of the tensor } \mathcal{M}^{(\ell)}_{i_1,\ldots,i_\ell}\rangle\!\rangle \\
&= \ell\mathcal{M}^{(\ell)}_{i_1,\ldots i_{\ell-1},j}\times x_{i_1}\cdots x_{i_{\ell-1}}\,,
\end{aligned}
\tag{S.29}
$$

$$
\begin{aligned}
\nabla^2 A &= \nabla_j(\nabla_j A) \\
&= \ell\mathcal{M}^{(\ell)}_{i_1,\ldots i_{\ell-1},j}\times\left(\delta_{j,i_1}x_{i_2}\cdots x_{i_{\ell-1}} + \cdots + \delta_{j,i_{\ell-1}}x_{i_1}\cdots x_{i_{\ell-2}}\right) \\
&= \sum_{n=1}^{\ell-1}\ell\mathcal{M}^{(\ell)}_{i_1,\ldots,i_{n-1},j,i_{n+1},\ldots i_{\ell-1},j}\times x_{i_1}\cdots x_{i_{n-1}}\,x_{i_{n+1}}\cdots x_{i_{\ell-1}} \\
&\quad\langle\!\langle\text{ by symmetry }\rangle\!\rangle \\
&= (\ell-1)\ell\mathcal{M}^{(\ell)}_{i_1,\ldots,i_{\ell-2},j,j}\times x_{i_1}\cdots x_{i_{\ell-2}} \\
&= (\ell-1)\ell\,\mathrm{tr}\big(\mathcal{M}^{(\ell)}\big)_{i_1,\ldots,i_{\ell-2}}\times x_{i_1}\cdots x_{i_{\ell-2}}\,.
\end{aligned}
\tag{S.30}
$$

Note: on the last line here, $\mathrm{tr}\big(\mathcal{M}^{(\ell)}\big)$ is a $(\ell-2)$ index totally symmetric tensor, which we then contract with $(\ell-2)$ copies of the $x_i$ vector to get a scalar.

Plugging the above formulae into the product rule (S.13), we see that the first two terms on the RHS cancel each other:

$$
\begin{aligned}
A\times\nabla^2 B + 2\nabla_k A\times\nabla_k B &= \left(\mathcal{M}^{(\ell)}_{i_1,\ldots,i_\ell}x_{i_i}\cdots x_{i_\ell}\right)\times\frac{(2\ell+1)(2\ell)}{r^{2\ell+3}} \\
&\quad + 2\left(\ell\,\mathcal{M}^{(\ell)}_{k,i_2,\ldots,i_\ell}x_{i_2}\cdots x_{i_\ell}\right)\times\frac{-(2\ell+1)x_k}{r^{2\ell+3}} \\
&= 0.
\end{aligned}
\tag{S.31}
$$

Thus, the overall Laplacian stems from the third term only, namely

$$\begin{aligned} 4\pi\epsilon_0\,\nabla^2\Phi &= (\nabla^2 A)\times B \\ &= \frac{\ell(\ell-1)\,\mathrm{tr}\big(\mathcal{M}^{(\ell)}\big)_{i_1,\ldots,i_{\ell-2}}\times x_{i_1}\cdots x_{i_{\ell-2}}}{r^{2\ell+1}} \\ &= \frac{\ell(\ell-1)\,\mathrm{tr}\big(\mathcal{M}^{(\ell)}\big)_{i_1,\ldots,i_{\ell-2}}\times n_{i_1}\cdots n_{i_{\ell-2}}}{r^{\ell+3}}\,. \end{aligned} \tag{S.32}$$

or in other words

$$\nabla^2\Phi(\mathbf{x}) = \frac{(\ell(\ell-1)}{4\pi\epsilon_0}\times\frac{\mathrm{tr}\left(\mathcal{M}^{(\ell)}\right)_{i_3,\ldots,i_\ell} n_{i_3}\cdots n_{i_\ell}}{r^{\ell+3}}\,. \tag{S.33}$$

To make this Laplacian vanish for all $\mathbf{x}\neq 0$, the numerator must vanish for all the unit direction vectors $\mathbf{n}$, which happens if and only if $\mathrm{tr}\left(\mathcal{M}^{(\ell)}\right)$ vanishes as a $(\ell-2)$–index symmetric tensor. In other words, the trace

$$\mathrm{tr}\left(\mathcal{M}^{(\ell)}\right)_{i_3,\ldots,i_\ell} = \sum_k \mathcal{M}^{(\ell)}_{k,k,i_3,\ldots,i_\ell} \tag{S.34}$$

must vanish for all values of the indices $i_3,\ldots,i_\ell$. And that's precisely what we mean by saying that the would-be multipole moment tensor $\mathcal{M}^{(\ell)}$ must be traceless. *Quod erat demonstrandum.*

Problem **2**(d):

The $2^\ell$–pole moment is a totally symmetric tensor with $\ell$ indices and zero trace tensor. Let's count the number of independent components of such a tensor.

But first, let's count the number of independent components of a totally symmetric tensor without the zero-trace condition. By symmetry, an independent component is characterized by the net numbers of its indices which happen to be take values $x$, $y$, or $z$, but we do not care about the ordering of such indices. Indeed, for any non-negative integers $a, b, c$ with $a+b+c=\ell$, any component with $a$ indices $= x$, $b$ indices $= y$, and $c$ indices $= z$ is related by symmetry to any other such component. Consequently, the number of independent components is equal to the number of partitions of $\ell$ into 3 non-negative integers $a, b, c$, specifically

$$\#\text{independent components} \;=\; N_\ell \;=\; \tfrac{1}{2}(\ell+1)(\ell+2). \tag{S.35}$$

Next, consider the zero trace condition for $\ell \geq 2$. The trace of an $\ell$–index totally symmetric tensor is an $(\ell-2)$–index totally symmetric tensor, so demanding this trace to vanish imposes $N_{\ell-2}$ independent linear constraints. Consequently, the number of independent components of a traceless symmetric tensor is

$$N_\ell - N_{\ell-2} \;=\; \tfrac{1}{2}(\ell+1)(\ell+2) - \tfrac{1}{2}(\ell-2)(\ell) \;=\; \tfrac{1}{2}(\ell^2+3\ell+2) - \tfrac{1}{2}(\ell^2-\ell) \;=\; \tfrac{1}{2}(4\ell+2) \;=\; 2\ell+1. \tag{S.36}$$

*Quod erat demonstrandum.*

Problem **3**(a):

The potential energy of a charged body with charge density $\rho(\mathbf{x})$ in an external potential $\Phi(\mathbf{x})$ is simply

$$U \;=\; \iiint\limits_{\text{body}} d^3\mathbf{x}\, \rho(\mathbf{x})\Phi(\mathbf{x}). \tag{S.37}$$

When the body in question has small size $a$ while the potential $\Phi(\mathbf{x})$ *slowly* varies with $\mathbf{x}$ on a much larger distance scale, we may expand the potential into a Taylor series around some point $\mathbf{x}_0$ inside the body, thus

$$\Phi(\mathbf{x}) \;=\; \Phi(\mathbf{x}_0) \;+\; (\mathbf{x}-\mathbf{x}_0)_i\, \nabla_i\Phi(\mathbf{x}_0) \;+\; \tfrac{1}{2}(\mathbf{x}-\mathbf{x}_0)_i(\mathbf{x}-\mathbf{x}_0)_j\, \nabla_i\nabla_j\Phi(\mathbf{x}_0) \;+\; \cdots. \tag{S.38}$$

Plugging this series into eq. (S.37) for the potential energy, we get

$$\begin{aligned} U \;&=\; \Phi(\mathbf{x}_0)\times\iiint d^3\mathbf{x}\,\rho(\mathbf{x}) \;+\; \nabla_i\Phi(\mathbf{x}_0)\times\iiint d^3\mathbf{x}\,\rho(\mathbf{x})\,(\mathbf{x}-\mathbf{x}_0)_i \;+\; \cdots \\ &=\; \Phi(\mathbf{x}_0)\times Q_{\text{net}} \;+\; \nabla_i\Phi(\mathbf{x}_0)\times p_i \;+\; \cdots \end{aligned} \tag{S.39}$$

where $p_i$ is the $i^{\text{th}}$ component of the body's electric dipole moment (evaluated relative to the $\mathbf{x}_0$), while $\cdots$ denote the terms related to the higher multipole moments, *cf.* part (c).

In particular, for a body of zero net charge, finite dipole moment, and negligible higher multipole moments, we have

$$\begin{aligned} U &= 0 + p_i \nabla_i \Phi(\mathbf{x}_0) + \text{negligible} \\ &= \mathbf{p} \cdot \nabla \Phi(\mathbf{x}_0) = -\mathbf{p} \cdot \mathbf{E}(\mathbf{x}_0), \end{aligned} \tag{S.40}$$

exactly as in eq. (12).

<u>Problem **3**(b):</u>
Under infinitesimal shifts of the body (but without rotations of deformations), we have

$$\mathbf{x}_0 \to \mathbf{x}_0 + \delta\mathbf{x}_0 , \quad \mathbf{p} \to \mathbf{p} + 0, \tag{S.41}$$

hence

$$\delta E_i(\mathbf{x}_0) = \nabla_j E_i(\mathbf{x}_0)\, (\delta\mathbf{x}_0)_j , \tag{S.42}$$

and

$$\delta U = -p_i\, \delta E_i(\mathbf{x}_0) = -p_i\, \nabla_j E_i(\mathbf{x}_0)\, (\delta\mathbf{x}_0)_j . \tag{S.43}$$

According to eq. (13), this infinitesimal change of the potential energy is related to the net force $\mathbf{F}$ on the dipole as

$$\delta U = -F_j\, (\delta\mathbf{x}_0)_j , \tag{S.44}$$

which means

$$F_j = +p_i \nabla_j E_i(\mathbf{x}_0), \tag{S.45}$$

which in vector notations becomes the first equality in eq. (14.a),

$$\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E})@\mathbf{x}_0 . \tag{S.46}$$

To get the second equality in eq. (14.a), we use

$$\nabla_j E_i = -\nabla_j \nabla_i \Phi = -\nabla_i \nabla_j \Phi = +\nabla_i E_j \tag{S.47}$$

(for the static electric field without a curl). Plugging this formula into eq. (S.45), we get

$$F_j = +p_i \nabla_i E_j(\mathbf{x}_0), \tag{S.48}$$

or in vector notations

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}(\mathbf{x}_0). \tag{S.49}$$

Now consider an infinitesimal rotation of the body around the point $\mathbf{x}_0$ through angle $\delta\vec{\alpha}$. (Note: in 3D, and infinitesimal angle is a vector but a finite angle is not.) Consequently, the dipole moment of the body rotates by $\delta\vec{\alpha}$, thus

$$\mathbf{p} \to \mathbf{p} + \delta\alpha \times \mathbf{p}, \tag{S.50}$$

while the location $\mathbf{x}_0$ — and hence the electric field $\mathbf{E}(\mathbf{x}_0)$ — remains unchanged. Consequently, the potential energy changes by

$$\delta U = -(\delta\mathbf{p}) \cdot \mathbf{E}(\mathbf{x}_0) = -(\delta\vec{\alpha} \times \mathbf{p}) \cdot \mathbf{E}(\mathbf{x}_0) = -\delta\vec{\alpha} \cdot (\mathbf{p} \times \mathbf{E}(\mathbf{x}_0)). \tag{S.51}$$

According to eq. (13), this infinitesimal charge of the potential energy is related to the net torque $\vec{\tau}$ on the body (relative to the pivot point $\mathbf{x}_0$) as

$$\delta U = -\delta\vec{\alpha} \cdot \vec{\tau}, \tag{S.52}$$

hence

$$\vec{\tau} = \mathbf{p} \times \mathbf{E}(\mathbf{x}_0), \tag{S.53}$$

exactly as in eq. (14.b). *Quod erat demonstrandum.*

Problem **3**(c):

Let's go back to the Taylor series (S.38), plug it into eq. (S.37) for the potential energy just as we did it in part (a), but instead of stopping after the first two terms in the series as in eq. (S.39), let's spell out the third term as well:

$$\begin{aligned}
U &= \Phi(\mathbf{x}_0) \times \iiint d^3\mathbf{x}\,\rho(\mathbf{x}) \;+\; \nabla_i\Phi(\mathbf{x}_0) \times \iiint d^3\mathbf{x}\,\rho(\mathbf{x})(\mathbf{x}-\mathbf{x}_0)_i \\
&\quad + \tfrac{1}{2}\nabla_i\nabla_j\Phi(\mathbf{x}_0) \times \iiint d^3\mathbf{x}\,\rho(\mathbf{x})(\mathbf{x}-\mathbf{x}_0)_i(\mathbf{x}-\mathbf{x}_0)_j \;+\; \cdots \\
&= \Phi(\mathbf{x}_0) \times Q_{\text{net}} \;+\; \nabla_i\Phi(\mathbf{x}_0) \times p_i \;+\; \tfrac{1}{2}\nabla_i\nabla_j\Phi(\mathbf{x}_0) \times T_{ij} \;+\; \cdots
\end{aligned} \tag{S.54}$$

where

$$T_{ij} \;=\; \iiint\limits_{\text{body}} d^3\mathbf{x}\,\rho(\mathbf{x})(\mathbf{x}-\mathbf{x}_0)_i(\mathbf{x}-\mathbf{x}_0)_j\,. \tag{S.55}$$

This $T_{ij}$ is a symmetric tensor related to the body's quadrupole moment tensor

$$Q_{ij} \;=\; \iiint\limits_{\text{body}} d^3\mathbf{x}\,\rho(\mathbf{x})\left[\tfrac{3}{2}(\mathbf{x}-\mathbf{x}_0)_i(\mathbf{x}-\mathbf{x}_0)_j \;-\; \tfrac{1}{2}\delta_{ij}(\mathbf{x}-\mathbf{x}_0)^2\right] \tag{S.56}$$

(relative to the point $\mathbf{x}_0$ inside the body), but unlike the quadrupole moment tensor, the $T_{ij}$ tensor is not traceless, $\operatorname{tr}(T) \neq 0$. However, the traceless part of the $T_{ij}$ tensor is proportional to the quadrupole moment, so we may write

$$T_{ij} \;=\; \tfrac{2}{3}Q_{ij} \;+\; \delta_{ij}S \tag{S.57}$$

for some scalar $S = \frac{1}{3}\operatorname{tr}(T)$.

Now let's plug eq. (S.57) into the third terms in the series (S.54) for the potential energy:

$$\begin{aligned}
U_{\text{3rd term}} &= \tfrac{1}{2}T_{ij}\nabla_i\nabla_j\Phi(\mathbf{x}_0) \\
&= \tfrac{1}{3}Q_{ij}\nabla_i\nabla_j\Phi(\mathbf{x}_0) \;+\; \tfrac{1}{2}S\delta_{ij}\nabla_i\nabla_j\Phi(\mathbf{x}_0) \\
&= \tfrac{1}{3}Q_{ij}\nabla_i\nabla_j\Phi(\mathbf{x}_0) \;+\; \tfrac{1}{2}S\,\nabla^2\Phi(\mathbf{x}_0).
\end{aligned} \tag{S.58}$$

Note that the $\Phi(x)$ here is the *external* potential which obeys the Laplace equation, $\nabla^2\Phi = 0$, so the second terms on the last line here vanishes, regardless of the value of the scalar moment

$S$. Instead, the third term in $U$ is completely determined by the quadrupole moment of the body, thus

$$U \;=\; \Phi(\mathbf{x}_0)\times Q_{\rm net} \;+\; \nabla_i\Phi(\mathbf{x}_0)\times p_i \;+\; \tfrac{1}{3}Q_{ij}\nabla_i\nabla_j\Phi(\mathbf{x}_0) \;+\; \cdots. \tag{S.59}$$

In particular, for the body which has zero net charge, zero net dipole moment, finite quadrupole moment, and negligible higher multipole moments, we have

$$U \;=\; \tfrac{1}{3}Q_{ij}\nabla_i\nabla_j\Phi(\mathbf{x}_0) \;=\; -\tfrac{1}{3}Q_{ij}\nabla_i\mathbf{E}_j(\mathbf{x}_0), \tag{S.60}$$

exactly as in eq. (15).

Problem **3**(d):

Let's proceed similar to part (b). The net force on the quadrupole follow from the variation of the potential energy (15) under infinitesimal shifts of the body (but without rotation or deformation). Under such shifts, the quadrupole moment tensor $Q_{ij}$ remains unchanged, while $\mathbf{x}_0 \to \mathbf{x}_0 + \delta\mathbf{x}_0$. Consequently,

$$\begin{aligned} \delta\Phi(\mathbf{x}_0) &= (\delta\mathbf{x}_0)_k\,\nabla_k\Phi(\mathbf{x}_0),\\ \delta\nabla_i\Phi(\mathbf{x}_0) &= (\delta\mathbf{x}_0)_k\,\nabla_k\nabla_i\Phi(\mathbf{x}_0),\\ \delta\nabla_i\nabla_j\Phi(\mathbf{x}_0) &= (\delta\mathbf{x}_0)_k\,\nabla_k\nabla_i\nabla_j\Phi(\mathbf{x}_0), \end{aligned} \tag{S.61}$$

and therefore

$$\delta U \;=\; \tfrac{1}{3}Q_{ij}\nabla_k\nabla_i\nabla_j\Phi(\mathbf{x}_0)\,(\delta\mathbf{x}_0)_k\,. \tag{S.62}$$

Interpreting this change as $-F_k(\delta\mathbf{x}_0)_k$, we find

$$\begin{aligned} F_k &= -\tfrac{1}{3}Q_{ij}\nabla_k\nabla_i\nabla_j\Phi(\mathbf{x}_0)\\ &= -\tfrac{1}{3}Q_{ij}\nabla_i\nabla_j\nabla_k\Phi(\mathbf{x}_0)\\ &= +\tfrac{1}{3}Q_{ij}\nabla_i\nabla_j E_k(\mathbf{x}_0), \end{aligned} \tag{S.63}$$

or in sort-of-vector notations

$$\mathbf{F} \;=\; \tfrac{1}{3}(Q_{ij}\nabla_i\nabla_j)\mathbf{E}(\mathbf{x}_0). \tag{S.64}$$

Next consider the torque on the body (relative to the pivot point $\mathbf{x}_0$). To find this torque, we rotate the body around the point $\mathbf{x}_0$ through infinitesimal angle $\delta\vec{\alpha}$; this leaves the $\mathbf{x}_0$ —

and hence the electric field and its gradient at $\mathbf{x}_0$ — unchanged, but the quadrupole moment tensor changes by

$$\delta Q_{ij} \;=\; \epsilon_{ikm}\delta\alpha_k Q_{mj} \;+\; \epsilon_{jkm}\delta\alpha_k Q_{im} \tag{S.65}$$

where $\epsilon_{ijk}$ is the Levi–Civita totally antisymmetric unit tensor. (See the Wikipedia article for details.) Note: under infinitesimal rotations, a scalar remains invariant, a vector such as $\mathbf{p}$ changes by $\delta\mathbf{p} = \delta\vec{\alpha}\times\mathbf{p}$ — or in index notations

$$\delta p_i \;=\; \epsilon_{ikm}\delta\alpha_k p_m, \tag{S.66}$$

while a tensor suffers similar changes for each of its indices, for example eq. (S.65) for a 2-index tensor $Q_{ij}$.

Anyhow, plugging the infinitesimal change (S.65) into eq. (15) for the potential energy, we have

$$\begin{aligned}
\delta U \;&=\; \tfrac{1}{3}\epsilon_{ikm}(\delta\alpha)_k Q_{mj}\nabla_i\nabla_j\Phi(\mathbf{x}_0) \;+\; \tfrac{1}{3}\epsilon_{jkm}(\delta\alpha)_k Q_{im}\nabla_i\nabla_j\Phi(\mathbf{x}_0) \\
&\quad \langle\!\langle\, \text{swapping } i\leftrightarrow j \text{ in the second term and using } \nabla_i\nabla_j=\nabla_j\nabla_i \,\rangle\!\rangle \\
&=\; \tfrac{1}{3}\epsilon_{ikm}(\delta\alpha)_k\big(Q_{mj}+Q_{jm}\big)\nabla_j\nabla_i\Phi(\mathbf{x}_0) \\
&\quad \langle\!\langle\, \text{by the symmetry } Q_{mj}=Q_{jm} \,\rangle\!\rangle \\
&=\; \tfrac{2}{3}\epsilon_{ikm}(\delta\alpha)_k Q_{mj}\nabla_j\nabla_i\Phi(\mathbf{x}_0) \\
&=\; -\tfrac{2}{3}\epsilon_{ikm}(\delta\alpha)_k Q_{mj}\nabla_j E_i(\mathbf{x}_0).
\end{aligned} \tag{S.67}$$

On the other hand, $\delta U = -(\delta\alpha)_k\tau_k$ where $\vec{\tau}$ is the torque on the body, thus

$$\tau_k \;=\; +\tfrac{2}{3}\epsilon_{ikm}Q_{mj}\nabla_j E_i(\mathbf{x}_0) \;=\; +\tfrac{2}{3}\epsilon_{kmi}(Q_{mj}\nabla_j)E_i(\mathbf{x}_0), \tag{S.68}$$

or in vector/tensor notations

$$\vec{\tau} \;=\; +\tfrac{2}{3}(Q\circ\nabla)\times\mathbf{E}(\mathbf{x}_0) \tag{S.69}$$

where $(Q\circ\nabla)$ is a vector with components $(Q\circ\nabla)_m = Q_{mj}\nabla_j$.

Problem **4**:

The hollow conductor in question can be thought as a linear superposition of two solid cylindrical conductors — one filling up the outer cylinder and the other filling up the hole — carrying uniform currents $\pm\mathbf{J}$ in opposite directions.

So let's consider the magnetic field inside a single solid cylinder of radius $R$ carrying a uniform current $\mathbf{J}$. By the rotational and the translational symmetries of the conductor and the current, the magnetic field's magnitude depends only on the distance from the axis while its direction is circular around the axis; in cylindrical coordinates $(z, s, \phi)$

$$\mathbf{B}(z,s,\phi) \;=\; B(s)\,\mathbf{n}_\phi\,. \qquad \text{(S.70)}$$

The $\mathbf{n}_\phi$ direction of the field follows from the reflection symmetry across the plane through the axis and the point $\mathbf{x}$ where you measure the field; note that the magnetic field is an axial vector rather than a polar vector. Alternatively, consider the vector potential $\mathbf{A}(\mathbf{x})$: for a current $\mathbf{J}$ flowing in a uniform direction, the vector potential must also point in the same direction, thus $\mathbf{A}(\mathbf{x}) = A(s)\,\mathbf{n}_z$, where the magnitude depends only on $s$ by the symmetries of the system. Consequently, $\mathbf{B} = \nabla\times\mathbf{A}$ points in the direction perpendicular to both $\mathbf{n}_s$ and $\mathbf{n}_z$ — *i.e.*, the circular direction $\mathbf{n}_\phi$.

Given the direction and the symmetry of the magnetic field (S.70), its magnitude $B(s)$ follows from the Ampere's Law: Take a circle of radius $s$ coaxial with the cylinder and use it for Ampere's loop:

$$\oint\limits_{\text{circle}} \mathbf{B}(x)\cdot d\mathbf{x} \;=\; 2\pi s\,B(s) \;=\; \mu_0\times I[\text{through the circle}]. \qquad \text{(S.71)}$$

For a circle wider than the conductor the current $I$ here is the net current through the cylinder, $I = \pi R^2\times J$, hence

$$B_{\text{outside}}(s) \;=\; \frac{\mu_0\times\pi R^2\times J}{2\pi s} \;=\; \frac{\mu_0 J R^2}{2s}\,, \qquad \text{(S.72)}$$

while for the circle embedded inside the cylinder only a part of the net current flows through the circle, thus $J = \pi s^2 \times J$ and hence

$$B_{\text{inside}}(s) \;=\; \frac{\mu_0 \times \pi s^2 \times J}{2\pi s} \;=\; \frac{\mu_0 J s}{2}\,, \tag{S.73}$$

In vector notations, the field inside the cylinder can be written as

$$\mathbf{B}_{\text{inside}}(\mathbf{x}) \;=\; \frac{\mu_0}{2}\,\mathbf{J}\times\mathbf{x}. \tag{S.74}$$

Indeed,

$$\mathbf{J}\times\mathbf{x} \;=\; (J\mathbf{n}_j)\times(z\mathbf{n}_z+s\mathbf{n}_s) \;=\; Jz\,(\mathbf{n}_z\times\mathbf{n}_z=0) \;+\; Js\,(\mathbf{n}_z\times\mathbf{n}_s=\mathbf{n}_\phi)) \;=\; Js\,\mathbf{n}_\phi\,. \tag{S.75}$$

Or in a more general coordinate frame,

$$\mathbf{B}_{\text{inside}}(\mathbf{x}) \;=\; \frac{\mu_0}{2}\,\mathbf{J}\times(\mathbf{x}-\mathbf{y}_0) \tag{S.76}$$

where $\mathbf{y}_0$ is some point on the axis of the cylindrical conductor.

Now let's go back to the hollow cylindrical conductor whose current uniform current $\mathbf{J}$ can be viewed as a superposition of (1) $\mathbf{J}_1 = +\mathbf{J}$ flowing through the solid outer cylinder (both the conductor and the hole in it), and (2) $\mathbf{J}_2 = -\mathbf{J}$ flowing through the cylindrical hole. The space inside the hole is inside both cylindrical currents, so the magnetic field inside the hole is

$$\mathbf{B}_{\text{hole}}(\mathbf{x}) \;=\; \mathbf{B}_1(\mathbf{x}) \;+\; \mathbf{B}_2(\mathbf{x}) \tag{S.77}$$

where both $\mathbf{B}_1$ and $\mathbf{B}_2$ are given by the appropriate eq. (S.76). Specifically,

$$\begin{aligned}\mathbf{B}_{\text{hole}}(\mathbf{x}) \;&=\; \frac{\mu_0}{2}\,\mathbf{J}_1\times(\mathbf{x}-\mathbf{y}_1) \;+\; \frac{\mu_0}{2}\,\mathbf{J}_2\times(\mathbf{x}-\mathbf{y}_2)\\ &=\; \frac{\mu_0}{2}\,\mathbf{J}\times\big((\mathbf{x}-\mathbf{y}_1) \;-\; (\mathbf{x}-\mathbf{y}_2)\big)\\ &=\; \frac{\mu_0}{2}\,\mathbf{J}\times(\mathbf{y}_2-\mathbf{y}_1).\end{aligned} \tag{S.78}$$

Note that the $\mathbf{x}$ cancels out from the last formula, so *the magnetic field inside the hole is uniform!*

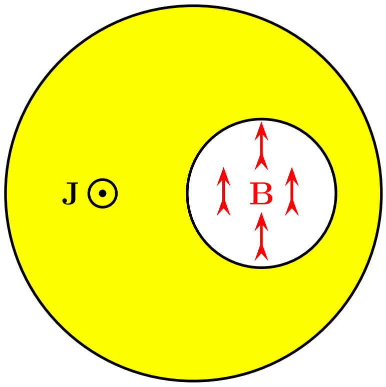
Finally, the $\mathbf{y}_2 - \mathbf{y}_1$ difference in eq. (S.78) is the displacement $\mathbf{d}$ of the hole's axis relative to the outer cylinder's axis; its direction is $\perp$ to both axes and hence to the current. Thus, the uniform field inside the hole is

$$\mathbf{B}_{\text{hole}} = \frac{\mu_0}{2} \mathbf{J} \times \mathbf{d}, \tag{S.79}$$

its magnitude is

$$B_{\text{inside}} = \tfrac{1}{2} \mu_0 J d, \tag{S.80}$$

while its direction is shown on the diagram below:




The current here flows out from the screen towards your face.