Problem **1**(a):

Note: the magnetization **M** inside the magnet stays constant only in co-moving coordinates, so if we allow the magnet to move, then $\mathbf{M}(\mathbf{x})$ at a fixed $\mathbf{x}$ may suddenly change as the magnet's edge goes through the point $\mathbf{x}$. To prevent this complication, let's assume the magnet stays fixed in place, but the coil can move around the magnet.

Let's calculate the net work — electric and mechanical — when the coil moves through an infinitesimal distance while the current changes by an infinitesimal amount δI. Both of these changes make the magnetic flux through the coil change by

$$\delta\Phi \;=\; \delta\oint\limits_{\text{coil}}\mathbf{A}\cdot d\mathbf{x} \;=\; \oint\limits_{\text{coil}}\delta\mathbf{A}(\mathbf{x})\cdot d\mathbf{x} \;+\; \oint\limits_{\delta\,\text{coil}}\mathbf{A}(\mathbf{x})\cdot d\mathbf{x}. \tag{S.1}$$

As I explained in class (see also my notes on magnetic energy) the electric work by the power supply needed to overcome the EMF induced by this $\delta\Phi$ is

$$\delta W_{\text{el}} \;=\; I\times\delta\Phi \;=\; \delta_1 W_{\text{el}} \;+\; \delta_2 W_{\text{el}} \tag{S.2}$$

where

$$\delta_1 W_{\text{el}} \;=\; \oint\limits_{\text{fixed coil}}\delta\mathbf{A}(\mathbf{x})\cdot I d\mathbf{x} \tag{S.3}$$

is due to the magnetic field changing at fixed $\mathbf{x}$ while

$$\delta_2 W_{\text{el}} \;=\; \oint\limits_{\delta\,\text{coil}}(\text{fixed }\mathbf{A}(\mathbf{x}))\cdot I\,d\mathbf{x}. \tag{S.4}$$

is due to the coil's displacement.

In addition to the net electric work (S.2), there is also the mechanical work δW_{mech} due to moving the coil against the Ampere forces on it. Microscopically, these Ampere forces stem from the Lorentz forces on the electrons flowing through the coil. When the coil itself is moving, these Lorentz forces on the electrons also generate the motional EMF in the coil, and hence are responsible for the $\delta_2 W_{\rm el}$ part of the electric work on the coil. Thus altogether,

$$\delta W_{\rm mech} \;+\; \delta_2 W_{\rm el} \;=\; \text{net work of the Lorentz forces on the electrons.} \tag{S.5}$$

But the Lorentz force on an electron has direction $\perp$ to its velocity, so its work is zero. Hence, the net work of the Lorentz forces on all the electrons is zero, so eq. (S.5) leads to

$$\delta W_{\rm mech} \;+\; \delta_2 W_{\rm el} \;=\; 0 \tag{S.6}$$

and therefore

$$\delta W_{\rm net} \;=\; W_{\rm mech} \;+\; \delta_1 W_{\rm el} \;+\; \delta_2 W_{\rm el} \;=\; \delta_1 W_{\rm el} \text{ only.} \tag{S.7}$$

Note: Macroscopically, the Lorentz forces lead to the motional EMF in moving wires — which can perform electric work — and also to the magnetic forces on the wires, which can perform mechanical work. But the net work of these two effects is zero since it amounts to the net work of all the microscopic Lorentz forces. Instead, the net effect of all the microscopic Lorentz forces is *converting the electric work into mechanical work or vice verse.*

For the problem at hand this means eq. (S.7) and hence

$$\delta W_{\rm net} \;=\; \delta_1 W_{\rm el} \;=\; \oint\limits_{\text{fixed coil}} \delta\mathbf{A}(\mathbf{x})\cdot I d\mathbf{x}. \tag{S.8}$$

Similarly to what I did in class, the RHS here can be expressed in terms of the magnetic $\mathbf{H}$ and $\mathbf{B}$ fields as

$$\oint\limits_{\text{coil}} \delta\mathbf{A}\cdot I d\mathbf{x} \;\longrightarrow\; \iiint \delta\mathbf{A}\cdot\mathbf{J}\, d^3\mathbf{x} \;=\; \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \delta\mathbf{B}\cdot\mathbf{H}\, d^3\mathbf{x}, \tag{S.9}$$

thus

$$\delta W_{\rm net} \;=\; \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \delta\mathbf{B}\cdot\mathbf{H}\, d^3\mathbf{x}. \tag{S.10}$$

Next, consider the relation between the magnetic fields $\mathbf{H}$ and $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$. When we vary the $\mathbf{H}(\mathbf{x})$ fields, the $\mathbf{B}(\mathbf{x})$ field varies by

$$\delta\mathbf{B}(\mathbf{x}) \;=\; \mu_0\delta\mathbf{H}(\mathbf{x}) \;+\; \mu_0\delta\mathbf{M}(\mathbf{x}) \tag{S.11}$$

where the second term obviously vanishes for $\mathbf{x}$ outside the magnet. For $\mathbf{x}$ inside a regular permanent magnet we might have $\delta\mathbf{M} \neq 0$ in response to the $\delta\mathbf{H}$, but the magnet in question is assumed to be so hard that its magnetization $\mathbf{M}$ remains constant despite the varying $\mathbf{H}$ field. Thus, for the problem at hand $\delta\mathbf{M}(\mathbf{x}) = 0$ for all $\mathbf{x}$ — both inside and outside the magnet — and therefore

$$\delta\mathbf{B}(\mathbf{x}) \;=\; \mu_0\delta\mathbf{H}(\mathbf{x}) \quad \text{at all } \mathbf{x}. \tag{S.12}$$

Consequently,

$$\delta W_{\text{net}} \;=\; \mu_0 \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{H}\cdot\delta\mathbf{H}\, d^3\mathbf{x} \;=\; \frac{\mu_0}{2} \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \delta(\mathbf{H}^2)\, d^3\mathbf{x} \;=\; \delta U \tag{S.13}$$

where

$$U \;=\; \frac{\mu_0}{2} \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{H}^2\, d^3\mathbf{x} \tag{S.14}$$

exactly as in eq. (2).

Finally, integrating eq. (S.13) over a finite process of changing the current and/or moving the coil (and hence changing the $\mathbf{H}$ field all over the place), we get

$$W^{\text{net}} \;=\; \Delta U. \tag{S.15}$$

Note that U is a function off the current state of the system but not of its past history, so for every process which begins exactly where it started $\Delta U = 0$ and hence net work is zero. By definition, this means that the work of changing the current near the magnet is *reversible* and that all of it goes towards changing the *magnetic energy* U.

Problem $\mathbf{1}$(b):

First, consider the system of a coil and a magnet as in part (a) but let's move the magnet instead of moving the coil. In this case, the direct calculation on the electric and mechanical work and relating them to the fields is more complicated, but by the *relative motion principle,* the net work should be exactly as if we were moving the coil relative to the fixed magnet. Indeed, by the Faraday's law the EMF and hence the electric work depends only on the $\delta\Phi$ which depends only on the relative motion of the coil and the magnet. As to the mechanical work, it also depends only on the relative motion as long as the forces between the coil and the magnet obey the third law of Newton. Altogether, *the magnetic energy (2) stores the net electric+mechanical work of the system, regardless of what's moving, the coil or the magnet.*

Moreover, as far as the permanent magnet is concerned, the coil is just an electromagnet, and we may just as well replace it with another permanent magnet, it won't affect the force or the torque on the original magnet. Thus, for a system of two permanent magnets (each having a constant magnetization), the magnetic energy (2) acts as the *potential energy* for their motion (including both the linear motion and the rotation).

Finally, thanks to the superposition principle for the $\mathbf{H}$ field, the potential energy of a system of several permanent magnets also have form (2). Indeed, let $\mathbf{H}_i(\mathbf{x})$ be the magnetic field generated by the magnet#i, then

$$\mathbf{H}_{\text{net}}(\mathbf{x}) = \sum_i \mathbf{H}_i(\mathbf{x}) \qquad \text{(S.16)}$$

and hence

$$\mathbf{H}_{\text{net}}^2(\mathbf{x}) = \sum_i \mathbf{H}_i^2(\mathbf{x}) + 2\sum_{i<j} \mathbf{H}_i(\mathbf{x}) \cdot \mathbf{H}_j(\mathbf{x}). \qquad \text{(S.17)}$$

In particular, for just 2 magnets

$$\mathbf{H}_{\text{net}}^2 = \mathbf{H}_1^2 + \mathbf{H}_2^2 + 2\mathbf{H}_1 \cdot \mathbf{H}_2$$

and hence the magnetic energy (2) becomes

$$U(1+2) = \frac{\mu_0}{2} \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{H}_1^2\, d^3\mathbf{x} + \frac{\mu_0}{2} \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{H}_2^2\, d^3\mathbf{x} + \mu_0 \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{H}_1 \cdot \mathbf{H}_2\, d^3\mathbf{x}. \qquad \text{(S.18)}$$

Moreover, the first two terms here do not depend on the relative positions of the two magnets, so as far as the two magnets motion (linear and rotational) is concerned, these two terms are constants, thus

$$U(1+2) \;=\; \text{const} \;+\; \mu_0 \iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{H}_1\cdot\mathbf{H}_2\,d^3\mathbf{x}. \tag{S.19}$$

Consequently, by the superposition principle, the potential energy of a system of $N>2$ permanent magnets obtains as

$$\begin{aligned} U(1+\cdots+N) \;&=\; \sum_{i<j}\mu_0 \iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{H}_i\cdot\mathbf{H}_j\,d^3\mathbf{x} \;+\; \text{const} \\ &=\; \frac{\mu_0}{2}\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{H}^2_{\text{net}}\,d^3\mathbf{x} \;+\; \text{const}. \end{aligned} \tag{S.20}$$

In other words, up to an irrelevant constant, the potential energy of N permanent magnets also obtains from eq. (2).

Problem $\mathbf{1}$(c):

Eq. (3) follows from integration by parts. Indeed,

$$\begin{aligned} \mathbf{H}\cdot\mathbf{B} \;&=\; \mathbf{H}\cdot(\nabla\times\mathbf{A}) \;=\; \nabla\cdot(\mathbf{A}\times\mathbf{H}) \;-\; \mathbf{A}\cdot(\mathbf{H}\times\overleftarrow{\nabla}) \\ &=\; \nabla\cdot(\mathbf{A}\times\mathbf{H}) \;+\; \mathbf{A}\cdot(\nabla\times\mathbf{H}) = \;\nabla\cdot(\mathbf{A}\times\mathbf{H}) \;+\; \mathbf{A}\cdot\mathbf{J}, \end{aligned} \tag{S.21}$$

hence for any volume $\mathcal{V}$ with boundary $\mathcal{S}$

$$\iiint\limits_{\mathcal{V}} \mathbf{H}\cdot\mathbf{B}\,d^3\mathbf{x} \;=\; \oint\!\!\!\oint\limits_{\mathcal{S}} (\mathbf{H}\times\mathbf{A})\cdot d^2\mathbf{a} \;+\; \iiint\limits_{\mathcal{V}} \mathbf{A}\cdot\mathbf{J}\,d^3\mathbf{x}. \tag{S.22}$$

When $\mathcal{V}$ extends to the whole space and its surface $\mathcal{S}$ recedes to infinity, the surface integral vanishes and we end up with

$$\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{H}\cdot\mathbf{B}\,d^3\mathbf{x} \;=\; \iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{J}\cdot\mathbf{A}\,d^3\mathbf{x}. \tag{S.23}$$

For the problem at hand, there are permanent magnets but no conduction currents anywhere in the system, $\mathbf{J} = 0$ at all $\mathbf{x}$, so the RHS in the last formula vanishes, thus

$$\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{H}\cdot\mathbf{B}\,d^3\mathbf{x} \;=\; 0. \tag{3}$$

Thanks to this formula, the magnetic energy (2) may be written in the form (4). Indeed, using

$$\mu_0\mathbf{H} \;=\; \mathbf{B} \;-\; \mu_0\mathbf{M} \tag{S.24}$$

we obtain

$$U \;-\; \text{const} \;=\; \frac{\mu_0}{2}\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{H}^2\,d^3\mathbf{x} \;=\; \frac{1}{2}\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{H}\cdot\mathbf{B}\,d^3\mathbf{x} \;-\; \frac{\mu_0}{2}\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{H}\cdot\mathbf{M}\,d^3\mathbf{x} \tag{S.25}$$

where the first term on the RHS is zero by eq. (3), thus

$$U \;=\; -\frac{\mu_0}{2}\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{H}\cdot\mathbf{M}\,d^3\mathbf{x} \;+\; \text{const.} \tag{S.26}$$

Formally, the integral here is over the whole space, but the $\mathbf{M}$ factor in the integrand vanishes outside the magnets. Consequently,

$$\begin{aligned} U \;-\; \text{const} \;&=\; -\frac{\mu_0}{2}\iiint\limits_{\text{all magnets}} \mathbf{H}\cdot\mathbf{M}\,d^3\mathbf{x} \\ &=\; -\frac{\mu_0}{2}\sum_{i}^{\text{magnets}}\iiint\limits_{\text{magnet}\#i} \mathbf{H}\cdot\mathbf{M}\,d^3\mathbf{x} \\ &=\; -\frac{\mu_0}{2}\sum_{i,j}^{\text{magnets}}\iiint\limits_{\text{magnet}\#i} \mathbf{H}_j(\mathbf{x})\cdot\mathbf{M}_i(\mathbf{x})\,d^3\mathbf{x} \end{aligned} \tag{S.27}$$

where $\mathbf{H}_j(\mathbf{x})$ denotes the magnetic field $\mathbf{H}(\mathbf{x})$ due to the magnet$\#j$ and likewise the magnetization $\mathbf{M}_i(\mathbf{x})$.

Finally, in the double sum on the last line of eq. (S.27) the terms with $i = j$ are self-interaction energies of the individual magnets. Such terms do not depend on the relative positions or orientations of the magnets, so as far as the net potential energy is concerned, they are constants. Adding these constants to all the other constants in U, we arrive at

$$U = -\frac{\mu_0}{2} \sum_{i \neq j}^{\text{magnets}} \iiint\limits_{\text{magnet}\#i} \mathbf{H}_j(\mathbf{x}) \cdot \mathbf{M}_i(\mathbf{x})\, d^3\mathbf{x} + \text{const.} \tag{4}$$

Quod erat demonstrandum.

Problem **1**(d):

When the distance between the two magnets is much larger than either magnet's size, the $\mathbf{H}_1(\mathbf{x})$ field of the first magnet does not change much within the second magnet, so we may approximate it as constant,

$$\text{for } \mathbf{x} \in \text{magnet}\#2, \quad \mathbf{H}_1(\mathbf{x}) \approx \mathbf{H}_1(2) = \text{const.} \tag{S.28}$$

Consequently, in eq. (4) for the two magnets

$$\iiint\limits_{\text{magnet}\#2} \mathbf{H}_1(\mathbf{x}) \cdot \mathbf{M}_2(\mathbf{x})\, d^3\mathbf{x} \approx \mathbf{H}_1(2) \cdot \iiint\limits_{\text{magnet}\#2} \mathbf{M}_2(\mathbf{x})\, d^3\mathbf{x} = \mathbf{H}_1(2) \cdot \mathbf{m}_2 \tag{S.29}$$

where $\mathbf{m}_2$ is the net magnetic moment of the second magnet. Likewise,

$$\iiint\limits_{\text{magnet}\#1} \mathbf{H}_2(\mathbf{x}) \cdot \mathbf{M}_1(\mathbf{x})\, d^3\mathbf{x} \approx \mathbf{H}_2(1) \cdot \mathbf{m}_1. \tag{S.30}$$

Therefore, eq. (4) for the two magnets becomes

$$U - \text{const} = -\frac{\mu_0}{2} \big(\mathbf{H}_1(2) \cdot \mathbf{m}_2 + \mathbf{H}_2(1) \cdot \mathbf{m}_1\big) = -\frac{1}{2} \big(\mathbf{B}_1(2) \cdot \mathbf{m}_2 + \mathbf{B}_2(1) \cdot \mathbf{m}_1\big). \tag{S.31}$$

Now let's take a closer look at the two terms in this formula. In the dipole approximation,

$$\mathbf{B}_1(2) = \frac{\mu_0}{4\pi r^3} \big(3(\mathbf{n} \cdot \mathbf{m}_1)\mathbf{n} - \mathbf{m}_1\big) \tag{S.32}$$

where r is the distance between the two magnets while $\mathbf{n}$ is the unit vector pointing from the first magnet towards the second. Consequently,

$$\mathbf{B}_1(2)\cdot\mathbf{m}_2 \;=\; \frac{\mu_0}{4\pi r^3}\bigl(3(\mathbf{n}\cdot\mathbf{m}_1)(\mathbf{n}\cdot\mathbf{m}_2)\;-\;(\mathbf{m}_1\cdot\mathbf{m}_2)\bigr), \tag{S.33}$$

and likewise

$$\mathbf{B}_2(1)\cdot\mathbf{m}_1 \;=\; \frac{\mu_0}{4\pi r^3}\bigl(3(\mathbf{n}'\cdot\mathbf{m}_1)(\mathbf{n}'\cdot\mathbf{m}_2)\;-\;(\mathbf{m}_1\cdot\mathbf{m}_2)\bigr), \tag{S.34}$$

where $\mathbf{n}'$ is the unit vector from the second magnets towards the first, thus $\mathbf{n}'=-\mathbf{n}$. But the RHS here is even with respect to $\mathbf{n}'$ so we may just as well reverse its direction and use $-\mathbf{n}'=+\mathbf{n}$ instead of the $\mathbf{n}'$. But then, the RHS of (S.34) becomes exactly as in eq. (S.33), thus

$$\mathbf{B}_2(1)\cdot\mathbf{m}_1 \;=\; \mathbf{B}_1(2)\cdot\mathbf{m}_2\,. \tag{S.35}$$

Combining this equality with eq. (S.31), we immediately obtain

$$U\;-\;\text{const}\;=\;-\mathbf{B}_2(1)\cdot\mathbf{m}_1\;=\;-\mathbf{B}_1(2)\cdot\mathbf{m}_2\,. \tag{5}$$

For completeness sake, let me write the magnetic energy in explicit form using eq. (S.33),

$$U\;=\;-\frac{\mu_0}{4\pi r^3}\bigl(3(\mathbf{n}\cdot\mathbf{m}_1)(\mathbf{n}\cdot\mathbf{m}_2)\;-\;(\mathbf{m}_1\cdot\mathbf{m}_2)\bigr)\;+\;\text{const}, \tag{S.36}$$

although for the purposes of this homework all we need is eq. (5).

Now consider the forces and the torques on the magnets stemming from the magnetic energy (5). Most generally, given the potential energy U as a function of the two bodies as a function of their positions and orientations, the forces follow as gradients

$$\mathbf{F}_1\;=\;-\nabla_{(1)}U(\mathbf{x}_1,\mathbf{x}_2,\text{orientations}),\quad \mathbf{F}_2\;=\;-\nabla_{(2)}U(\mathbf{x}_1,\mathbf{x}_2,\text{orientations}), \tag{S.37}$$

where the gradient $\nabla_{(1)}$ is taken WRT the $\mathbf{x}_1$ while the $\mathbf{x}_2$ and both orientations are held fixed, and likewise for the $\nabla_{(2)}$. For the potential energy (5), this means

$$\mathbf{F}_1\;=\;+\nabla_{(1)}\bigl(\mathbf{m}_1\cdot\mathbf{B}_2(\mathbf{x}_1)\bigr),\quad \mathbf{F}_2\;=\;+\nabla_{(2)}\bigl(\mathbf{m}_2\cdot\mathbf{B}_1(\mathbf{x}_2)\bigr), \tag{S.38}$$

in perfect agreement with eq. (6) for the force on a magnetic dipole.

Finally, the torques on the magnets follow from the variation of the potential energy under infinitesimal rotations of the respective magnets. In particular, to get the torque on the first magnet, we rotate it through infinitesimal angle $\delta\vec{\alpha}$ while keeping the orientation of the second magnets and both magnet's positions fixed. Thus

$$\text{for } \delta\mathbf{m} = \delta\vec{\alpha}\times\mathbf{m},\ \delta\mathbf{m}_2 = 0,\ \delta\mathbf{x}_1 = \delta\mathbf{x}_2 = 0, \quad \delta U \;=\; -\delta\vec{\alpha}\cdot\vec{\tau}_1\,, \tag{S.39}$$

and likewise for the torque on the second magnet. For the potential energy U, infinitesimally rotating the first magnet while keeping everything else fixed results in

$$\delta U \;=\; \delta\big(-\mathbf{B}_2(1)\cdot\mathbf{m}_1\big) \;=\; -\mathbf{B}_2(1)\cdot\big(\delta\vec{\alpha}\times\mathbf{m}_1\big) \;=\; -\delta\vec{\alpha}\cdot\big(\mathbf{m}_1\times\mathbf{B}_2(1)\big), \tag{S.40}$$

which gives us the torque

$$\vec{\tau}_1 \;=\; \mathbf{m}_1\times\mathbf{B}_2(1). \tag{S.41}$$

Likewise, the torque on the second magnet is

$$\vec{\tau}_2 \;=\; \mathbf{m}_2\times\mathbf{B}_1(2), \tag{S.42}$$

and both of these torques are in perfect agreement with eq. (6) for the torque on a magnetic dipole. *Quod erat demonstrandum.*

Problem **2**(a-b):

Let's assume all features of an ideal transformer — no hysteresis in the core, no eddy currents, no ohmic resistance in the wires — except for $k = 1$. Instead, let's allow for any magnetic coupling coefficient k between 0 and 1. Consequently,

$$\frac{M_{21}}{L_2} \;=\; \frac{M_{21}}{\sqrt{L_1L_2}}\times\sqrt{\frac{L_1}{L_2}} \;=\; k\times\frac{1}{n} \quad\text{and}\quad \frac{M_{12}}{L_1} \;=\; \frac{M_{12}}{\sqrt{L_1L_2}}\times\sqrt{\frac{L_2}{L_1}} \;=\; k\times n. \tag{S.43}$$

We shall use these ratios in a moment, but first we need the relations between the currents and the voltages in the two coils of the transformer.

By the Faraday law and Lenz rule, the electromotive forces in two magnetically coupled coils are

$$\mathcal{E}_1 \;=\; -L_1\times\frac{dI_1}{dt} \;+\; M_{12}\times\frac{dI_2}{dt}\,, \quad \mathcal{E}_2 \;=\; -L_2\times\frac{dI_2}{dt} \;+\; M_{21}\times\frac{dI_1}{dt}\,. \tag{S.44}$$

where the plus signs of the mutual inductance terms reflect opposite directions of the currents I_1 and I_2 in the two coils. Also, the primary coil acts as a load to the current I_1 while the secondary coil acts as a power supply to the current I_2, hence

$$V_1 \;=\; -\mathcal{E}_1 \quad \text{but} \quad V_2 \;=\; +\mathcal{E}_2\,, \tag{S.45}$$

and therefore

$$\begin{aligned} V_1 \;&=\; -\mathcal{E}_1 \;=\; +L_1\times\frac{dI_1}{dt} \;-\; M_{12}\times\frac{dI_2}{dt}\,,\\ V_2 \;&=\; +\mathcal{E}_2 \;=\; -L_2\times\frac{dI_2}{dt} \;+\; M_{21}\times\frac{dI_2}{dt}\,. \end{aligned} \tag{S.46}$$

For the AC currents and voltages which depend on time according to

$$I(t) \;=\; \mathrm{Re}\left(I\times e^{j\omega t}\right), \quad V(t) \;=\; \mathrm{Re}\left(V\times e^{j\omega t}\right), \tag{S.47}$$

with *complex amplitudes* I or V, eqs. (S.46) become

$$V_1 \;=\; +j\omega L_1\times I_1 \;-\; j\omega M_{12}\times I_2\,, \tag{S.48}$$

$$V_2 \;=\; -j\omega L_2\times I_2 \;+\; j\omega M_{21}\times I_1\,. \tag{S.49}$$

Also, for a linear load of impedance Z_2,

$$V_2 \;=\; Z_2\times I_2\,. \tag{S.50}$$

Everything else follows from solving the linear equations (S.48), (S.49), and (S.50).

Combining eqs. (S.49) and (S.50), we obtain

$$Z_2 \times I_2 \;=\; V_2 \;=\; -j\omega L_2 \times I_2 \;+\; j\omega M_{21} \times I_1 = \;-j\omega L_2 \times I_2 \;+\; j\omega L_2 \times \frac{k}{n} \times I_1 \quad \text{(S.51)}$$

and hence the current ratio

$$\frac{I_2}{I_1} \;=\; \frac{k}{n} \times \frac{j\omega L_2}{j\omega L_2 + Z_2}\,. \qquad \text{(S.52)}$$

As to the voltages,

$$V_2 \;=\; Z_2 \times I_2 \;=\; Z_2 \times \frac{j\omega L_2}{j\omega L_2 + Z_2} \times \frac{k}{n} \times I_1 \;=\; kn \times \frac{Z_2}{j\omega L_2 + Z_2} \times j\omega L_1\, I_1 \qquad \text{(S.53)}$$

while

$$\begin{aligned} V_1 \;&=\; +j\omega L_1 \times I_1 \;-\; j\omega M_{12} \times I_2 \;=\; j\omega L_1 \times I_1 \times \left(1 \;-\; \frac{M_{12}}{L_1} \times \frac{I_2}{I_1}\right) \\ &=\; \left(1 \;-\; kn \times \frac{k}{n} \times \frac{j\omega L_2}{j\omega L_2 + Z_2} \;=\; \frac{Z_2 \;+\; (1-k^2) \times j\omega L_2}{j\omega L_2 + Z_2}\right) \times j\omega L_1\, I_1 \end{aligned} \qquad \text{(S.54)}$$

and hence voltage ratio

$$\frac{V_2}{V_1} \;=\; kn \times \frac{Z_2}{Z_2 \;+\; (1-k^2) \times j\omega L_2}\,. \qquad \text{(S.55)}$$

Eqs. (S.52) and (S.55) for the ratios of currents and voltages in the two coils are valid for any k and any load impedance Z_2. Now let's suppose $k \approx 1$ so that $1 - k^2 \ll 1$. Then, for low enough load impedance $|Z_2| \ll \omega L_2$, the current ratio (S.52) becomes

$$\frac{I_2}{I_1} \;\approx\; \frac{k}{n} \;\approx\; \frac{1}{n}\,. \qquad \text{(S.56)}$$

Also, if the impedance Z_2 is not too low, $|Z_2| \gg (1-k^2) \times \omega L_2$, then the voltage ratio (S.55) becomes

$$\frac{V_2}{V_1} \;\approx\; kn \;\approx\; n. \qquad \text{(S.57)}$$

Note that the two conditions (10) on the load impedance Z_2 are compatible only for $k \approx 1$ so that $(1 - k^2) \ll 1$.

Finally, for an ideal transformer with $k = 1$, the voltage ratio (S.55) becomes n regardless of the impedance load, while the current ratio is as in eq. (7), and for $|Z_2| \ll \omega L_2$ becomes approximately $1/n$.

Problem **2**(c):

Consider a toroidal core with two coils would around it. For the sake of definiteness, suppose each coils is densely wound and each covers the entire surface area of the core, with the primary coil being wound around the core itself while the secondary coil is wound around the primary coil. Since the wires in the coils have finite diameter, the secondary coil has slightly wider diameter than the primary coil, which is in turn slightly wider than the ferromagnetic core itself. For the future reference, let R be the the long radius of the toroidal core, and let's define the following integrals over the cross-sections of the core and the coils:

$$\frac{a_c}{2\pi R} = \iint\limits_{\text{core}} \frac{d^2a}{2\pi s}, \quad \frac{a_1}{2\pi R} = \iint\limits_{\text{gap\#1}} \frac{d^2a}{2\pi s}, \quad \frac{a_2}{2\pi R} = \iint\limits_{\text{gap\#2}} \frac{d^2a}{2\pi s}, \tag{S.58}$$

where s is the distance from the symmetry axis of the toroid, gap#1 is the space between the core and the primary coil, and gap#2 is space between the primary and the secondary coils. For a long thin torus, a_c, a_1, and a_3 are simply the cross-sectional areas of — respectively — the core, the first gap, and the second gap. For a shorter fatter torus we need more complicated formulae, but for the problem at hand we do not need their details, all we need are the notations a_c, a_1, and a_2 for the integrals (S.58).

Now let's turn on the current I_1 in the primary coil while the secondary coil has no current. By the Ampere's Law and the rotational symmetry of the torus, the **H** field inside the primary coil is

$$\mathbf{H} = \frac{N_1 I_1}{2\pi s}\,\mathbf{n}_{\text{toroidal}} \tag{S.59}$$

while outside the primary coil $\mathbf{H} = 0$. As to the magnetic induction field **B**, it's $\mu\mu_0\mathbf{H}$ in the core but only $\mu_0\mathbf{H}$ outside the core. Consequently, the magnetic flux through the core of the transformer is

$$\Phi_{\text{core}} = \iint\limits_{\text{core}} \mathbf{B}\cdot d^2\mathbf{a} = \mu\mu_0 N_1 I_1 \iint\limits_{\text{core}} \frac{d^2a}{2\pi s} = \mu\mu_0 N_1 I_1 \times \frac{a_g}{2\pi R} \tag{S.60}$$

while there is a much smaller flux through the gap between the core and the primary coil,

$$\Phi_{\text{gap}\#1} = \iint\limits_{\text{gap}\#1} \mathbf{B}\cdot d^2\mathbf{a} = \mu_0 N_1 I_1 \iint\limits_{\text{gap}\#1} \frac{d^2a}{2\pi s} = \mu_0 N_1 I_1 \times \frac{a_1}{2\pi R}. \tag{S.61}$$

Both of these fluxes go through both coils, N_1 times through the primary coil and N_2 times through the secondary coil, and there is no extra flux through the gap#2 between the coils. Consequently

$$\Phi_1 = N_1(\Phi_{\text{core}} + \Phi_{\text{gap}\#1}), \quad \Phi_2 = N_2(\Phi_{\text{core}} + \Phi_{\text{gap}\#1}),$$

and hence

$$L_1 = \frac{\Phi_1}{I_1} = N_1 \times \frac{\mu a_g + a_1}{2\pi L}\,\mu_0 N_1\,, \tag{S.62}$$

$$M_{21} = \frac{\Phi_2}{I_1} = N_2 \times \frac{\mu a_g + a_1}{2\pi L}\,\mu_0 N_1\,. \tag{S.63}$$

In particular,

$$\frac{M_{21}}{L_1} = \frac{N_2}{N_1}\,. \tag{S.64}$$

Now let's turn off the current in the primary core and turn on the current I_2 in the secondary core. This time, the magnetic intensity field inside the secondary coil — including both the core and the two gaps (between the coils, and between the primary and the core) — is

$$\mathbf{H} = \frac{N_2 I_2}{2\pi s}\,\mathbf{n}_{\text{toroidal}}\,, \tag{S.65}$$

while the magnetic induction field $\mathbf{B}$ is $\mu\mu_0\mathbf{H}$ inside the core but only $|mu_0\mathbf{H}$ outside it.

Consequently, the magnetic flux through the core is

$$\Phi_{\text{core}} = \iint\limits_{\text{core}} \mathbf{B} \cdot d^2\mathbf{a} = \mu\mu_0 N_2 I_2 \iint\limits_{\text{core}} \frac{d^2a}{2\pi s} = \mu\mu_0 N_2 I_2 \times \frac{a_c}{2\pi R}, \tag{S.66}$$

the flux through the gap#1 between the core and the primary coil is much smaller

$$\Phi_{\text{gap\#1}} = \iint\limits_{\text{gap\#1}} \mathbf{B} \cdot d^2\mathbf{a} = \mu_0 N_2 I_2 \iint\limits_{\text{gap\#1}} \frac{d^2a}{2\pi s} = \mu_0 N_2 I_2 \times \frac{a_1}{2\pi R}, \tag{S.67}$$

and there is also a similarly small flux through the gap#2 between the coils,

$$\Phi_{\text{gap\#2}} = \iint\limits_{\text{gap\#2}} \mathbf{B} \cdot d^2\mathbf{a} = \mu_0 N_2 I_2 \iint\limits_{\text{gap\#2}} \frac{d^2a}{2\pi s} = \mu_0 N_2 I_2 \times \frac{a_1}{2\pi R}. \tag{S.68}$$

This time, the fluxes through the core and through the gap#1 go through both coils, but the flux through the gap#2 goes only through the secondary coil. Consequently,

$$\Phi_1 = N_1 \times (\Phi_{\text{core}} + \Phi_{\text{gap\#1}}) \quad \text{but} \quad \Phi_2 = N_2 \times (\Phi_{\text{core}} + \Phi_{\text{gap\#1}} + \Phi_{\text{gap\#2}}), \tag{S.69}$$

and hence

$$L_2 = \frac{\Phi_2}{I_2} = N_2 \times \frac{\mu a_g + a_1 + a_2}{2\pi R}\,\mu_0 N_2, \tag{S.70}$$

$$M_{12} = \frac{\Phi_1}{I_2} = N_1 \times \frac{\mu a_g + a_1}{2\pi R}\,\mu_0 N_2. \tag{S.71}$$

Note that eqs. (S.63) and (S.71) yield exactly the same mutual conductivities $M_{12} = M_{21}$. On the other hand, instead of

$$\frac{M_{12}}{L_2} = \frac{N_1}{N_2} \tag{S.72}$$

similarly to eq. (S.64), this time we get

$$\frac{M_{12}}{L_2} = \frac{N_1}{N_2} \times \frac{\mu a_c + a_1}{\mu a_c + a_1 + a_2}. \tag{S.73}$$

In terms of the magnetic coupling coefficient k, this means that

$$k^2 \;=\; \frac{M_{12}}{L_2}\times\frac{M_{21}}{L_1} \;=\; \frac{\mu a_c + a_1}{\mu a_c + a_1 + a_2}\,. \tag{S.74}$$

For a ferromagnetic core with high permeability $\mu \gg 1$, we have

$$1\,-\,k^2 \;=\; \frac{a_2}{\mu a_c + a_1 + a_2} \;\approx\; \frac{a_2}{a_c}\times\frac{1}{\mu} \;\longrightarrow\; 0 \quad \text{for } \mu\to\infty \tag{S.75}$$

and hence $k \to 1$. But a small portion of the magnetic flux through the outer secondary coil which misses the inner primary coil leads to k being not quite 1, just close to it.

For other geometries of the two coils on the same ferromagnetic core — for example, each coil covering only a segment of the core — the exact analysis is different, but the overall picture is similar: A small fraction of the magnetic flux created by the current in one coil passes outside the other coil, and that's why

$$\frac{M_{21}}{L_1} \;<\; \frac{N_2}{N_1} \quad \text{or} \quad \frac{M_{12}}{L_2} \;<\; \frac{N_1}{N_2} \quad \text{or both,} \tag{S.76}$$

and hence $k < 1$. But for a high-permeability core, most of the flux goes through the core and hence through both coils. Only an $O(1/\mu)$ fraction of the flux goes comes from outside the core, and only a part of that outside-the-core flux misses the other coil. Consequently,

$$1\,-\,k^2 \;=\; O(1/\mu) \;\to\; 0 \;\Longrightarrow\; k \;\to\; 1 \quad \text{for } \mu\to\infty. \tag{S.77}$$

Finally, consider the stepping ratio n. For the geometry where the secondary coil is would around the primary coil, we have eq. (S.64) for

$$\frac{M_{21}}{L_1} \;=\; \frac{M_{21}}{\sqrt{L_1L_2}}\times\sqrt{\frac{L_2}{L_1}} \;=\; k\times n \tag{S.78}$$

and hence

$$n \;=\; \frac{N_2}{N_1}\times\frac{1}{k} \;\approx\; \frac{N_2}{N_1}\,. \tag{S.79}$$

Likewise, for the geometry where the primary coils is wound around the secondary coil we would get

$$n = \frac{N_2}{N_1} \times k \approx \frac{N_2}{N_1}. \tag{S.80}$$

Other geometries might yield other exact formulae, but in the $\mu \to \infty$ and hence $k \to 1$ limit they all yield

$$n = \frac{N_2}{N_1}. \tag{S.81}$$

Quod erat demonstrandum.