Problem **1**(a):

In terms of the 3D retarded Green's function

$$G_R(\mathbf{x}-\mathbf{y},t_x-t_y) \;=\; \frac{\delta(t_x-t_y-r/c)}{4\pi r} \quad \text{where } r=|\mathbf{x}-\mathbf{y}|, \tag{S.1}$$

the wave generated by an instant line source at time $t_y=0$ is simply

$$\Psi(\mathbf{x},t) \;=\; \int\limits_{\text{line}} d\ell_y\, G(\mathbf{x}-\mathbf{y},t). \tag{S.2}$$

For our purposes, let the source line be the whole x_3 axis, then

$$\Psi(\mathbf{x},t) \;=\; \int\limits_{-\infty}^{+\infty} dy_3\,\frac{\delta(t-r/c)}{4\pi r} \;=\; \sum_{\substack{\text{points } y_3\\ \text{where } r=ct}} \frac{c}{4\pi r}\Bigg/\left|\frac{\partial r}{\partial y_3}\right|. \tag{S.3}$$

The solutions to the

$$r \;=\; \sqrt{x_1^2+x_2^2+(x_3-y_3)^2} \;=\; ct \tag{S.4}$$

condition depend on $r_{2d}=\sqrt{x_1^2+x_2^2}$: For $r_{2d}>ct$ there are no solutions, while for $r_{2d}<ct$ there are two solutions at $y_3=x_3\pm\sqrt{c^2t^2-r_{2d}^2}$, at which points

$$\frac{\partial r}{\partial y_3} \;=\; \frac{(y_3-x_3)}{r} \;=\; \pm\frac{\sqrt{c^2t^2-r_{2d}^2}}{r=ct} \quad\Longrightarrow\quad \frac{c}{4\pi r}\Bigg/\left|\frac{\partial r}{\partial y_3}\right| \;=\; \frac{c}{4\pi\sqrt{c^2t^2-r_{2d}^2}}. \tag{S.5}$$

Consequently,

$$\Psi(\mathbf{x},t) \;=\; \frac{2c\Theta(ct-r_{2d})}{4\pi\sqrt{c^2t^2-r_{2d}^2}}\,, \tag{S.6}$$

in perfect agreement with eq. (2) for the 2D wave of a point source.

Problem **1**(b):

This time, the instant source spans an entire plain, which we take to be the (y_2, y_3) plane, so the wave generated by this source is

$$\Psi(\mathbf{x},t) \;=\; \iint dy_2\, dy_3\, \frac{\delta(t-r/c)}{4\pi r}\,, \tag{S.7}$$

or in polar coordinates (s,ϕ) centered at (x_2,x_3)

$$\Phi(x_1,t) \;=\; \int\limits_0^\infty ds\, s \int\limits_0^{2\pi} d\phi\, \frac{c\delta(ct-r(s))}{4\pi r(s)} \;=\; \int\limits_0^\infty ds\, 2\pi s\, \frac{c\delta(ct-r(s))}{4\pi r(s)} \tag{S.8}$$

$$\text{for } r(s) \;=\; \sqrt{s^2+x_1^2}\,. \tag{S.9}$$

For $|x_1| > ct$ satisfying the condition $r(s) = ct$ is impossible, hence $\Psi(x_1,t) = 0$. On the other hand, for $|x_1| < ct$ there is a whole ring of solutions in the (y_2,y_3) plane corresponding to $s = +\sqrt{c^2t^2 - x_1^2}$, thus

$$\Psi \;=\; \frac{2\pi sc}{4\pi r} \Bigg/ \frac{\partial r}{\partial s} \;=\; \frac{sc}{2r} \Bigg/ \frac{s}{r} \;=\; \frac{c}{2}\,. \tag{S.10}$$

Altogether,

$$\Psi \;=\; \frac{c}{2}\,\Theta(ct-|x_1|), \tag{S.11}$$

in perfect agreement with eq. (3) for the 1D wave of a point source.

Problem **2**(a):

The densities (6) of charges and currents trivially obey the continuity equation:

$$\nabla\cdot\mathbf{J} \;=\; \delta'(t)\,\nabla\cdot\big(\mathbf{p}\,\delta^{(3)}(\mathbf{x})\big) \;=\; \delta'(t)\,(\mathbf{p}\cdot\nabla)\delta^{(3)}(\mathbf{x}) \;=\; -\frac{\partial\rho}{\partial t}\,. \tag{S.12}$$

As to the scalar potential in the Coulomb gauge, the formal solution of the first eq. (4) is

$$\Phi \;=\; \frac{1}{\epsilon_0}\,\frac{-1}{\nabla^2}\rho \;=\; -\frac{1}{\epsilon_0}\,\delta(t)\,\frac{-1}{\nabla^2}\,(\mathbf{p}\cdot\nabla)\delta^{(3)}(\mathbf{x}). \tag{S.13}$$

Since the operators $(\mathbf{p}\cdot\nabla)$ and $(-1/\nabla^2)$ commute with each other, we have

$$\frac{-1}{\nabla^2}\,(\mathbf{p}\cdot\nabla)\delta^{(3)}(\mathbf{x}) \;=\; (\mathbf{p}\cdot\nabla)\,\frac{-1}{\nabla^2}\,\delta^{(3)}(\mathbf{x}) \;=\; (\mathbf{p}\cdot\nabla)\,\frac{1}{4\pi r} \;=\; -\frac{\mathbf{p}\cdot\mathbf{n}}{4\pi r^2} \qquad \text{(S.14)}$$

and consequently

$$\Phi(\mathbf{x},t) \;=\; +\delta(t)\,\frac{\mathbf{p}\cdot\mathbf{n}}{4\pi\epsilon_0\,r^2}\,. \qquad \text{(S.15)}$$

Problem **2**(b):

For the current density as in eq. (6),

$$\nabla\left(\frac{-1}{\nabla^2}\,(\nabla\cdot\mathbf{J})\right) \;=\; \delta'(t)\,\nabla\,\frac{-1}{\nabla^2}\,(\mathbf{p}\cdot\nabla)\delta^{(3)}(\mathbf{x}), \qquad \text{(S.16)}$$

where all three operators — ∇, $(\mathbf{p}\cdot\nabla)$, and $(-1/\nabla^2)$ — commute with each other. Consequently,

$$\nabla\,\frac{-1}{\nabla^2}\,(\mathbf{p}\cdot\nabla)\delta^{(3)}(\mathbf{x}) \;=\; \nabla(\mathbf{p}\cdot\nabla)\,\frac{-1}{\nabla^2}\,\delta^{(3)}(\mathbf{x}) \;=\; \nabla(\mathbf{p}\cdot\nabla)\,\frac{1}{4\pi r}\,, \qquad \text{(S.17)}$$

and hence eq. (8) for the transverse current. As to eq. (9), it follows from eq. (8) and

$$\nabla_i\nabla_j\,\frac{1}{4\pi r} \;=\; \frac{3n_in_j-\delta_{ij}}{4\pi r^3} \;-\; \frac{\delta_{ij}}{3}\,\delta^{(3)}(\mathbf{x}).$$

Problem **2**(c):

Let's start with the first lemma (10). In spherical coordinates (r,θ,ϕ) for $\mathbf{z}$, the LHS of eq. (10) — which I am going to denote L_1 — becomes

$$\begin{aligned} L_1 \;&\stackrel{\text{def}}{=}\; \iiint\limits_{\substack{\text{whole}\\ \text{space}}} d^3\mathbf{z}\,\frac{\delta'(t-|\mathbf{z}|/c)}{|\mathbf{z}|}\times F(\mathbf{z}) \;=\; \int\limits_0^\infty dr\,r^2 \oint\!\!\!\oint\limits_{4\pi} d^2\Omega(\theta,\phi)\,\frac{\delta'(t-r/c)}{r}\times F(r,\theta,\phi) \\ &=\; \int\limits_0^\infty dr\,\delta'(t-r/c)\times r\,G(r) \end{aligned} \qquad \text{(S.18)}$$

where

$$G(r) \stackrel{\text{def}}{=} \oiint\limits_{4\pi} d^2\Omega(\theta,\phi)\, F(r,\theta,\phi). \tag{S.19}$$

Next,

$$\delta'(t - r/c) \;=\; -c^2\delta'(r - ct), \tag{S.20}$$

and consequently

$$L_1 \;=\; -c^2 \int\limits_0^\infty dr\, \delta'(r - ct) \times r\, G(r) \;=\; +c^2 \int\limits_0^\infty dr\, \delta(r - ct) \times \frac{d}{dr}\big(rG(r)\big). \tag{S.21}$$

where the second equality obtains from integration by parts, — which is the standard procedure for handling the derivatives of the delta functions. The remaining integral involving the ordinary delta function $\delta(r - ct)$ yields

$$L_1 \;=\; +c^2 \left[\frac{d}{dr}\big(rG(r)\big) \;=\; \left(1 + r\frac{d}{dr}\right) G(r)\right]_{r=ct} \tag{S.22}$$

provided $r = ct$ is within the integration range $0 < ct < \infty$, but zero otherwise. In other words,

$$L_1 \;=\; c^2\Theta(t) \left[\left(1 + r\frac{d}{dr}\right) G(r)\right]_{r=ct} \tag{S.23}$$

where $\Theta(t)$ is the step-function: 1 for $t > 0$ but 0 for $t < 0$. Finally, plugging eq. (S.19) for $G(r)$ into eq. (S.23) completes the proof of the first Lemma (10).

As to the second Lemma (11), it's the good old Mean Value Theorem of electrostatics: Averaging a Coulomb potential of a point charge over a spherical surface yields the potential at the sphere's center provided the charge is outside the sphere. For the charge inside the sphere, the averaging yields a potential of a similar charge moved to the center of the sphere.

The simplest way to prove the mean value theorem is via the multipole expansion (once you know how it works),

$$\frac{1}{|\mathbf{x}+R\mathbf{n}|} = \sum_{\ell=0}^{\infty} \frac{[\min(|\mathbf{x}|,R)]^{\ell}}{[\max(|\mathbf{x}|,R)]^{\ell+1}} \times P_{\ell}(-\cos\theta) \tag{S.24}$$

where θ is the angle between the unit vector $\mathbf{n}$ and the vector $\mathbf{x}$. Let's plug the expansion (S.24) into the angular integral (10) and integrate term by term: without the radial factor,

$$\frac{1}{4\pi} \oiint\limits_{4\pi} d^2\Omega_n P_{\ell}(-\cos\theta) = \frac{1}{2}\int\limits_{-1}^{+1} d\cos\theta P_{\ell}(-\cos\theta) = \delta_{\ell,0}\,, \tag{S.25}$$

so only the $\ell = 0$ term contributes to the net integral. Consequently,

$$\frac{1}{4\pi} \oiint\limits_{4\pi} d^2\Omega_n \frac{1}{|\mathbf{x}+R\mathbf{n}|} = \frac{[\min(|\mathbf{x}|,R)]^{0}}{[\max(|\mathbf{x}|,R)]^{1}} = \frac{1}{\max(|\mathbf{x}|,R)}\,. \tag{S.26}$$

which completes the proof of the mean value theorem (11).

Problem **2**(d):

The formal solution of the wave equation (4) for the vector potential obtains via the retarded Green's function as

$$\mathbf{A}(\mathbf{x},t_x) = \frac{\mu_0}{4\pi} \iiint\limits_{\substack{\text{whole}\\ \text{space}}} d^3\mathbf{y} \int dt_y \, \frac{\delta(t_x - t_y - |\mathbf{x}-\mathbf{y}|/c)}{|\mathbf{x}-\mathbf{y}|} \, \mathbf{J}_T(\mathbf{y},t_y) \tag{S.27}$$

where $\mathbf{J}_T$ is the transverse current (8). Since the time-dependence and the $\mathbf{y}$ dependence of this transverse current factorize as

$$\mathbf{J}_T(\mathbf{y},t_y) = \delta'(t_y) \times \mathbf{J}_0(\mathbf{y}), \tag{S.28}$$

$$\mathbf{J}_0(\mathbf{y}) = \mathbf{p}\,\delta^{(3)}(\mathbf{y}) + \nabla(\mathbf{p}\cdot\nabla)\,\frac{1}{4\pi|\mathbf{y}|}\,. \tag{S.29}$$

the time integral in eq. (S.27) yields

$$\int dt_y \, \delta(t_x - t_y - |\mathbf{x}-\mathbf{y}|/c) \times \delta'(t_y) \;=\; \delta'(t_x - |\mathbf{x}-\mathbf{y}|/c) \tag{S.30}$$

and hence

$$\mathbf{A}(\mathbf{x},t) \;=\; \frac{\mu_0}{4\pi} \iiint\limits_{\substack{\text{whole}\\\text{space}}} d^3\mathbf{y}\, \frac{\delta'(t_x - |\mathbf{y}-\mathbf{x}|/c)}{|\mathbf{y}-\mathbf{x}|}\, \mathbf{J}_0(\mathbf{y}). \tag{S.31}$$

Now consider the two terms in the current (S.29):

$$\begin{aligned} \mathbf{J}_0(\mathbf{y}) &= \mathbf{J}_1(\mathbf{y}) \;+\; \mathbf{J}_2(\mathbf{y}),\\ \mathbf{J}_1(\mathbf{y}) &= \mathbf{p}\,\delta^{(3)}(\mathbf{y}),\\ \mathbf{J}_2(\mathbf{y}) &= \nabla(\mathbf{p}\cdot\nabla)\,\frac{1}{4\pi|\mathbf{y}|}\,. \end{aligned} \tag{S.32}$$

Plugging them into the integral (S.31) leads to

$$\mathbf{A}(\mathbf{x},t) \;=\; \mathbf{A}_1(\mathbf{x},t) \;+\; \mathbf{A}_2(\mathbf{x},t) \tag{S.33}$$

where the first term is

$$\begin{aligned} \mathbf{A}_1 &= \frac{\mu_0}{4\pi} \iiint\limits_{\substack{\text{whole}\\\text{space}}} d^3\mathbf{y}\, \frac{\delta'(t_x - |\mathbf{y}-\mathbf{x}|/c)}{|\mathbf{y}-\mathbf{x}|}\, \mathbf{J}_1(\mathbf{y})\\ &= \frac{\mu_0\mathbf{p}}{4\pi} \iiint\limits_{\substack{\text{whole}\\\text{space}}} d^3\mathbf{y}\, \frac{\delta'(t_x - |\mathbf{y}-\mathbf{x}|/c)}{|\mathbf{y}-\mathbf{x}|}\, \delta^{(3)}(\mathbf{y})\\ &= \frac{\mu_0\mathbf{p}}{4\pi|\mathbf{x}|}\,\delta'(t_x - |\mathbf{x}|/c), \end{aligned} \tag{S.34}$$

a flash spreading out in all directions at the speed of light. This flash vanishes for $t < |\mathbf{x}|/c$, so we may ignore it for this part of the problem, although we will need it later in part (f).

The second term in the vector potential (S.33) is

$$\begin{aligned}
\mathbf{A}_2 &= \frac{\mu_0}{4\pi} \iiint\limits_{\substack{\text{whole}\\\text{space}}} d^3\mathbf{y}\, \frac{\delta'(t_x - |\mathbf{y}-\mathbf{x}|/c)}{|\mathbf{y}-\mathbf{x}|}\, \mathbf{J}_2(\mathbf{y}) \\
&= \frac{\mu_0}{4\pi} \iiint\limits_{\substack{\text{whole}\\\text{space}}} d^3\mathbf{y}\, \frac{\delta'(t_x - |\mathbf{y}-\mathbf{x}|/c)}{|\mathbf{y}-\mathbf{x}|}\, \nabla_y(\mathbf{p}\cdot\nabla_y)\, \frac{1}{4\pi|\mathbf{y}|} \\
&= \frac{\mu_0}{4\pi} \iiint\limits_{\substack{\text{whole}\\\text{space}}} d^3\mathbf{z}\, \frac{\delta'(t_x - |\mathbf{z}|/c)}{|\mathbf{z}|}\, \nabla_y(\mathbf{p}\cdot\nabla_y)\, \frac{1}{4\pi|\mathbf{x}+\mathbf{z}|}
\end{aligned} \tag{S.35}$$

where on the last line I have changed the integration variable from $\mathbf{y}$ to $\mathbf{z} = \mathbf{x}-\mathbf{y}$. Formally, the space derivatives inside this integral are WRT $\mathbf{y} = \mathbf{x}+\mathbf{z}$, but since they act on a function which depends only on the $\mathbf{y} = \mathbf{x}+\mathbf{z}$, we may change them to $\mathbf{x}$-derivatives for a fixed $\mathbf{z}$,

$$\left[\nabla_y(\mathbf{p}\cdot\nabla_y)\, \frac{1}{4\pi|\mathbf{y}|}\right]_{\mathbf{y}=\mathbf{x}+\mathbf{z}} = \nabla_x(\mathbf{p}\cdot\nabla_x)\, \frac{1}{4\pi|\mathbf{x}+\mathbf{z}|}\,. \tag{S.36}$$

Consequently, the integral (S.35) becomes

$$\begin{aligned}
\mathbf{A}_2(\mathbf{x}, t_x) &= \frac{\mu_0}{4\pi} \iiint\limits_{\substack{\text{whole}\\\text{space}}} d^3\mathbf{z}\, \frac{\delta'(t_x - |\mathbf{z}|/c)}{|\mathbf{z}|}\, \nabla_x(\mathbf{p}\cdot\nabla_x)\, \frac{1}{4\pi|\mathbf{x}+\mathbf{z}|} \\
&= \frac{\mu_0}{4\pi}\, \nabla_x(\mathbf{p}\cdot\nabla_x) \iiint\limits_{\substack{\text{whole}\\\text{space}}} d^3\mathbf{z}\, \frac{\delta'(t_x - |\mathbf{z}|/c)}{|\mathbf{z}|}\, \frac{1}{4\pi|\mathbf{x}+\mathbf{z}|}\,.
\end{aligned} \tag{S.37}$$

The remaining integral on the second line here looks like the LHS of the Lemma (10) for

$$F(\mathbf{z}) = \frac{1}{4\pi|\mathbf{x}+\mathbf{z}|}\,, \tag{S.38}$$

hence

$$\iiint\limits_{\substack{\text{whole}\\\text{space}}} d^3\mathbf{z}\, \frac{\delta'(t_x - |\mathbf{z}|/c)}{|\mathbf{z}|}\, \frac{1}{4\pi|\mathbf{x}+\mathbf{z}|} = c^2\Theta(t) \left[\left(1 + r\frac{\partial}{\partial r}\right) \oiint d^2\Omega_n\, \frac{1}{4\pi|\mathbf{x}+r\mathbf{n}|}\right]_{r=ct}. \tag{S.39}$$

Note: in this formula r and $\mathbf{n}$ are the magnitude and the direction of $\mathbf{z}$ rather than $\mathbf{x}$.

Next, the angular integral in eq. (S.39) obtains from the Lemma (11):

$$\oiint d^2\Omega_n \, \frac{1}{4\pi|\mathbf{x}+r\mathbf{n}|} \;=\; \frac{1}{\max(r,|\mathbf{x}|)} \;=\; \begin{cases} (1/r) & \text{for } r>|\mathbf{x}|, \\ (1/|\mathbf{x}|) & \text{for } r<|\mathbf{x}|. \end{cases} \tag{S.40}$$

Consequently,

for $r = ct > |\mathbf{x}|$,

$$\left(1 + r\frac{\partial}{\partial r}\right) \oiint d^2\Omega_n \, \frac{1}{4\pi|\mathbf{x}+r\mathbf{n}|} \;=\; \left(1 + r\frac{\partial}{\partial r}\right)\frac{1}{r} \;=\; 0, \tag{S.41}$$

for $r = ct < |\mathbf{x}|$,

$$\left(1 + r\frac{\partial}{\partial r}\right) \oiint d^2\Omega_n \, \frac{1}{4\pi|\mathbf{x}+r\mathbf{n}|} \;=\; \left(1 + r\frac{\partial}{\partial r}\right)\frac{1}{|\mathbf{x}|} \;=\; \frac{1}{|\mathbf{x}|}, \tag{S.42}$$

and therefore

$$\iiint\limits_{\substack{\text{whole}\\ \text{space}}} d^3\mathbf{z} \, \frac{\delta'(t_x - |\mathbf{z}|/c)}{|\mathbf{z}|} \, \frac{1}{4\pi|\mathbf{x}+\mathbf{z}|} \;=\; c^2\Theta(t)\Theta(|\mathbf{x}|-ct)\, \frac{1}{|\mathbf{x}|}\,. \tag{S.43}$$

Plugging this formula back into eq. (S.37), we arrive at

$$\mathbf{A}_2(\mathbf{x},t) \;=\; \frac{c^2\mu_0}{4\pi}\,\Theta(t)\,\nabla(\mathbf{p}\cdot\nabla)\,\frac{\Theta(|\mathbf{x}|-ct)}{|\mathbf{x}|} \tag{S.44}$$

where $c^2\mu_0 = 1/\epsilon_0$.

For the later use in part (f), let me write down the entire vector potential for all times $t = t_x$,

$$\mathbf{A}(\mathbf{x},t) \;=\; \mathbf{A}_1(\mathbf{x},t) \;+\; \mathbf{A}_2(\mathbf{x},t) \;=\; \frac{\mu_0}{4\pi|\mathbf{x}|}\,\mathbf{p}\,\delta'(t-|\mathbf{x}|/c) \;+\; \frac{\Theta(t)}{4\pi\epsilon_0}\,\nabla(\mathbf{p}\cdot\nabla)\left(\frac{\Theta(|\mathbf{x}|-ct)}{|\mathbf{x}|}\right). \tag{S.45}$$

But for the current part (d) we assume $t < |\mathbf{x}|/c$, so the vector potential simplifies to

$$\mathbf{A}(\mathbf{x},t) \;=\; \frac{\Theta(t)}{4\pi\epsilon_0}\,\nabla(\mathbf{p}\cdot\nabla)\,\frac{1}{|\mathbf{x}|} \;=\; \Theta(t)\,\frac{3(\mathbf{p}\cdot\mathbf{n}_x)\mathbf{n}_x \,-\, \mathbf{p}}{4\pi\epsilon_0\,|\mathbf{x}|^3}\,. \tag{S.46}$$

Problem **2**(e):

At times $t < |\mathbf{x}|/c$ — before the light pulse from the dipole flash (6) reaches the point $\mathbf{x}$, — the vector potential (S.46) is a pure gradient of some scalar field,

$$\mathbf{A} = \nabla\left(\frac{\Theta(t)}{4\pi\epsilon_0}\,\frac{-(\mathbf{p}\cdot\mathbf{n}_x)}{|\mathbf{x}|^2}\right), \tag{S.47}$$

so its curl $\mathbf{B} = \nabla\times\mathbf{A}$ must vanish, $\mathbf{B} = 0$. Thus, the magnetic field does not propagate faster than light.

As to the electric field

$$\mathbf{E} = -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A}, \tag{S.48}$$

the vector potential (S.46) is a step function of time: It turns on at $t = 0$ and then stays constant until the light pulse of the dipole flash reaches the point $\mathbf{x}$. Consequently, at $t < |\mathbf{x}|/c$,

$$\frac{\partial\mathbf{A}}{\partial t} = +\delta(t)\,\nabla(\mathbf{p}\cdot\nabla)\,\frac{1}{4\pi\epsilon_0\,|\mathbf{x}|}\,. \tag{S.49}$$

At the same time, in part (a) we found the scalar potential to be

$$\Phi(\mathbf{x},t) = -\delta(t)\,(\mathbf{p}\cdot\nabla)\,\frac{1}{4\pi\epsilon_0\,|\mathbf{x}|}\,, \tag{S.50}$$

hence

$$\mathbf{E} = -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A} = 0. \tag{S.51}$$

Thus, the superluminal terms in the scalar and the vector potentials cancel each other from the electric field! Consequently, the electric field — just like the magnetic field — does not propagate faster than light.

Problem **2**(f):

The scalar potential $\Phi(\mathbf{x},t)$ flashes at $t=0$ and then vanishes, so at all later times $t>0$ both magnetic and the electric field obtain solely from the vector potential $\mathbf{A}(\mathbf{x},t)$. In the solutions to part (d), I have written down eq. (S.45) for the vector potential at all times, both before the light front passes through the point in question and afterward. To simplify the notations in that formula, let me redefine $r=|\mathbf{x}|$ (instead of $r=|\mathbf{z}|$ we have used in part (d)), then eq. (S.45) becomes

$$\mathbf{A}(\mathbf{x},t) \;=\; \frac{\mu_0}{4\pi\, r}\,\mathbf{p}\,\delta'(t-r/c) \;+\; \frac{\Theta(t)}{4\pi\epsilon_0}\,\nabla(\mathbf{p}\cdot\nabla)\left(\frac{\Theta(r-ct)}{r}\right). \tag{S.52}$$

From this formula we see that for $t>r/c$ — after the light front has moved on — the vector potential drops to zero, so *there are no EM fields left over behind the light front,*

$$\text{for } t>r/c,\quad \mathbf{E}(\mathbf{x},t) \;=\; 0 \text{ and } \mathbf{B}(\mathbf{x},t) \;=\; 0. \tag{S.53}$$

On the other hand, the vector potential (S.52) is non-zero — and quite singular — right at the light front $r=t/c$. To get all the singularities right, we should remember that the space-derivative operator $\nabla(\mathbf{p}\cdot\nabla)$ acts not only on the $1/r$ factor but also on the step function $\Theta(r-ct)$ in the numerator, thus

$$\nabla_i\left(\frac{\Theta(r-ct)}{r}\right) \;=\; n_i\left(\frac{\delta(r-ct)}{r} \;-\; \frac{\Theta(r-ct)}{r^2}\right), \tag{S.54}$$

$$\begin{aligned}
\nabla_j\nabla_i\left(\frac{\Theta(r-ct)}{r}\right) \;&=\; \frac{\delta_{ij}-n_jn_i}{r}\left(\frac{\delta(r-ct)}{r} \;-\; \frac{\Theta(r-ct)}{r^2}\right)\\
&\quad+\; n_jn_i\left(\frac{\delta'(r-ct)}{r} \;-\; \frac{2\delta(r-ct)}{r^2} \;+\; \frac{2\Theta(r-ct)}{r^3}\right)\\
&=\; n_in_j\,\frac{\delta'(r-ct)}{r}\\
&\quad-\; (3n_in_j-\delta_{ij})\left(\frac{\delta(r-ct)}{r^2} \;-\; \frac{\Theta(r-ct)}{r^3}\right),
\end{aligned} \tag{S.55}$$

$$\nabla(\mathbf{p}\cdot\nabla)\left(\frac{\Theta(r-ct)}{r}\right) \;=\; \mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,\frac{\delta'(r-ct)}{r} \;-\; (3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p})\left(\frac{\delta(r-ct)}{r^2}\;-\;\frac{\Theta(r-ct)}{r^3}\right). \tag{S.56}$$

At the same time,

$$\frac{\mu_0}{4\pi\,r}\,\mathbf{p}\,\delta'(t-r/c) \;=\; -\frac{\mu_0 c^2}{4\pi r}\,\mathbf{p}\,\delta'(r-ct) \;=\; -\frac{\mathbf{p}}{4\pi\epsilon_0}\,\delta'(r-ct), \tag{S.57}$$

so putting all terms together, *at the light front* $r=ct$,

$$\mathbf{A}(\mathbf{x},t) \;=\; \frac{1}{4\pi\epsilon_0}\left[(\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p})\frac{\delta'(r-ct)}{r} \;+\; (3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p})\left(\frac{\delta(r-ct)}{r^2}\;-\;\frac{\Theta(r-ct)}{r^3}\right)\right]. \tag{S.58}$$

Problem **2**(g):

We saw in part (e) that before the light front $\mathbf{E}=\mathbf{B}=0$, and in part (f) we saw that after the light front $\mathbf{A}=0$ and hence also $\mathbf{E}=\mathbf{B}=0$. Thus, the electric and the magnetic fields of the instant dipole flash exist only at the light front $r=ct$. To find them, we simply need to take one more space or time derivative of the vector potential (S.58).

In particular, the time derivative is rather simple:

$$-\frac{\partial}{\partial t}\Theta(r-ct) \;=\; c\delta(r-ct), \quad -\frac{\partial}{\partial t}\delta(r-ct) \;=\; c\delta'(r-ct), \quad -\frac{\partial}{\partial t}\delta'(r-ct) \;=\; c\delta''(r-ct), \tag{S.59}$$

hence the electric field is

$$\mathbf{E}(\mathbf{x},t) \;=\; -\frac{\partial\mathbf{A}}{\partial t} \;=\; \frac{c}{4\pi\epsilon_0}\left[\begin{array}{c} (\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p})\,\dfrac{\delta''(r-ct)}{r} \\ +\;(3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p})\left(\dfrac{\delta'(r-ct)}{r^2}\;-\;\dfrac{\delta(r-ct)}{r^3}\right)\end{array}\right]. \tag{S.60}$$

As to the magnetic field, we can simplify taking the curl of the vector potential (S.58) by noting that

$$\frac{\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p}}{r} \;=\; \nabla(\mathbf{n}\cdot\mathbf{p}) \quad\Longrightarrow\quad \nabla\times\frac{\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p}}{r} \;=\; 0, \tag{S.61}$$

hence

$$\begin{aligned}\nabla\times\left(\frac{\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p}}{r}\,\delta'(r-ct)\right) \;&=\; \nabla\big(\delta'(r-ct)\big)\times\frac{\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p}}{r}\\ &=\; \delta''(r-ct)\,\mathbf{n}\times\frac{\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p}}{r} \;=\; \frac{\mathbf{p}\times\mathbf{n}}{r}\,\delta''(r-ct).\end{aligned} \tag{S.62}$$

Likewise,

$$\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p}}{r^3} \;=\; \nabla\left(-\frac{\mathbf{n}\cdot\mathbf{p}}{r^2}\right) \quad\Longrightarrow\quad \nabla\times\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p}}{r^3} \;=\; 0, \tag{S.63}$$

hence

$$\begin{aligned}\nabla\times&\left(\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p}}{r^3}\Big(r\delta(r-ct)\,-\,\Theta(r-ct)\Big)\right) \;=\\ &=\; \nabla\Big(r\delta(r-ct)\,-\,\Theta(r-ct)\Big)\times\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p}}{r^3}\\ &=\; \Big(r\delta'(r-ct)\Big)\mathbf{n}\times\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})\,-\,\mathbf{p}}{r^3}\\ &=\; \frac{\mathbf{p}\times\mathbf{n}}{r^2}\,\delta'(r-ct).\end{aligned} \tag{S.64}$$

Altogether,

$$\mathbf{B} \;=\; \nabla\times\mathbf{A}[\text{from eq. (S.58)}] \;=\; \frac{\mathbf{p}\times\mathbf{n}}{4\pi\epsilon_0}\left[\frac{\delta''(r-ct)}{r}\,+\,\frac{\delta'(r-ct)}{r^2}\right]. \tag{S.65}$$

Problem **3**:

The Poynting theorem (20) for the energy works similarly for both uniform and non-uniform media, as long as they are perfectly linear and the permittivity ϵ and the permeability μ do not depend on time or frequency. Indeed, for $\mathbf{D} = \epsilon\epsilon_0\mathbf{D}$ with a time- or frequency-independent ϵ,

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\mathbf{E}\cdot\mathbf{D}\right) \;=\; \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} \;=\; \mathbf{D}\cdot\frac{\partial\mathbf{E}}{\partial t}\,, \tag{S.66}$$

and likewise for $\mathbf{B} = \mu\mu_0\mathbf{H}$ with a time- or frequency-independent μ,

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\mathbf{H}\cdot\mathbf{B}\right) \;=\; \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} \;=\; \mathbf{B}\cdot\frac{\partial\mathbf{H}}{\partial t}\,, \tag{S.67}$$

Thus, the time derivative of the EM energy (16) is

$$\begin{aligned}\frac{\partial u}{\partial t} &= \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} \;+\; \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\\ &= \mathbf{E}\cdot\big(\nabla\times\mathbf{H} - \mathbf{J}\big) \;+\; \mathbf{H}\cdot\big(-\nabla\times\mathbf{E}\big)\\ &= -\mathbf{J}\cdot\mathbf{E} \;-\; \nabla\cdot(\mathbf{E}\times\mathbf{H}),\end{aligned} \tag{S.68}$$

hence for the energy flow density $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ as in eq. (17) and the EM power density $P = \mathbf{J}\cdot\mathbf{E}$ as in eq. (14), we have

$$\frac{\partial u}{\partial t} \;+\; \nabla\cdot\mathbf{S} \;+\; P \;=\; 0. \tag{S.69}$$

Quod erat demonstrandum.

As to the Poynting-like theorem (21) for the momentum, the momentum density (18) and the stress tensor (19) work for both uniform and non-uniform media (as long as they are perfectly linear), but the EM force density (15) needs to be modified for a non-uniform media.

To see how this works, let's take the time derivative of the momentum density (18):

$$\begin{aligned}
\frac{\partial}{\partial t}\big(\mathbf{g}\,=\,\mathbf{D}\times\mathbf{B}\big) \;&=\; \frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B}\;+\;\mathbf{D}\times\frac{\partial\mathbf{B}}{\partial t}\\
&=\;\big(\nabla\times\mathbf{H}\,-\,\mathbf{J})\times\mathbf{B}\;+\;\mathbf{D}\times\big(-\nabla\times\mathbf{E}\big)\\
&=\;-\mathbf{J}\times\mathbf{B}\;-\;\mathbf{B}\times(\nabla\times\mathbf{H})\;-\;\mathbf{D}\times(\nabla\times\mathbf{E}).
\end{aligned}\tag{S.70}$$

In components,

$$\begin{aligned}
\big[-\mathbf{B}\times(\nabla\times\mathbf{H})\big]_i \;&=\; -\epsilon_{ijk}\epsilon_{k\ell m}B_j\nabla_\ell H_m \;=\; -(\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell})B_j\nabla_\ell H_m\\
&=\; -B_j\nabla_i H_j\;+\;B_j\nabla_j H_i\,,
\end{aligned}\tag{S.71}$$

where

$$\begin{aligned}
B_j\nabla_i H_j \;&=\; \mu\mu_0 H_j\nabla_i H_j \;=\; \tfrac{1}{2}\mu\mu_0\nabla_i(\mathbf{H}^2)\\
&=\; \nabla_i(\tfrac{1}{2}\mu\mu_0\cdot\mathbf{H}^2)\;-\;\tfrac{1}{2}\mu_0\mathbf{H}^2\nabla_i(\mu)\\
&=\; \nabla_i(\tfrac{1}{2}\mathbf{B}\cdot\mathbf{H})\;-\;\tfrac{1}{2}\mu_0\mathbf{H}^2\nabla_i(\mu),
\end{aligned}\tag{S.72}$$

while

$$B_j\nabla_j H_i \;=\; \nabla_j(B_jH_i)\;-\;H_i\big(\nabla_jB_j=0\big)\;=\;\nabla_j(B_jH_i),\tag{S.73}$$

so together

$$\big[-\mathbf{B}\times(\nabla\times\mathbf{H})\big]_i \;=\; -\nabla_i(\tfrac{1}{2}\mathbf{B}\cdot\mathbf{H})\;+\;\tfrac{1}{2}\mu_0\mathbf{H}^2\nabla_i\mu\;+\;\nabla_j(B_jH_i).\tag{S.74}$$

In a similar way,

$$\begin{aligned}
\big[-\mathbf{D}\times(\nabla\times\mathbf{E})]_i \;&=\; -D_j\nabla_i E_j\;+\;D_j\nabla_j E_i\\
&=\; -\nabla_i\big(\tfrac{1}{2}\mathbf{D}\cdot\mathbf{E}\big)\;+\;\tfrac{1}{2}\epsilon_0\mathbf{E}^2\nabla_i\epsilon\\
&\qquad+\;\nabla_j(D_jE_i)\;-\;E_i\big(\nabla_jD_j\;=\;\rho).
\end{aligned}\tag{S.75}$$

Plugging these formulae back into eq. (S.70), we arrive at

$$\begin{aligned}
\frac{\partial g_i}{\partial t}\;&=\;-\big(\rho\mathbf{E}\,+\,\mathbf{J}\times\mathbf{B}\big)_i\;+\tfrac{1}{2}\mu_0\mathbf{B}^2\nabla_i\mu\;+\tfrac{1}{2}\epsilon_0\mathbf{E}^2\nabla_i\epsilon\\
&\quad+\;\nabla_j\big(B_jH_i\,+\,D_jE_i\big)\;-\;\nabla_i\big(\tfrac{1}{2}\mathbf{B}\cdot\mathbf{H}\,+\,\mathbf{D}\cdot\mathbf{E}\big).
\end{aligned}\tag{S.76}$$

The last line in this formula is obviously the "divergence" of the stress tensor,

$$\nabla_j\Big(T_{ij} \;=\; H_i B_j \;+\; E_i D_j \;-\; \tfrac{1}{2}\big(\mathbf{H}\cdot\mathbf{B}+\mathbf{E}\cdot\mathbf{D}\big)\delta_{ij}\Big), \tag{S.77}$$

so if we want the momentum conservation equation

$$\frac{\partial g_i}{\partial t} \;-\; \nabla_j T_{ij} \;+\; f_i \;=\; 0 \tag{21}$$

to work, then (minus) the top line of eq. (S.76) should be identified as the *force density*

$$\mathbf{f} \;=\; \rho\mathbf{E} \;+\; \mathbf{J}\times\mathbf{B} \;-\; \tfrac{1}{2}\mu_0\mathbf{H}^2\,\nabla(\mu) \;-\; \tfrac{1}{2}\epsilon_0\mathbf{E}^2\,\nabla(\epsilon). \tag{S.78}$$

In a uniform medium, the last two terms in the force (S.78) disappear, and the remaining force density is precisely as in eq. (15). This completes part (a) of the problem.

For a non-uniformed medium, we just saw that the EM momentum density and the EM stress tensor work unmodified, but the force density should be modified according to eq. (S.78). Physically, the extra terms stem from the attraction of dielectrics to the regions of strong electric fields, and likewise of the attraction of the ferromagnetic or paramagnetic materials to the strong magnetic fields. Indeed, earlier in class we saw that the net force on a piece of dielectric is

$$\mathbf{F} \;=\; \frac{\epsilon_0}{2}\iiint\limits_{\text{dielectric}} (\epsilon(\mathbf{x})-1)\nabla(\mathbf{E}^2(\mathbf{x}))\,d^3\mathbf{x}. \tag{S.79}$$

The integral here can be extended to the integral over the whole space, which allows us to integrate by parts, thus

$$\mathbf{F} \;=\; -\frac{\epsilon_0}{2}\iiint \mathbf{E}^2(\mathbf{x})\nabla(\epsilon(\mathbf{x}))\,d^3\mathbf{x}\,, \tag{S.80}$$

in perfect agreement with the third term in the force density (S.78). Likewise, the net force on a piece of magnetic material is

$$\begin{aligned}\mathbf{F} &= \frac{\mu_0}{2} \iiint\limits_{\text{piece}} (\mu(\mathbf{x}) - 1)\nabla(\mathbf{H}^2(\mathbf{x}))\, d^3\mathbf{x} \;=\; \frac{\mu_0}{2} \iiint\limits_{\substack{\text{whole}\\ \text{space}}} (\mu(\mathbf{x}) - 1)\nabla(\mathbf{H}^2(\mathbf{x}))\, d^3\mathbf{x} \\ &= -\frac{\mu_0}{2} \iiint \mathbf{H}^2(\mathbf{x})\, \nabla(\mu(\mathbf{x}))\, d^3\mathbf{x}\end{aligned} \tag{S.81}$$

in perfect agreement with the fourth term in the force density (S.78).