Problem **1**(a):

For the sake of definiteness, lets assume the toroid lies in the horizontal plane, with its center at the coordinate origin, and that the current in the coil flows up on the inner side of the toroid and down on the outer side. Then the magnetic field inside the coil is

$$\mathbf{H} \;=\; \frac{IN}{2\pi s}\,\mathbf{n}_\phi \;\approx\; \frac{IN}{2\pi R}\,\mathbf{n}_\phi \tag{S.1}$$

while the magnetic field outside the coil is negligibly small. (For an infinitely dense coil it would be exactly zero.) At the same time, the electric field of the point charge at the center is

$$\mathbf{E} \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{\mathbf{n}_r}{r^2}\,, \tag{S.2}$$

so within the coil where $r \approx R$ and $\mathbf{n}_r \approx \mathbf{n}_s$ (the direction horizontally away from the z axis), we have

$$\mathbf{E} \;\approx\; \frac{Q}{4\pi\epsilon_0\,R^2}\,\mathbf{n}_s\,. \tag{S.3}$$

Consequently, the Poynting vector is

$$\mathbf{S} \;=\; \mathbf{E}\times\mathbf{H} \;\approx\; \frac{INQ}{8\pi^2\epsilon_0 R^3}\,(\mathbf{n}_s\times\mathbf{n}_\phi) \;=\; \frac{INQ}{8\pi^2\epsilon_0 R^3}\,\mathbf{n}_z \tag{S.4}$$

inside the coil, while outside the coil $\mathbf{S} = 0$.

The Poynting vector — or rather $(1/c^2)\mathbf{S}$ — is the momentum density of the EM fields. For the system at hand, this momentum density is uniform inside the coil and zero outside it, hence the net EM momentum is simply

$$P_{\text{net}} \;=\; \frac{1}{c^2}\,S[\text{inside}]\times\text{toroid's volume} \;=\; \epsilon_0\mu_0\times\frac{INQ}{8\pi^2\epsilon_0 R^3}\times 2\pi RA \;=\; \frac{\mu_0 INQA}{4\pi R^2}, \tag{S.5}$$

or in vector notations

$$\mathbf{P}_{\text{net}} \;=\; \frac{\mu_0 INQA}{4\pi R^2}\,\mathbf{n}_z\,. \tag{S.6}$$

Note the direction of this momentum: Up along the symmetry axis of the toroid.

Problem $\mathbf{1}$(b):

Let's make the toroid's cross-section $A = 1\ \text{cm}^2$ and large radius $R = 10$ cm, while the coil wounds $N = 1000$ turns around the toroid. With this geometry, the wire has length $L \sim 35$ m, while its diameter should be smaller that $2\pi R/N$, about 0.63 mm. Assuming copper wire of diameter 0.5 mm, we have its ohmic resistance about 3 Ω. If we try to run the $I = 10$ A current through this wire, it would generate 300 Watt's of heat, which would make it a fire danger. To keep the heat generation down to a much safer 3 Watts, let's take $I = 1$ A.

As to the static charge Q at the center of the toroid, the electric field near its surface should not exceed the dielectric strength of the air — about $3\cdot 10^6$ V/m — or else the charge would leak out via sparks. Emulating a 'point' charge with e metal ball of radius $r = 3$ cm, we are limited to

$$\frac{Q}{4\pi\epsilon_0 r^2} \;\le\; E_{\max} \quad\Longrightarrow\quad Q \;\le\; 0.3\ \mu\text{C}, \tag{S.7}$$

so to be safe let's take $Q = 0.1\ \mu$C.

Plugging all these data into eq. (S.6), we find the EM momentum to be only $P = 10^{-13}$ Ns. By everyday standards, this is a very small momentum, even a 1 mg ant lazily crawling at 1 cm/s has momentum $p = 10^{-8}$ Ns, five orders of magnitude larger then the EM fields' momentum in our example! A better comparison to the EM momentum (S.7) would be a smaller and slower creature, such as a 0.4 microgram amoeba crawling at 250 microns per second.

For another comparison, the LHC accelerates protons up to energy 6.5 TeV (per proton) so each proton has momentum 6.5 TeV/c, or in conventional units, $3.5\cdot 10^{-15}$ Ns. Thus, the EM field in our example has a s much mechanical momentum as 28 LHC protons.

Problem $\mathbf{1}$(c):

The electric field induced by the time-changing magnetic field is governed by the equations

$$\nabla\cdot\mathbf{E}_{\text{induced}} \;=\; 0, \quad \nabla\times\mathbf{E}_{\text{induced}} \;=\; -\frac{\partial\mathbf{B}}{\partial t} \tag{S.8}$$

which are mathematically similar to

$$\nabla\cdot\mathbf{B} \;=\; 0, \quad \nabla\times\mathbf{B} \;=\; \mu_0\mathbf{J}. \tag{S.9}$$

Consequently, the solutions to these equations are also mathematically similar, namely the Biot–Savart–Laplace–like formula for the $\mathbf{E}_{\rm induced}$ where $-\partial\mathbf{B}/\partial t$ plays the role of the electric current,

$$\mathbf{E}_{\rm induced}(\mathbf{x}) \;=\; -\frac{1}{4\pi}\iiint d^3\mathbf{y}\,\frac{\partial\mathbf{B}}{\partial t}\times\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\,. \tag{S.10}$$

In particular, the electric field induced at the center of the toroid is mathematically similar to the magnetic field in the center of a circular current loop,

$$\mathbf{E}_{\rm induced}(\text{center}) \;=\; -\frac{\mathbf{n}_z}{2R}\,\frac{d\Phi}{dt} \tag{S.11}$$

where Φ is the magnetic flux in the toroid,

$$\Phi \;=\; A\times\mathbf{B} \;=\; \frac{\mu_0 INA}{2\pi R}\,. \tag{S.12}$$

The electric field (S.11) pushes the charge Q with the force

$$\mathbf{F} \;=\; Q\mathbf{E}_{\rm induced}(\text{center}) \;=\; -\frac{Q\mu_0 NA}{4\pi R^2}\,\frac{dI}{dt}\,\mathbf{n}_z\,, \tag{S.13}$$

and the net impulse of this force while the current changes by ΔI is

$$\Delta\mathbf{p} \;=\; \int\mathbf{F}\,dt \;=\; -\frac{Q\mu_0 NA}{4\pi R^2}\,\Delta I\,\mathbf{n}_z\,. \tag{S.14}$$

In particular, for the current which drops from the original current I down to zero, the impulse is

$$\Delta\mathbf{p} \;=\; +\frac{QI\mu_0 NA}{4\pi R^2}\,\mathbf{n}_z\,. \tag{S.15}$$

Comparing this formula to eq. (S.6), we immediately see that the net impulse on the charge Q is precisely the EM momentum the system had before we turned off the current.

Problem $\mathbf{2}$(b):

Since the problem is based on the Jackson's textbook, in these solutions I follow the textbook notations rather than notations of my own notes. In particular, i is the incidence angle, r is the reflections angle, the electric amplitudes of the incident, the refracted, and the reflected waves are respectively E_0, E'_0, and E''_0, and likewise ϵ, μ, and n are for the incident wave's side while ϵ', μ', and n' are for the other side.

First of all, the wave impedance in a medium that's both dielectric and magnetic is

$$Z = \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}} = \sqrt{\frac{\mu}{\epsilon}} \times Z_0 = \frac{\mu}{n} \times Z_0\,. \tag{S.16}$$

With this relation in mind, the expression $\sqrt{\mu\epsilon'/\mu'\epsilon}$ in eqs. (7.42) is simply the impedance ratio Z/Z'. Consequently, eqs. (7.42) for the waves of normal incidence become

$$\frac{E'_0}{E_0} = \frac{2}{(Z/Z')+1} = \frac{2Z'}{Z+Z'}\,, \tag{7.42.a}$$

$$\frac{E''_0}{E_0} = \frac{(Z/Z')-1}{(Z/Z')+1} = \frac{Z-Z'}{Z+Z'}\,. \tag{7.42.b}$$

Note that these transmission and reflection coefficients depends only on the wave impedances of the two media and do not care for the refraction indices n and n'. In particular, if the two media happen to have equal impedances $Z' = Z$ there would be no reflection at all, even if the refraction indices of the two media are different, $n' \neq n$. For example, suppose $\epsilon = \mu = 1$ while $\epsilon' = \mu' = 2$, then $Z' = Z = Z_0$ and hence no reflection despite $n = 1$ while $n' = 2$.

For the waves which strike the boundary at non-zero incidence angle i we have more complicated formulae (7.39) and (7.41), depending on the polarization. Nevertheless, we can re-express them in terms of the impedance ratio Z'/Z instead of the refraction indices and the μ'/μ; however, we would also need the cosine ratio

$$\frac{\cos(r)}{\cos(i)} \tag{S.17}$$

which implicitly depends on the refraction indices. Indeed, the expression $\sqrt{n'^2 - n^2\sin^2(i)}$

in eqs. (7.39) and (7.41) is nothing but

$$\sqrt{n'^2 - n^2 \sin^2(i)} \;=\; \sqrt{n'^2 - n'^2 \sin^2(r)} \;=\; n' \cos(r). \tag{S.18}$$

Consequently, eqs. (7.39) for the waves polarized $\perp$ to the plane of incidence become

$$\begin{aligned}
\frac{E_0'}{E_0} &= \frac{2n\cos(i)}{n\cos(i) + (\mu/\mu') \times n'\cos(r)} = \frac{2(n/\mu)\cos(i)}{(n/\mu)\cos(i) + (n'/\mu')\cos(r)} \\
&= \frac{2(Z_0/Z)\cos(i)}{(Z_0/Z)\cos(i) + (Z_0/Z')\cos(r)} = \frac{2Z'\cos(i)}{Z'\cos(i) + Z\cos(r)} \\
&= \frac{2Z'/\cos(r)}{(Z'/\cos(r)) + (Z/\cos(i))}
\end{aligned} \tag{7.39.a}$$

and likewise

$$\frac{E_0''}{E_0} = \frac{(Z'/\cos(r)) - (Z/\cos(i))}{(Z'/\cos(r)) + (Z/\cos(i))}. \tag{7.39.b}$$

Again, the reflection and the refraction of the EM waves is governed by the impedance ratio of the two media, except that for the wave crossing the boundary at non-zero angles i and r, — and polarized $\perp$ to the plane of incidence — the effective impedances are

$$Z_{\text{eff}} = \frac{Z}{\cos(i)} \quad \text{and} \quad Z'_{\text{eff}} = \frac{Z'}{\cos(r)} \tag{S.19}$$

instead of Z and Z' themselves. If these effective impedances happen to match for some incident angle i, there would be no reflection despite different refraction indices. However, this is possible only for magnetic materials with

$$\frac{\mu'}{\mu} > \frac{\epsilon'}{\epsilon} > \frac{\mu}{\mu'} \quad \text{or} \quad \frac{\mu'}{\mu} < \frac{\epsilon'}{\epsilon} < \frac{\mu}{\mu'}, \tag{S.20}$$

and that's why I did not mention this possibility in class.

Finally, consider the EM waves polarized within the plane of incidence. In this case, eqs. (7.41) become

$$\begin{aligned}
\frac{E_0'}{E_0} &= \frac{2nn'\cos(i)}{(\mu/\mu')\times n'^2\cos(i)\,+\,nn'\cos(r)} = \frac{2(n/\mu)\cos(i)}{(n'/\mu')\cos(i)\,+\,(n/\mu)\cos(r)}\\
&= \frac{2(Z_0/Z)\cos(i)}{(Z_0/Z')\cos(i)\,+\,(Z_0/Z)\cos(r)} = \frac{2Z'\cos(i)}{Z\cos(i)\,+\,Z'\cos(r)}\\
&= \frac{Z'}{Z}\times\frac{2Z\cos(i)}{Z\cos(i)\,+\,Z'\cos(r)}
\end{aligned}\tag{7.41.a}$$

and likewise

$$\frac{E_0''}{E_0} = \frac{Z\cos(i)\,-\,Z'\cos(r)}{Z\cos(i)\,+\,Z'\cos(r)}\,.\tag{S.21}$$

Again, reflection and refraction coefficients follow from the ratio of effective impedances, but for the wave polarized within the incidence plane, the effective impedances are

$$Z_{\text{eff}} = Z\cos(i) \quad \text{and} \quad Z'_{\text{eff}} = Z'\cos(r)\tag{S.22}$$

instead of (S.19). At the Brewster angle i_b, these effective impedances happen to match, $Z_{\text{eff}} = Z'_{\text{eff}}$, and there is no reflection. (Of the wave polarized within the incidence plane.) The general formula for the Brewster angle is

$$\sin^2(i_b) = \frac{(\epsilon'/\epsilon)-(\mu'/\mu)}{(\epsilon'/\epsilon)-(\epsilon/\epsilon')}\,,\tag{S.23}$$

and for magnetic media we generally have $i_b + r_b \neq 90^\circ$. Only in the non-magnetic media eq. (S.23) reduces to the simple geometric condition for the Brewster angle, In non-magnetic media, this formula reduces to

$$\begin{aligned}
\sin^2(i_b) = \frac{\epsilon'}{\epsilon'+\epsilon} = \frac{n'^2}{n'^2+n^2} \quad &\Longrightarrow \quad \sin^2(r_b) = \frac{n^2}{n'^2+n^2} = 1\,-\,\sin^2(i_b)\\
&\Longrightarrow \quad i_b\,+\,r_b = 90^\circ.
\end{aligned}\tag{S.24}$$

Problem $\mathbf{3}$(a):

Let's use a rotated coordinate system

$$x' = \mathbf{m}_1 \cdot \mathbf{x}, \quad y' = y, \quad z' = \mathbf{n}_1 \cdot \mathbf{x}, \tag{S.25}$$

where z' runs along the incident wave beam while x' and y' run across the beam. In these coordinates, eq. (3) for the electric field factorizes to

$$\mathbf{E}_1(\mathbf{x},t) = \mathcal{E}_0\,\mathbf{e}_1 \exp(-i\omega t) * \exp(+ik_0\tilde{z}) * 1(\tilde{y}) * \exp\left(-\frac{\tilde{x}^2}{2a^2}\right), \tag{S.26}$$

so we may Fourier transform each factor by itself:

$$\int\! dx'\, \exp(-ix'k_{x'}) \times \exp\left(-\frac{x'^2}{2a^2}\right) = \sqrt{2\pi}a\, \exp\bigl(-\tfrac{1}{2}ak_{x'}^2\bigr), \tag{S.27}$$

$$\int\! dy'\, \exp(-iy'k_{y'}) \times 1(y') = 2\pi\delta(k_{y'}), \tag{S.28}$$

$$\int\! dz'\, \exp(-iz'k_{z'}) \times \exp(+ik_0z') = 2\pi\delta(k_{z'} - k_0), \tag{S.29}$$

hence altogether

$$\begin{aligned}\tilde{\mathbf{E}}(\mathbf{k},t) &= \iiint\! d^3\mathbf{x}\, \exp(-i\mathbf{k}\cdot\mathbf{x}) * \mathbf{E}(\mathbf{x},t)\\ &= \mathcal{E}_0\,\mathbf{e}_1 \exp(-i\omega t) * \sqrt{2\pi}a\, \exp\bigl(-\tfrac{1}{2}ak_{x'}^2\bigr) * 2\pi\delta(k_{y'}) * 2\pi\delta(k_{z'} - k_0)\\ &= (2\pi)^{5/2}a\,\mathcal{E}_0\,\mathbf{e}_1\, \exp(-i\omega t) * \exp\bigl(-\tfrac{1}{2}a^2(\mathbf{m}_1\cdot\mathbf{k})^2\bigr) * \delta(k_y)\delta(\mathbf{n}_1\cdot\mathbf{k} - k_0),\end{aligned} \tag{S.30}$$

in perfect agreement with eq. (6).

Next, under total internal reflection the incident plane wave turns into the reflected wave with amplitude $\vec{\mathcal{E}}_3 = \vec{\mathcal{E}}_1 \times \exp(i\phi)$ — where the phase ϕ depends on the direction of the incident wave — while the wave vector is reflected:

$$\mathbf{k}_1 = (k_x, k_y, k_z) \ \to \mathbf{k}_3 = (+k_x, +k_y, -k_z) \quad \text{[in the original coordinates } (x,y,z)\text{]}, \tag{S.31}$$

and the polarization vector is reflected in a similar manner. For the not-quite-plane waves, we may use the superposition principle for their Fourier transforms, for which the incident and the reflected waves are related just like the plane waves on the $\mathbf{k}$ by $\mathbf{k}$ manner, thus

$$\tilde{\mathbf{E}}_3(\mathbf{k},t) \;=\; e^{i\phi}\,\tilde{\mathbf{E}}_1^{\rm refl}(\mathbf{k}^{\rm refl},t). \tag{S.32}$$

In particular, for the incident wave (6), the reflected wave is

$$\tilde{\mathbf{E}}(\mathbf{k},t) \;=\; e^{i\phi} * (2\pi)^{5/2} a\,\mathcal{E}_0\,\mathbf{e}_3\,\exp(-i\omega t) * \exp\bigl(-\tfrac{1}{2}a^2(\mathbf{m}_1\cdot\mathbf{k}^{\rm refl})^2\bigr) * \delta(k_y^{\rm refl})\delta(\mathbf{n}_1\cdot\mathbf{k}^{\rm refl}-k_0), \tag{S.33}$$

where

$$\begin{aligned} \mathbf{m}_1\cdot\mathbf{k}^{\rm refl} \;&=\; \cos\alpha(k_x^{\rm refl}=k_x) \;-\; \sin\alpha(k_z^{\rm refl}=-k_z) \;=\; \mathbf{m}_3\cdot\mathbf{k}, \\ k_y^{\rm refl} \;&=\; k_y\,, \\ \mathbf{n}_1\cdot\mathbf{k}^{\rm refl} \;&=\; \sin\alpha(k_x^{\rm refl}=k_x) \;+\; \cos\alpha(k_z^{\rm refl}=-k_z) \;=\; \mathbf{n}_3\cdot\mathbf{k}, \end{aligned} \tag{S.34}$$

thus

$$\tilde{\mathbf{E}}(\mathbf{k},t) \;=\; e^{i\phi} * (2\pi)^{5/2} a\,\mathcal{E}_0\,\mathbf{e}_3\,\exp(-i\omega t) * \exp\bigl(-\tfrac{1}{2}a^2(\mathbf{m}_3\cdot\mathbf{k})^2\bigr) * \delta(k_y) * \delta(\mathbf{n}_3\cdot\mathbf{k}-k_0), \tag{S.35}$$

exactly as in eq. (7). *Quod erat demonstrandum.*

Problem **3**(b):
To Fourier transform the reflected wave (7) back to the coordinate space, we again introduce the rotated coordinates, but now in the directions across or along the reflected beam rather than the incident beam, thus

$$x'' \;=\; \mathbf{x}\cdot\mathbf{m}_3\,, \quad y'' \;=\; y, \quad z'' \;=\; \mathbf{x}\cdot\mathbf{n}_3\,. \tag{S.36}$$

In these coordinates, eq. (7) for the reflected beam factorizes to

$$\tilde{\mathbf{E}}(\mathbf{k},t) \;=\; e^{i\phi} * (2\pi)^{5/2} a\,\mathcal{E}_0\,\mathbf{e}_3\,\exp(-i\omega t) * \exp\bigl(-\tfrac{1}{2}a^2k_{x''}^2\bigr) * \delta(k_{y''}) * \delta(k_{z''}-k_0), \tag{S.37}$$

so we may immediately Fourier transform the $k_{y''}$ and the $k_{z''}$ components of $\mathbf{k}$ to the coordinate space:

$$\int\frac{dk_{y''}}{2\pi}\,\exp(+ik_{y''}y'')\times(2\pi)\delta(k_{y''}) \;=\; 1 \quad [\text{for any } y], \tag{S.38}$$

$$\int \frac{dk_{z''}}{2\pi}\,\exp(+ik_{z''}z'') \times (2\pi)\delta(k_{y''}-k_0) \;=\; \exp(ik_0 z'') \;=\; \exp(ik_0\mathbf{n}_3\cdot\mathbf{x}), \qquad \text{(S.39)}$$

hence

$$\begin{aligned}\mathbf{E}(\mathbf{x},t) \;&=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\exp(+i\mathbf{k}\cdot\mathbf{x}) * \tilde{\mathbf{E}}(\mathbf{k},t)\\ &=\; \mathcal{E}_0\,\mathbf{e}_3\,\exp(-i\omega t + ik_0 z'') \int \frac{a\,dk_{x''}}{\sqrt{2\pi}}\,\exp(ik_{x''}x'') * \exp\bigl(-\tfrac{1}{2}a^2k_{x''}^2\bigr) * \exp(i\phi).\end{aligned} \qquad \text{(S.40)}$$

This formula becomes the top line of eq. (9) once we identify the remaining integration variable $k_{x''}$ with the $k_\perp$ in eq. (9).

Note that we cannot pull out the $e^{i\phi}$ factor out of the integral in eq. (S.40) because it depends on the angle of incidence α, or equivalently on the angle of reflection $\gamma = \alpha$. For the not-quite-plane wave in eq. (S.40), the wave vector of the reflected wave is

$$\mathbf{k} \;=\; k_{x''}\mathbf{m}_3 \;+\; k_{y''}(0,1,0) \;+\; k_{z''}\mathbf{n}_3 \;=\; k_{x''}\mathbf{m}_3 \;+\; k_0\mathbf{n}_3\,, \qquad \text{(S.41)}$$

so it's direction depends on the integration variable $k_{x''}$, and that's what makes the $e^{i\phi}$ factor depend on the $k_{x''}$.

Fortunately, the direction of the reflected wave varies within a rather small angle. Indeed, due to the Gaussian factor $\exp(-\frac{1}{2}a^2k_{x''}^2)$ inside the integral (S.40), the integral is dominated by

$$|k_{x''}| \;\le\; O(1/a) \;\ll\; k_0\,, \qquad \text{(S.42)}$$

so in this effective integration range we may approximate the angle of reflection as

$$\gamma \;=\; \arctan\frac{k_x}{-k_z} \;=\; \gamma_0 \;+\; \arctan\frac{k_{x''}}{k_0} \;\approx\; \gamma_0 \;+\; \frac{k_{x''}}{k_0}\,, \qquad \text{(S.43)}$$

where the second term is a small correction to the first. Consequently, we may approximate the direction-dependent phase ϕ of the total internal reflection as

$$\phi(\gamma) \;\approx\; \phi(\gamma_0) \;+\; \frac{d\phi}{d\gamma}\times\frac{k_{x''}}{k_0}\,, \tag{S.44}$$

or equivalently

$$\phi(k_{x''}) \;\approx\; \phi(\alpha_0) \;+\; \frac{k_{x''}}{k_0}\times\frac{d\phi}{d\alpha}\,. \tag{S.45}$$

Or in terms of D defined as in eq. (10),

$$\phi(k_{x''}) \;\approx\; \phi(\alpha_0) \;-\; D\times k_{x''}\,. \tag{S.46}$$

Now let's plug this formula into the remaining Fourier integral in eq. (S.40):

$$\begin{aligned}
&\int\frac{a\,dk_{x''}}{\sqrt{2\pi}}\,\exp(ik_{x''}x'')\times\exp\bigl(-\tfrac{1}{2}a^2k_{x''}^2\bigr)*\exp(i\phi)\\
&\qquad=\; e^{i\phi(\alpha_0)}\int\frac{a\,dk_{x''}}{\sqrt{2\pi}}\,\exp\Bigl(-\tfrac{1}{2}a^2k_{x''}^2\;+\;ix''k_{x''}\;-\;iDk_{x''}\Bigr)\\
&\qquad=\; e^{i\phi(\alpha_0)}\exp\left(-\frac{(x''-D)^2}{2a^2}\right),
\end{aligned} \tag{S.47}$$

hence

$$\mathbf{E}(x'',y'',z'',t) \;=\; e^{i\phi(\alpha_0)}\mathcal{E}_0\,\mathbf{e}_3\,\exp(-i\omega t+ik_0z'')*\exp\left(-\frac{(x''-D)^2}{2a^2}\right), \tag{S.48}$$

or in vector notations

$$\mathbf{E}(\mathbf{x},t) \;=\; e^{i\phi(\alpha_0)}\mathcal{E}_0\,\mathbf{e}_3\,\exp\bigl(ik_0\mathbf{n}_3\cdot\mathbf{x}-i\omega t\bigr)*\exp\left(-\frac{(\mathbf{m}_3\cdot\mathbf{x}-D)^2}{2a^2}\right), \tag{S.49}$$

exactly as on the bottom line of eq. (9). In other words, the reflected beam is shifted through distance D in the direction $\mathbf{m}_3$ that's perpendicular too the beam. And the shift distance D is exactly as in eq. (10), *cf.* eqs. (S.45) and (S.46). *Quod erat demonstrandum.*

Problem **2**(c):

The phase shift $\phi(\alpha)$ of the totally-reflected wave is calculated in my notes on refraction and reflection, with the answers given in eqs. (69) and (70) on the last page. Specifically, for the wave polarized normally to the plane of incidence

$$\phi_\perp(\alpha) \;=\; -2\arctan\left(\frac{\sqrt{\sin^2\alpha\,-\,(n_2/n_1)^2}}{\cos\alpha}\right), \tag{S.50}$$

while for the wave polarized parallel to the plane of incidence

$$\phi_\|(\alpha) \;=\; -2\arctan\left(\frac{\sqrt{\sin^2\alpha\,-\,(n_2/n_1)^2}}{(n_2/n_1)^2\cos\alpha}\right). \tag{S.51}$$

To calculate the Goos–Hänchen displacement D of the reflected waves, all we need is to take the derivatives of these phase shifts WRT to the angle of incidence α.

To simplify the algebra, let's rephrase eqs. (S.50) and (S.51) as

$$\phi_\perp \;=\; -2\arctan\bigl(\sqrt{g(\alpha)}\bigr),\qquad \phi_\| \;=\; -2\arctan\bigl((n_1/n_2)^2\sqrt{g(\alpha)}\bigr), \tag{S.52}$$

where

$$g(\alpha) \;=\; \frac{\sin^2\alpha\,-\,(n_2/n_1)^2}{\cos^2\alpha}. \tag{S.53}$$

Taking the derivatives, we find

$$\frac{dg}{d\alpha} \;=\; \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha}\;+\;\frac{2\sin\alpha(\sin^2\alpha\,-\,(n_2/n_1)^2)}{\cos^3\alpha} \;=\; 2(1-(n_2/n_1)^2)\,\frac{\sin\alpha}{\cos^3\alpha}. \tag{S.54}$$

$$\begin{aligned} -\frac{d\phi_\perp}{dg} &= +2\times\frac{1}{1+(\sqrt{g})^2}\times\frac{1}{2\sqrt{g}} \;=\; \frac{1}{\sqrt{g}(1+g)} \\ &= \frac{\cos\alpha}{\sqrt{\sin^2\alpha\,-\,(n_2/n_1)^2}}\times\frac{\cos^2\alpha}{1\,-\,(n_2/n_1)^2}, \end{aligned} \tag{S.55}$$

$$
\begin{aligned}
-\frac{d\phi_\parallel}{dg} &= +2 \times \frac{(n_1/n_2)^2}{1 \,+\, (n_1/n_2)^4 g} \times \frac{1}{2\sqrt{g}} \\
&= \frac{\cos\alpha}{\sqrt{\sin^2\alpha \,-\, (n_2/n_1)^2}} \times \frac{(n_2/n_1)^2 \cos^2\alpha}{(n_2/n_1)^4 \cos^2\alpha \,+\, \sin^2\alpha \,-\, (n_2/n_1)^2} \\
&= \frac{\cos\alpha}{\sqrt{\sin^2\alpha \,-\, (n_2/n_1)^2}} \times \frac{\cos^2\alpha}{1 \,-\, (n_2/n_1)^2} \times \frac{1}{(1 + (n_1/n_2)^2)\sin^2\alpha \,+\, 1} \\
&= -\frac{d\phi_\perp}{dg} \times \frac{1}{(1 + (n_1/n_2)^2)\sin^2\alpha \,+\, 1}\,,
\end{aligned} \tag{S.56}
$$

and consequently

$$
D_\perp \;=\; -\frac{1}{k}\,\frac{d\phi_\perp}{dg} \times \frac{dg}{d\alpha} \;=\; \frac{2}{k} \times \frac{\sin\alpha}{\sqrt{\sin^2\alpha \,-\, (n_2/n_1)^2}}\,, \tag{S.57}
$$

exactly as in eq. (11), and

$$
D_\parallel \;=\; -\frac{1}{k}\,\frac{d\phi_\parallel}{dg} \times \frac{dg}{d\alpha} \;=\; D_\perp \times \frac{1}{(1 + (n_1/n_2)^2)\sin^2\alpha \,+\, 1}\,, \tag{S.58}
$$

exactly as in eq. (12). *Quod erat demonstrandum.*