Problem **1**(a):

For the electromagnetic fields as in eqs. (1), the space derivative ∇ acts as $i(k+i\kappa)\mathbf{n}_z$ while the time derivative acts as $-i\omega$. Consequently, the Gauss law equations

$$\nabla\cdot\mathbf{D} \;=\; \nabla\cdot\mathbf{B} \;=\; 0 \quad\Longrightarrow\quad \nabla\cdot\mathbf{E} \;=\; \nabla\cdot\mathbf{H} \;=\; 0 \tag{S.1}$$

imply

$$i(k+i\kappa)\mathbf{n}_z\cdot\vec{\mathcal{E}} \;=\; i(k+i\kappa)\mathbf{n}_z\cdot\vec{\mathcal{H}} \;=\; 0, \tag{S.2}$$

which means that both of the amplitude vectors $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ must lie in the (x,y) plane transverse to the wave's direction. Next, the induction law

$$\nabla\times\mathbf{E} \;=\; -\frac{\partial\mathbf{B}}{\partial t} \;=\; \mu_0\,\frac{\partial\mathbf{H}}{\partial t} \tag{S.3}$$

leads to

$$i(k+i\kappa)\mathbf{n}_z\times\vec{\mathcal{E}} \;=\; +i\omega\mu_0\vec{\mathcal{H}} \quad\Longrightarrow\quad \vec{\mathcal{H}} \;=\; \frac{(k+i\kappa)}{\mu_0\omega}\,\mathbf{n}_z\times\vec{\mathcal{E}}. \tag{S.4}$$

Likewise, the Maxwell–Ampere equation

$$\nabla\times\mathbf{H} \;=\; \mathbf{J} \;+\; \frac{\partial\mathbf{D}}{\partial t} \;=\; \sigma\mathbf{E} \;+\; \epsilon\epsilon_0\,\frac{\partial\mathbf{E}}{\partial t} \tag{S.5}$$

leads to

$$i(k+i\kappa)\mathbf{n}_2\times\vec{\mathcal{H}} \;=\; \sigma\vec{\mathcal{E}} \;-\; i\omega\epsilon\epsilon_0\vec{\mathcal{E}} = \Big(\sigma \;-\; i\omega\epsilon\epsilon_0\Big)\vec{\mathcal{E}} \tag{S.6}$$

and hence

$$\vec{\mathcal{E}} \;=\; -\frac{(k+i\kappa)}{(\omega\epsilon\epsilon_0 \;+\; i\sigma)}\,\mathbf{n}_z\times\vec{\mathcal{H}}. \tag{S.7}$$

To make sure eqs. (S.4) and (S.7) are consistent with each other, we need

$$\vec{\mathcal{E}} \;=\; -\mathbf{n}_z\times(\mathbf{n}_z\times\vec{\mathcal{E}}) \;=\; -\frac{\mu_0\omega}{(k_i\kappa)}\,\mathbf{n}_z\times\vec{\mathcal{H}} \;=\; +\,\frac{\mu_0\omega\times(\omega\epsilon\epsilon_0 \;+\; i\sigma)}{(k+i\kappa)^2}\,\vec{\mathcal{E}} \tag{S.8}$$

and therefore

$$(k + i\kappa)^2 \;=\; \mu_0\omega \times (\omega\epsilon\epsilon_0 \,+\, i\sigma)\,. \tag{S.9}$$

For convenience, let's define the complex refraction index according to

$$n(\omega) \;=\; \sqrt{\epsilon_{\rm eff} \;=\; \epsilon \,+\, \frac{i\sigma}{\epsilon_0\omega}}\,. \tag{S.10}$$

Then in terms of this complex index, eq. (S.9) becomes

$$(k + i\kappa)^2 \;=\; \frac{n^2(\omega)\omega^2}{c^2} \quad\Longrightarrow\quad k \;=\; \frac{\omega}{c} \times \mathrm{Re}\,n(\omega), \quad \kappa \;=\; \frac{\omega}{c} \times \mathrm{Im}\,n(\omega). \tag{S.11}$$

Also, the relation (S.4) between the electric and the magnetic amplitudes of the wave becomes

$$\vec{\mathcal{H}} \;=\; \frac{n(\omega)}{Z_0}\,\mathbf{n}_z \times \vec{\mathcal{E}} \tag{S.12}$$

where $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx\ 377\ \Omega$ is the vacuum impedance.

Problem **1**(b):

Fresnel equations give us the reflection coefficient

$$r \;=\; \frac{\mathcal{E}_{\rm reflected}}{\mathcal{E}_{\rm incident}} \tag{S.13}$$

as an analytic function of the incidence angle and the refraction indices. In particular, for an EM wave striking the boundary head on from the vacuum side, the Fresnel equation gives us

$$r \;=\; \frac{n-1}{n+1}\,, \tag{S.14}$$

and this formula is valid for any refraction index n of the second material, be it real or complex.

- When n happens to be real, the reflection coefficient r is also real, which means that the reflected wave has either the same phase or exactly opposite phase as the incident wave.

- When n happens to be purely imaginary, the reflection coefficient r is a unimodular complex number, $|r| = 1$. Physically, this means 100% *reflectivity* $R = |r|^2 = 1$, but the phase of the reflected wave is shifted by $\arg(r)$ relative to the incident wave's phase.

⋆ Generically, n is neither real nor imaginary. For such generic complex n, the reflectivity is

$$R \;=\; |r|^2 \;=\; \frac{|n|^2 + 1 \,-\, 2\,\mathrm{Re}(n)}{|n|^2 + 1 + 2\,\mathrm{Re}(n)}\,. \tag{S.15}$$

In particular, for the complex n from eq. (S.10),

$$\frac{\mathrm{Re}(n)}{|n|} \;=\; \cos\left(\mathrm{phase}(n) \;=\; \tfrac{1}{2}\,\mathrm{phase}(n^2) \;=\; \tfrac{1}{2}\arccos\frac{\epsilon}{|n|^2}\right) \;=\; \sqrt{\frac{1}{2} \,+\, \frac{\epsilon}{2|n|^2}}\,, \tag{S.16}$$

hence

$$2\,\mathrm{Re}(n) \;=\; \sqrt{2(\epsilon + |n|^2)} \tag{S.17}$$

and therefore reflectivity

$$R \;=\; \frac{|n|^2 \,+\, 1 \,-\, \sqrt{2(\epsilon + |n|^2)}}{|n|^2 \,+\, 1 \,+\, \sqrt{2(\epsilon + |n|^2)}} \;=\; 1 \,-\, \frac{2\sqrt{2(\epsilon + |n|^2)}}{|n|^2 \,+\, 1 \,+\, \sqrt{2(\epsilon + |n|^2)}}\,. \tag{S.18}$$

Problem **1**(c):

For good conductors, the imaginary part of n^2,

$$\mathrm{Im}(n^2) \;=\; \frac{\sigma}{\epsilon_0\omega} \tag{S.19}$$

is very large for any radio-wave or microwave frequencies. For example, for the copper at $\omega = 2\pi \times 100$ GHz we have $\mathrm{Im}(n^2) \approx 10^7$, and even at the visible-light frequencies we would get $\mathrm{Im}(n^2) \sim 1700$. However, at the infrared and higher frequencies we would need a frequency-dependent formula for the conductivity, so eq. (S.10) would no longer be valid. So for the purposes of this problem, let's limit ourselves to the radio-wave or microwave frequencies, and for these frequencies $\mathrm{Im}(n^2) \gg 1$ for any metal, semimetal, or even a good electrolyte like the sea water.

Anyway, a good conductor has a very large $\mathrm{Im}(n^2)$, hence

$$|n|^2 \;\approx\; \mathrm{Im}(n^2) \;\gg\; \sqrt{2(\epsilon + |n|^2)} \;\gg\; 1, \tag{S.20}$$

so we may approximate the RHS of eq. (S.18) for the reflectivity as

$$R \;\approx\; 1 \;-\; \frac{2\sqrt{2|n|^2}}{|n|^2} \;=\; 1 \;-\; \frac{2\sqrt{2}}{|n|} \;\approx\; 1 \;-\; \frac{2\sqrt{2}}{\sqrt{\mathrm{Im}(n^2)}}\,. \tag{S.21}$$

The RHS here is related to the ratio of the conductor's skin depth

$$\delta \;=\; \sqrt{\frac{2}{\mu_0\sigma\omega}} \tag{S.22}$$

and the wavelength in vacuum

$$\lambda_0 \;=\; \frac{2\pi c}{\omega}\,, \tag{S.23}$$

both at the same frequency ω. Indeed,

$$\frac{\lambda_0^2}{\delta^2} \;=\; \frac{4\pi^2c^2}{\omega^2} \times \frac{\mu_0\sigma\omega}{2} \;=\; \frac{2\pi^2\sigma}{\omega} \times \left(c^2\mu_0 \;=\; \frac{1}{\epsilon_0}\right) \;=\; \frac{2\pi^2\sigma}{\epsilon_0\omega} \;=\; 2\pi^2\,\mathrm{Im}(n^2). \tag{S.24}$$

Consequently, in eq. (S.21) $\mathrm{Im}(n^2)$ becomes $\lambda_0^2/2\pi^2\delta^2$ and therefore

$$R \;\approx\; 1 \;-\; \frac{2\sqrt{2}}{\sqrt{\lambda_0^2/2\pi^2\delta^2}} \;=\; 1 \;-\; \frac{4\pi\delta}{\lambda_0}\,, \tag{S.25}$$

exactly as in eq. (2)

Problem **1**(d):

The USW frequency band reserved for the FM radio broadcasts all over the world is from 88 to 108 MHZ, so let work with $(\omega/2\pi) = 100$ MHz. At this frequency, the vacuum wavelength is $\lambda_0 = 3$ meters, while the skin depth is sea water of conductivity $\sigma = 5\,\mho/\mathrm{m}$ is only

$$\delta \;=\; \sqrt{\frac{2}{\mu_0\sigma\omega}} \;=\; 0.7\ \mathrm{mm}. \tag{S.26}$$

Consequently,

$$1 \;-\; R \;\approx\; \frac{4\pi\delta}{\lambda_0} \;=\; 3\times 10^{-3}, \tag{S.27}$$

thus the sea water reflects 99.7% of the radio-wave's energy.

Problem **2**(a):

Fourier transforming the macroscopic Maxwell equations from time to frequency gives us

$$\begin{aligned}
\nabla\cdot\mathbf{D}(\mathbf{x},\omega) \;&=\; \rho(\mathbf{x},\omega), && \text{(S.28)}\\
\nabla\times\mathbf{E}(\mathbf{x},\omega) \;&=\; +i\omega\mathbf{B}(\mathbf{x},\omega), && \text{(S.29)}\\
\nabla\cdot\mathbf{B}(\mathbf{x},\omega) \;&=\; 0, && \text{(S.30)}\\
\nabla\times\mathbf{H}(\mathbf{x},\omega) \;&=\; \mathbf{J}_{\rm net}(\mathbf{x},\omega) \;=\; \mathbf{J}_{\rm cond}(\mathbf{x},\omega) \;+\; \mathbf{J}_{\rm disp}(\mathbf{x},\omega) && \text{(S.31)}\\
&=\; \sigma_c(\omega)\mathbf{E}(\mathbf{x},\omega). && (3)
\end{aligned}$$

Let's rewrite eq. (3) for the net conduction + displacement current in tersm of the electric displacement field $\mathbf{D}$ rather than the tension field $\mathbf{E}$,

$$\mathbf{J}_{\rm net} \;=\; \sigma_c(\omega)\mathbf{E} \;=\; \frac{\sigma_c(\omega)}{\epsilon(\omega)\epsilon_0}\,\mathbf{D} \tag{S.32}$$

and then take the divergenses of both sides of this equation:

$$\nabla\cdot\mathbf{J}_{\rm net}(\mathbf{x},\omega) \;=\; \frac{\sigma_c(\omega)}{\epsilon(\omega)\epsilon_0}\,\nabla\cdot\mathbf{D}(\mathbf{x},t). \tag{S.33}$$

On the RHS here, $\nabla\cdot\mathbf{D} = \rho$ by the Gauss Law (S.28), while the LHS must vanish by the

Maxwell–Ampere Law (S.31),

$$\nabla \cdot \mathbf{J}_{\rm net} \;=\; \nabla \cdot (\nabla \times \mathbf{H}) \;=\; 0, \tag{S.34}$$

thus eq. (S.33) becomes an equation for the electric charge density, or rather for its Fourier transform $\rho(\mathbf{x},\omega)$:

$$\frac{\sigma_c(\omega)}{\epsilon(\omega)\epsilon_0}\,\rho(\mathbf{x},\omega) \;=\; 0. \tag{S.35}$$

Finally, multiplying this equation by the non-singular $\epsilon(\omega)\epsilon_0$, we arrive at

$$\text{for all } \mathbf{x} \text{ and all } \omega, \quad \sigma_c(\omega)\rho(\mathbf{x},\omega) \;=\; 0. \tag{5}$$

Quod erat demonstrandum.

Problem **2**(b):

Strictly speaking, eq. (6) is valid only for ω's which are small compared to the frequencies of the electronic resonances in the ion cores stripped of their conduction electrons. But for the present exercise I am treating eq. (6) as exact.

Given eq. (6) for the complex conductivity $\sigma_c(\omega)$, eq. (5) for the (Fourier-transformed) charge density perturbations becomes

$$\frac{\epsilon_b\epsilon_0}{\gamma_0 - i\omega}(\omega_p^2 \,-\, i\gamma_0\omega \,-\, \omega^2)\rho(\mathbf{x},\omega) \;=\; 0. \tag{S.36}$$

and hence

$$(\omega_p^2 \,-\, i\gamma_0\omega \,-\, \omega^2)\rho(\mathbf{x},\omega) \;=\; 0. \tag{7}$$

Note: this is an algebraic equation for the ω-dependence of the Fourier-transformed charge density, and it holds for every point $\mathbf{x}$ regardless of any other point $\mathbf{y}$. In terms of the original time-dependent charge density oscillation $\rho(\mathbf{x},t)$, eq. (S.36) becomes a second-order ordinary differential equation

$$\left(\frac{\partial^2}{\partial t^2} \,+\, \gamma_0\frac{\partial}{\partial t} \,+\, \omega_p^2\right)\rho(\mathbf{x},t) \;=\; 0. \tag{8}$$

Problem **2**(c):

Eq. (8) governs the time dependence of $\rho(\mathbf{x},t)$ at each $\mathbf{x}$, independently from all other locations in space. Specifically, for each $\mathbf{x}$ we have the damped harmonic oscillator equation of the form

$$\ddot{\psi}(t) \;+\; \gamma_0\dot{\psi}(t) \;+\; \omega_p^2\psi(t) \;=\; 0 \tag{S.37}$$

for $\psi(t)=\rho(\mathbf{x},t)$. The general solution of this equation is

$$\psi(t) \;=\; \mathrm{Re}\left[\psi_0 e^{-i\omega' t}e^{-\gamma_0 t/2}\right], \tag{S.38}$$

oscillations with frequency

$$\omega' \;=\; \sqrt{\omega_p^2 - \tfrac{1}{4}\gamma_0^2} \;\approx\; \omega_p\,, \tag{S.39}$$

dumping rate γ_0, and initial complex amplitude ψ_0. Consequently, solving eq. (8) for the charge density oscillations gives us

$$\rho(\mathbf{x},t) \;=\; \mathrm{Re}\Big[\rho_0(\mathbf{x})\,e^{-i\omega' t}\,e^{-\gamma_0 t/2}\Big] \tag{S.40}$$

for some initial complex amplitude $\rho_0(\mathbf{x})$. Thus, the charge density perturbations oscillate in place with frequency $\omega'\approx\omega_p$ while their intensity $|\rho|^2$ decays at the rate γ_0. *Quod erat demonstrandum.*.

Problem **3**(a):

Let $\Psi(x,t)$ be a component of the electric or the magnetic field in a 1D wave propagating through some linear and uniform but dispersive medium. The Fourier transform $\psi(x,\omega)$ of this wave obeys the second-order equation

$$\left(\frac{\omega^2 n^2(\omega)}{c^2} \;+\; \frac{\partial^2}{\partial x^2}\right)\psi(x,\omega) \;=\; 0, \tag{S.41}$$

whose general solution at any fixed ω is a superposition of sine and cosine waves of wave number $k=n\omega/c$, or equivalently

$$\psi(x,\omega) \;=\; A(\omega)\times\exp(+i\omega n(\omega)x/c) \;+\; B(\omega)\times\exp(-i\omega n(\omega)x/c). \tag{S.42}$$

for some arbitrary coefficients $A(\omega)$ and $B(\omega)$.

Eq. (9) follows from eq. (S.42) via Fourier transform from $\psi(x,\omega)$ back to $\psi(x,t)$.

Problem **3**(b):

Given eq. (9) for $\psi(x,t)$, let's take its complex conjugate:

$$\psi(x,t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \exp(-i\omega t) \begin{pmatrix} A(\omega)\times\exp(+i\omega n(\omega)x/c) \\ +\,B(\omega)\times\exp(-i\omega n(\omega)x/c) \end{pmatrix}, \tag{6}$$

$$\begin{aligned}
\psi^*(x,t) &= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \exp(+i\omega t) \begin{pmatrix} A^*(\omega)\times\exp(-i\omega n^*(\omega)x/c) \\ +\,B^*(\omega)\times\exp(+i\omega n^*(\omega)x/c) \end{pmatrix} \\
&\quad \langle\!\langle \text{ changing int. variable } \omega = -\omega' \,\rangle\!\rangle \\
&= \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi}\, \exp(-i\omega' t) \begin{pmatrix} A^*(-\omega')\times\exp(+i\omega' n^*(-\omega')x/c) \\ +\,B^*(-\omega')\times\exp(-i\omega' n^*(-\omega')x/c) \end{pmatrix}, \qquad \text{(S.43)} \\
&\quad \langle\!\langle \text{ rename } \omega' \to \omega \,\rangle\!\rangle \\
\psi^*(x,t) &= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \exp(-i\omega t) \begin{pmatrix} A^*(-\omega)\times\exp(+i\omega n^*(-\omega)x/c) \\ +\,B^*(-\omega)\times\exp(-i\omega n^*(-\omega)x/c) \end{pmatrix}. \qquad \text{(S.44)}
\end{aligned}$$

Requiring $\psi(x,t)$ to be real for all x and all t means that

$$\forall x,t: \quad \psi^*(x,t)[\text{from eq. (S.44)}] \;=\; \psi(x,t)[\text{from eq. (9)}], \tag{S.45}$$

and by inspection of these two formulae, having

$$\forall\omega: \quad n^*(-\omega) \;=\; n(+\omega), \quad A^*(-\omega) \;=\; A(+\omega), \quad B^*(-\omega) \;=\; B(+\omega), \tag{S.46}$$

is a sufficient and necessary condition. *Quod erat demonstrandum.*

Problem **3**(c):

Given eq. (9) for $\psi(x,t)$, the ψ itself and its x derivative at $x=0$ are given by

$$\psi(0,t) \;=\; \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \exp(-i\omega t)\times\big(A(\omega)+B(\omega)\big), \tag{S.47}$$

$$\frac{\partial}{\partial x}\Psi(0,t) \;=\; \int\limits_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\,\exp(-i\omega t)\times\frac{i\omega n(\omega)}{c}\times\big(A(\omega)-B(\omega)\big). \tag{S.48}$$

Inverting these Fourier transforms gives us

$$\big(A(\omega)+B(\omega)\big) \;=\; \int\limits_{-\infty}^{+\infty} dt\,\exp(+i\omega t)\times\psi(0,t), \tag{S.49}$$

$$\frac{i\omega n(\omega)}{c}\times\big(A(\omega)-B(\omega)\big) \;=\; \int\limits_{-\infty}^{+\infty} dt\,\exp(+i\omega t)\times\frac{\partial}{\partial x}\psi(0,t), \tag{S.50}$$

and hence eqs. (10) for the $A(\omega)$ and $B(\omega)$.

<u>Problem **4**</u>:

To prove that under the assumptions at hand, the group velocity of an EM wave is less than c, I am going to show that

$$\begin{aligned} \text{(a)}\quad & v_{\text{group}} \;<\; v_{\text{phase}}\,, \\ \text{and (b)}\quad & v_{\text{group}}\times v_{\text{phases}} \;<\; c^2. \end{aligned} \tag{S.51}$$

Together, these two points immediately lead to $v_{\text{group}} < c$.

My starting point is the refraction coefficient $n(\omega) = \sqrt{\epsilon(\omega)\mu(\omega)}$ and its frequency dependence. Given eq. (11) and $\mu \approx 1$, we have

$$n^2(\omega) \;\approx\; 1 \;+\; \frac{Ne^2}{m_e\epsilon_0}\sum_i \frac{f_i}{\omega_i^2-\omega^2-i\omega\gamma_i}\,, \tag{S.52}$$

which at frequencies not too close to any of the resonances ω_i — *i.e.*, in the regime of normal dispersion — becomes

$$n^2(\omega) \;\approx\; 1 \;+\; \frac{Ne^2}{m_e\epsilon_0}\sum_i \frac{f_i}{\omega_i^2-\omega^2}\,. \tag{S.53}$$

As I have explained in class, the phase velocity and the group velocity of the EM waves follow from this refraction index according to

$$\begin{aligned} v_{\text{phase}} &= \frac{\omega}{k} = \frac{c}{n}, \\ v_{\text{group}} &= \frac{d\omega}{dk} = \frac{c}{n + \omega(dn/d\omega)}. \end{aligned} \tag{S.54}$$

In the normal dispersion regime, the refraction index has positive derivative,

$$\frac{dn^2}{d\omega} = \frac{Ne^2}{m_e\epsilon_0}\sum_i \frac{2f_i\omega}{(\omega_i^2-\omega^2)^2} > 0, \tag{S.55}$$

hence

$$\frac{c}{v_{\text{group}}} = n + \omega\frac{dn}{d\omega} > n = \frac{c}{v_{\text{phase}}} \tag{S.56}$$

and therefore $v_{\text{group}} < v_{\text{phase}}$. This proves point (a).

To prove point (b), consider the product

$$\frac{c}{v_{\text{phase}}} \times \frac{c}{v_{\text{group}}} = n \times \left(n + \omega\frac{dn}{d\omega}\right) = n^2 + \frac{\omega}{2}\times\frac{dn^2}{d\omega}. \tag{S.57}$$

For the refraction index (S.53),

$$\begin{aligned} \left(1 + \frac{\omega}{2}\frac{d}{d\omega}\right) n^2(\omega) &= 1 + \frac{Ne^2}{m_e\epsilon_0}\sum_i f_i\left(1+\frac{\omega}{2}\frac{d}{d\omega}\right)\frac{1}{\omega_i^2-\omega^2} \\ &= 1 + \frac{Ne^2}{m_e\epsilon_0}\sum_i f_i\left(\frac{1}{\omega_i^2-\omega^2} + \frac{\omega^2}{(\omega_i^2-\omega^2)^2}\right) \\ &= 1 + \frac{Ne^2}{m_e\epsilon_0}\sum_i f_i\,\frac{\omega_i^2}{(\omega_i^2-\omega^2)^2} \\ &= 1 + \sum(\text{positive terms}) > 1, \end{aligned} \tag{S.58}$$

hence

$$\frac{c}{v_{\text{phase}}} \times \frac{c}{v_{\text{group}}} > 1 \quad\Longrightarrow\quad v_{\text{group}} \times v_{\text{phase}} < c^2. \tag{S.59}$$

This proves point (b) and hence $v_{\text{group}} < c$.