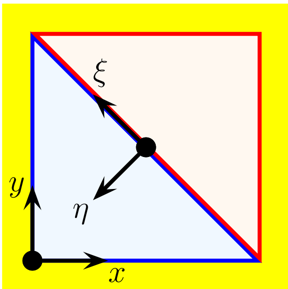
Problem **2**, preamble:

In this problem, we would need two coordinate systems for the two transverse dimensions of the waveguide. First, $(x, y)$ with the coordinate axes along the two short sides of the triangle. Second, $(\xi, \eta)$ where $\xi$ axis is parallel to the triangle;s long side and $\eta$ axis is normal to it. Here is the diagram of the triangle and its mirror image showing the two coordinate systems:




(S.1)

In terms of $(\xi, \eta)$ coordinates, the mirror reflection off the long side of the triangle acts as

$$\xi_{\text{image}} = +\xi_{\text{orig}}, \qquad \eta_{\text{image}} = -\eta_{\text{orig}}, \tag{S.2}$$

while in terms of $(x, y)$ coordinates, it works according to

$$x_{\text{image}} = a - y_{\text{orig}}, \qquad y_{\text{image}} = a - x_{\text{orig}}. \tag{S.3}$$

Problem **2**(a):

Suppose $\psi(x, y)$ — or equivalently $\psi(\xi, \eta)$ — obeys the eigenvalue equation $(\nabla^2 + \Gamma)^2\psi = 0$ everywhere inside the original triangle as well as Neumann or Dirichlet conditions on the 3 sides of the triangle. Then obviously it's extension to the image triangle according to eq. (2) — or in terms of $(\xi, \eta)$ coordinates

$$\psi(\xi, \eta < 0) = \pm\psi(+\xi, -\eta > 0), \tag{S.4}$$

— also obeys the eigenvalue equation $(\nabla^2 + \Gamma)^2\psi = 0$ everywhere inside the image triangle as well as Neumann or Dirichlet conditions on its sides. In terms of the square made from both triangles, this means $\psi$ obeys the Neumann or Dirichlet conditions on all 4 sides of the square as well as $(\nabla^2 + \Gamma)^2\psi = 0$ everywhere inside the square, except maybe at the $\eta = 0$ diagonal line separating the two triangles.

Thus, to make sure the extended $\psi$ is a proper eigenstate of the $-\nabla^2$ operator for the whole square, we must make sure that $\nabla^2\psi$ does not have $\delta$-like singularities along the diagonal line, which means that $\psi$ itself and its first derivatives $\partial\psi/\partial\xi$ and $\partial\psi/\partial\eta$ must be continuous at $\eta = 0$. Once this condition is satisfied, all the higher derivatives would also be continuous — this is automatic for any $\psi$ obeying $(\nabla^2 + \Gamma)^2\psi = 0$ on both sides of the diagonal.

So let's check the continuity of $\psi(\xi, \eta)$ and its first derivatives for both types of boundary conditions. In the Dirichlet case, the boundary condition on the diagonal of the original triangle is $\psi_{\text{orig}}(\xi, \eta = 0) = 0$, hence in the image triangle

$$\psi_{\text{image}}(\xi, \eta) \;=\; -\psi_{\text{orig}}(\xi, -\eta) \;\to\; 0 \text{ for } \eta \to 0. \tag{S.5}$$

so on both sides of the diagonal $\psi$ vanishes for $\eta \to \pm 0$. This makes $\psi$ itself continuous along the diagonal, hence its derivative WRT $\xi$ is also continuous. As to the normal derivative $\partial\psi\partial\eta$, the minus sign in eq. (S.5) means

$$\frac{\partial}{\partial\eta}\psi_{\text{image}}(\xi, \eta) \;=\; +\frac{\partial}{\partial\eta}\psi_{\text{orig}}(\xi, -\eta), \tag{S.6}$$

so both $\psi_{\text{image}}$ and $\psi_{\text{orig}}$ have the same value at $\eta = 0$. In other words, $\partial\psi\partial\eta$ is also continuous across the diagonal.

Now consider the Neumann boundary conditions. In this case, the plus sign in eq. (2), — or equivalently

$$\psi_{\text{image}}(\xi, \eta) \;=\; +\psi_{\text{orig}}(\xi, -\eta), \tag{S.7}$$

— immediately leads to the continuity of $\psi$ itself across the $\eta = 0$ diagonal, and hence to the continuity of the tangent derivative $\partial\psi\partial\xi$. As to the normal derivative, the Neumann boundary conditions for the original triangle include

$$\frac{\partial}{\partial\eta}\psi_{\rm orig}(\xi,\eta)\ \to\ 0 \text{ for } \eta\ \to +0, \tag{S.8}$$

while eq. (2) leads to

$$\frac{\partial}{\partial\eta}\psi_{\rm image}(\xi,\eta)\ =\ -\frac{\partial}{\partial\eta}\psi_{\rm orig}(\xi,-\eta)\ \to\ 0 \text{ for } \eta\ \to -0. \tag{S.9}$$

So the normal derivative is also continuous across the diagonal.

Thus, for both types of boundary conditions, the extended $\psi$ and its first derivatives are continuous across the diagonal, which makes it obey the $(\nabla^2+\Gamma^2)\psi=0$ equation not only on both sides of the diagonal but also on the diagonal itself. Altogether, the extended $\psi$ obeys $(\nabla^2+\Gamma^2)\psi=0$ across the whole square, and also obeys the appropriate boundary conditions on all 4 sides of the squatre. Physically, this means that the eigenstate of the triangle extended to the whole square by the mirror reflection (2) is indeed an eigenstate of the whole square. *Quod erat demonstrandum.*

Problem **2**(b):

In part (a) we saw that an eigenstate of the triangle extended to the whole square becomes an eigenstate of the whole square for the same eigenvalue $\Gamma^2$. On the other hand, an eigenstate of the whole square reduced to the original triangle becomes an eigenstate of that triangle only if it happens to obey the boundary conditions on the triangle's long side. Fortunately, this condition can always be satisfied by superimposing two degenerate eigenstates of the square related by the mirror symmetry (2). Specifically, let $\psi(x,y)$ be an eigenstate of the square, then

$$\psi'(x,y)\ =\ \psi(a-y,a-x) \tag{S.10}$$

is also an eigenstate with the same eigenvalue, and

$$\Psi(x,y)\ =\ \psi(x,y)\ \pm\ \psi'(x,y) \tag{S.11}$$

obeys the Neumann conditions on the diagonal for the + sign and the Dirichlet condition for the $-$ sign, so unless $\Psi\equiv 0$ then it is truly an eigenstate of the triangle.

This gives us a way to construct all of the triangle's eigenstates starting from the square's eigenstates, so let's do it first for the Neumann boundary conditions and the for the Dirichlet.

Neumann boundary conditions:

The eigenstates of the square with Neumann BC are products of cosine waves in $x$ and $y$ directions,

$$\begin{aligned}\psi^N_{m,n} &= \cos\frac{m\pi x}{a} \times \cos\frac{n\pi y}{a}\,,\\ \Gamma^2_{m,n} &= \frac{\pi^2}{a^2}(m^2+n^2),\end{aligned}$$

where $m$ and $n$ are independent non-negative integers $m,n = 0,1,2,3,\ldots$. The mirror reflection (S.3) swaps $m \leftrightarrow n$, and it also multiplies the cosine product by the overall sign $(-1)^{n+m}$,

$$\begin{aligned}\psi^N_{m,n}(a-y,a-x) &= \cos\frac{m\pi(a-y)}{a} \times \cos\frac{n\pi(a-x)}{a}\\ &= (-1)^{m+n} \times \cos\frac{n\pi x}{a} \times \cos\frac{m\pi y}{a}\\ &= (-1)^{m+n} \times \psi^N_{n,m}\,,\end{aligned} \tag{S.12}$$

so the triangle's eigenstates are

$$\Psi^N_{m,n} = \cos\frac{m\pi x}{a} \times \cos\frac{n\pi y}{a} + (-1)^{m+n} \times \cos\frac{n\pi x}{a} \times \cos\frac{m\pi y}{a}\,. \tag{S.13}$$

However, one has to be careful counting such eigenstates for the triangle. For $m \neq n$, the square's eigenstates $\psi^N_{m,n}$ and $\psi^N_{n,m}$ are different states (albeit with the same eigenvalue), but the corresponding triangle's eignestates (S.13) become identical, $\Psi^N_{m,n} = \pm\Psi^N_{n,m}$, so they should be counted as one eigenstate rather than two. Thus, to avoid the double-counting of the triangle's eigenstates, we must restrict the list of $\Psi^N_{m,n}$ to $m \leq n$ only, thus

$$\psi^N_{0,0}\,, \quad \psi^N_{0,1}\,, \quad \psi^N_{1,1}\,, \quad \psi^N_{0,2}\,, \quad \psi^N_{1,2}\,, \quad \psi^N_{2,2}\,, \quad \psi^N_{3,0}\,, \quad \ldots. \tag{S.14}$$

Dirichlet boundary conditions:

The eigenstates of the square with Dirichlet BC are products of sine waves in $x$ and $y$ directions,

$$\begin{aligned}\psi^D_{m,n} &= \sin\frac{m\pi x}{a}\times\sin\frac{n\pi y}{a}\,,\\ \Gamma^2_{m,n} &= \frac{\pi^2}{a^2}(m^2+n^2),\end{aligned}$$

where $m$ and $n$ are independent positive integers $m,n=\cancel{0},1,2,3,4,\ldots$. The mirror reflection (S.3) swaps $m\leftrightarrow n$, and it also multiplies the sine product by the overall sign $(-1)^{n+m}$,

$$\begin{aligned}\psi^D_{m,n}(a-y,a-x) &= \sin\frac{m\pi(a-y)}{a}\times\sin\frac{n\pi(a-x)}{a}\\ &= (-1)^{m+n}\times\sin\frac{n\pi x}{a}\times\sin\frac{m\pi y}{a}\\ &= (-1)^{m+n}\times\psi^D_{n,m}\,,\end{aligned}\tag{S.15}$$

so the triangle's eigenstates are

$$\Psi^D m,n \;=\; \sin\frac{m\pi x}{a}\times\sin\frac{n\pi y}{a}\;-\;(-1)^{m+n}\times\sin\frac{n\pi x}{a}\times\sin\frac{m\pi y}{a}\,.\tag{S.16}$$

However, the difference here vanishes for $m=n$ so the triangle eigenstates obtain only for $m\neq n$. Also, for $m\neq n$ the square has two different (albeit degenerate) eigenstates $\psi^D_{m,n}$ and $\psi^D_{n,m}$, but for the triangle they are identical up to a sign. So to avoid double-counting of the triangle's eigenstates, we must restrict the list of $\psi^D_{m,n}$ to $m<n$ only, thus

$$\psi^D_{1,2}\,,\quad \psi^D_{1,3}\,,\quad \psi^D_{2,3}\,,\quad \psi^D_{4,1}\,,\quad \psi^D_{4,2}\,,\quad \psi^D_{4,3}\,,\quad \psi^D_{5,0}\,,\quad \ldots.\tag{S.17}$$

Problem **2**(c):

The TE waves have $E_z\equiv 0$ while the $H_z(x,y)$ is an eigenstate of the two-dimensional $-\nabla^2$ operator for the Neumann boundary conditions at the triangle sides. As we saw in part (b), such eigenstates have form

$$H_z(x,y) \;=\; H_0\,\cos\frac{m\pi x}{a}\,\cos\frac{n\pi y}{a} \;+\; (-1)^{m+n} H_0\,\cos\frac{n\pi x}{a}\,\cos\frac{m\pi y}{a} \tag{S.18}$$

for integers $m$ and $n$ such that $n \geq m \geq 0$, with the corresponding eigenstates being

$$\Gamma^2_{m,n} \;=\; (m^2+n^2)\times\frac{\pi^2}{a^2}\,. \tag{S.19}$$

For the TE waves, this translates to $\mathrm{TE_{m,n}}$ waves for integer $m, n \geq 0$ and $m \leq n$; however, we cannot have $H_z = \mathrm{const} \neq 0$, so the $TE_{0,0}$ wave does not exist. But all other combinations of $n \geq m \geq 0$ are allowed, so the list of TE modes goes

$$\mathrm{TE}_{0,1}\,,\quad \mathrm{TE}_{1,1}\,,\quad \mathrm{TE}_{0,2}\,,\quad \mathrm{TE}_{1,2}\,,\quad \mathrm{TE}_{2,2}\,,\quad \mathrm{TE}_{0,3}\,,\quad \ldots. \tag{S.20}$$

As to the cutoff frequencies of all these modes, they follow from the eigenvalues $\Gamma_{m,n}$:

$$\omega_{\min}(\mathrm{TE}_{m,n}) \;=\; c\Gamma_{m,n} \;=\; \frac{\pi c}{a}\times\sqrt{m^2+n^2}\,. \tag{S.21}$$

Now, the TM waves have $H_z \equiv 0$ while the $E_z(x,y)$ is an eigenstate of the two-dimensional $-\nabla^2$ operator for the Dirichlet boundary conditions at the triangle sides. As we saw in part (b), such eigenstates have form

$$E_z(x,y) \;=\; E_0\,\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{a} \;-\; (-1)^{m+n} E_0\,\sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{a} \tag{S.22}$$

for positive integers $m$ and $n$ with $n > m > 0$. Thus, the TM modes are $\mathrm{TM_{m,n}}$ are restricted to these combinations of $m$ and $n$, so the list of TM modes goes

$$\mathrm{TM}_{1,2}\,,\quad \mathrm{TM}_{1,3}\,,\quad \mathrm{TM}_{2,3}\,,\quad \mathrm{TM}_{1,4}\,,\quad \mathrm{TM}_{2,4}\,,\quad \mathrm{TM}_{3,4}\,,\quad \mathrm{TM}_{1,5}\,,\quad \ldots. \tag{S.23}$$

In terms of $m$ and $n$, the eigenvalues $\Gamma_{m,n}$ are exactly the same

$$\Gamma^2_{m,n} \;=\; (m^2+n^2)\times\frac{\pi^2}{a^2} \tag{S.19}$$

as for the TE modes, hence similar cutoff frequencies

$$\Omega(\mathrm{TM}_{m,n}) \;=\; c\Gamma_{m,n} \;=\; \frac{\pi c}{a}\times\sqrt{m^2+n^2}\,. \tag{S.24}$$

For completeness sake (although it was not a requires part of this problem), let me list the first baker's dozen of modes in the order of increasing cutoff frequencies, or rather the first 13 levels of $\Omega = \omega_{\text{min}}$:

| modes | $\Omega$ in units of $(\pi c/a)$ |
|---|---|
| $\text{TE}_{0,1}$ | 1 |
| $\text{TE}_{1,1}$ | $\sqrt{2} \approx 1.414$ |
| $\text{TE}_{0,2}$ | 2 |
| $\text{TE}_{1,2},\ \text{TM}_{1,2}$ | $\sqrt{5} \approx 2.236$ |
| $\text{TE}_{2,2}$ | $\sqrt{8} \approx 2.828$ |
| $\text{TE}_{0,3}$ | 3 |
| $\text{TE}_{1,3},\ \text{TM}_{1,3}$ | $\sqrt{10} \approx 3.162$ |
| $\text{TE}_{2,3},\ \text{TM}_{2,3}$ | $\sqrt{13} \approx 3.606$ |
| $\text{TE}_{0,4}$ | 4 |
| $\text{TE}_{1,4},\ \text{TM}_{1,4}$ | $\sqrt{17} \approx 4.123$ |
| $\text{TE}_{3,3}$ | $\sqrt{18} \approx 4.243$ |
| $\text{TE}_{2,4},\ \text{TM}_{2,4}$ | $\sqrt{20} \approx 4.472$ |
| $\text{TE}_{3,4},\ \text{TM}_{3,4},\ \text{TE}_{0,5}$ | 5 |

Problem **3**, preamble:

The attenuation rate of any particular mode obtains as

$$\alpha = \frac{\text{(power loss)/length}}{\text{(net power)}} \tag{S.25}$$

where

$$\text{(net power)} = \frac{k\omega}{2\Gamma^2} \int\limits_{\substack{\text{cross}\\ \text{section}}} \left(\epsilon_0 |E_z|^2 \text{ or } \mu_0 |H_z|^2\right) dx\,dy \tag{S.26}$$

(assuming no dielectric inside the waveguide, just vacuum or air), and

$$\frac{\text{(power loss)}}{\text{length}} = \frac{R_s}{2} \oint\limits_{\text{perimeter}} |\mathbf{H}|^2\, d\ell. \tag{S.27}$$

In these formulae,

$$R_s = \frac{1}{\sigma\delta} \tag{S.28}$$

is the surface resistivity of the waveguide walls,

$$\frac{k\omega}{\Gamma^2} = c \times \frac{\omega\sqrt{\omega^2 - \Omega^2}}{\Omega^2}, \tag{S.29}$$

and

$$c \times \mu_0 = Z_0, \qquad c \times \epsilon_0 = \frac{1}{Z_0}, \tag{S.30}$$

where $Z_0 = 377\ \Omega$ is the wave impedance of the vacuum. Altogether, all these formulae lead to

$$\alpha = \frac{R_s}{Z_0} \times \frac{\Omega^2}{\omega\sqrt{\omega^2 - \Omega^2}} \times \frac{\oint |\mathbf{H}^2|\, d\ell}{\iint \big(|H_z|^2 \text{ or } |E_z|^2/Z_0^2\big)\, dx\, dy} \tag{S.31}$$

Also, the simplest way to integrate $|H_z|^2$ or $|E_z|^2$ over the triangle is to extend the integrand to the square (1) using the mirror reflection (2), integrate over the whole square, and then divide by 2, thus

$$\begin{aligned} \iint\limits_{\text{triangle}} \big(|\mathbf{H}|^2 \text{ or } |E_z|^2/Z_0^2\big)\, dx\, dy &= \frac{1}{2} \iint\limits_{\text{square}} \big(|\mathbf{H}|^2 \text{ or } |E_z|^2/Z_0^2\big)\, dx\, dy \\ &= \frac{1}{2} \int_0^a dx \int_0^a dy\, \big(|\mathbf{H}|^2 \text{ or } |E_z|^2/Z_0^2\big). \end{aligned} \tag{S.32}$$

Indeed, by the mirror symmetry, the integral over the image triangle is exactly equal to the integral over the original triangle, so the over the whole square is simply $2 \times$ the integral over the original triangle.

Problem $\mathbf{3}$(a):

As we saw in problem 1, the lowest cutoff frequency among the TE waves belongs to the $\mathrm{TE}_{0,1}$ mode with

$$\begin{aligned} H_z &= H_0 \cos\frac{\pi y}{a} - H_0 \cos\frac{\pi x}{a}\,, \qquad (S.33)\\ \Gamma &= \frac{\pi}{a}\,, \qquad \Omega = \frac{\pi c}{a}\,, \qquad (S.34)\end{aligned}$$

and hence

$$\begin{aligned} H_x &= -\frac{ika}{\pi}\,H_0 \sin\frac{\pi x}{a}\,, \qquad (S.35)\\ H_y &= +\frac{ika}{\pi}\,H_0 \sin\frac{\pi y}{a}\,. \qquad (S.36)\end{aligned}$$

For this mode,

$$\begin{aligned} \int_0^a dx \int_0^a dy\,|H_z|^2 &= |H_0|^2 \int_0^a dx \int_0^a dy\left(\cos^2\frac{\pi x}{a} + \cos^2\frac{\pi y}{a} - 2\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\right)\\ &= |H_0|^2\left(\frac{a^2}{2} + \frac{a^2}{2} - 2\times 0\right)\\ &= |H_0|^2 a^2 \end{aligned} \qquad (S.37)$$

and therefore

$$\iint\limits_{\text{triangle}} dx\,dy\,|H_z|^2 = \frac{|H_0|^2 a^2}{2}\,. \qquad (S.38)$$

Next, let's integrate $|\mathbf{H}|^2$ over the perimeter, *i.e.* over the 3 sides of the triangle. Over the vertical side at $x=0$ we have

$$H_x = 0, \quad H_y = +\frac{ikaH_0}{\pi}\sin\frac{\pi y}{a}\,, \quad H_z = H_0\left(\cos\frac{\pi y}{a} - 1\right), \qquad (S.39)$$

hence

$$\begin{aligned}\int_0^a dy\,|\mathbf{H}|^2 &= \frac{k^2a^2|H_0|^2}{\pi^2}\times\int_0^a dy\,\sin^2\frac{\pi y}{a}\\ &\quad + |H_0|^2\times\int_0^a dy\left[\left(\cos\frac{\pi y}{a}-1\right)^2 = \cos^2\frac{\pi y}{a}-2\cos\frac{\pi y}{a}+1\right]\\ &= \frac{k^2a^2|H_0|^2}{\pi^2}\times\frac{a}{2}+|H_0|^2\times\left[\frac{a}{2}-2\times 0+a\right]\\ &= a|H_0|^2\times\left(\frac{3}{2}+\frac{k^2a^2}{2\pi^2}\right).\end{aligned} \tag{S.40}$$

Likewise, for the horizontal side of the triangle at $y=0$ we also get

$$\int_0^a dx\,|\mathbf{H}|^2 = a|H_0|^2\times\left(\frac{3}{2}+\frac{k^2a^2}{2\pi^2}\right). \tag{S.41}$$

Finally, along the diagonal side of the triangle

$$y = a - x,\quad d\ell = \sqrt{2}\,dx,$$

while

$$H_z = -2H_0\cos\frac{\pi x}{a}, \tag{S.42}$$

$$H_x = -H_y = \frac{ikaH_0}{\pi}\sin\frac{\pi x}{a}, \tag{S.43}$$

$$|\mathbf{H}|^2 = |H_0|^2\times\left(4\cos^2\frac{\pi x}{a}+\frac{2k^2a^2}{\pi^2}\sin^2\frac{\pi x}{a}\right), \tag{S.44}$$

which integrates to

$$\begin{aligned}\int|\mathbf{H}|^2\,d\ell &= \sqrt{2}|H_0|^2\times\int_0^a dx\left(4\cos^2\frac{\pi x}{a}+\frac{2k^2a^2}{\pi^2}\sin^2\frac{\pi x}{a}\right)\\ &= \sqrt{2}|H_0|^2\times\left(4\times\frac{a}{2}+\frac{2k^2a^2}{\pi^2}\times\frac{a}{2}\right).\end{aligned} \tag{S.45}$$

Altogether, the perimeter integral evaluates to

$$\oint |\mathbf{H}|^2\,d\ell \;=\; a|H_0|^2\times\left((3+2\sqrt{2})\;+\;(1+\sqrt{2})\times\frac{k^2a^2}{\pi^2}\right). \tag{S.46}$$

In this formula, $(3+2\sqrt{2})=(1+\sqrt{2})^2$ while

$$\frac{k^2a^2}{\pi^2}\;=\;\frac{k^2}{\Gamma^2}\;=\;\frac{\omega^2-\Omega^2}{\Omega^2}\,, \tag{S.47}$$

so we end up with

$$\oint |\mathbf{H}|^2\,d\ell \;=\; a|H_0|^2\times(1+\sqrt{2})\times\frac{\omega^2+\sqrt{2}\Omega^2}{\Omega^2}\,. \tag{S.48}$$

Finally, plugging the perimeter integral (S.48) and the area integral (S.38) into eq. (S.31) for the attenuation rate, we get

$$\begin{aligned}\alpha \;&=\; \frac{R_s}{Z_0}\times\frac{\Omega^2}{\omega\sqrt{\omega^2-\Omega^2}}\times\frac{\oint|\mathbf{H}^2|\,d\ell}{\iint|H_z|^2\,dx\,dy}\\ &=\; \frac{R_s}{Z_0}\times\frac{\Omega^2}{\omega\sqrt{\omega^2-\Omega^2}}\times a|H_0|^2\times(1+\sqrt{2})\times\frac{\omega^2+\sqrt{2}\Omega^2}{\Omega^2}\;\Big/\;\frac{|H_0|^2a^2}{2}\\ &=\; 2(\sqrt{2}+1)\,\frac{R_s}{aZ_0}\times\frac{\omega^2+\sqrt{2}\Omega^2}{\omega\sqrt{\omega^2-\Omega^2}}\,.\end{aligned} \tag{S.49}$$

Problem **3**(b):

The lowest cutoff frequency among the TM waves belongs to the $\mathrm{TM}_{1,2}$ mode with

$$E_z \;=\; E_0\,\sin\frac{\pi x}{a}\,\sin\frac{2\pi y}{x}\;+\;E_0\,\sin\frac{2\pi x}{a}\,\sin\frac{\pi y}{a}\,, \tag{S.50}$$

$$\Gamma \;=\; \frac{\sqrt{5}\pi}{a}\,,\qquad \Omega\;=\;\frac{\sqrt{5}\pi c}{a}\,, \tag{S.51}$$

and hence

$$H_x \;=\; -iH_1\left(2\sin\frac{\pi x}{a}\,\cos\frac{2\pi y}{a}\;+\;\sin\frac{2\pi x}{a}\,\cos\frac{\pi y}{a}\right), \tag{S.52}$$

$$H_y \;=\; +iH_1\left(\cos\frac{\pi x}{a}\,\sin\frac{2\pi y}{a}\;+\;2\cos\frac{2\pi x}{a}\,\sin\frac{\pi y}{a}\right). \tag{S.53}$$

$$H_z\;\equiv 0, \tag{S.54}$$

for

$$H_1 \;=\; \frac{\omega\mu_0(\pi/a)}{\Gamma^2}\,E_0 \;=\; \frac{E_0}{Z_0}\times\frac{\omega}{\sqrt{5}\Omega}\,. \tag{S.55}$$

For this wave, the area integral evaluates to

$$\begin{aligned}
\iint\limits_{\text{square}} |E_z|^2\,dx\,dy \;&=\; |E_0|^2 \int\limits_0^a dx \int\limits_0^a dy\,\left(\sin\frac{\pi x}{a}\,\sin\frac{2\pi y}{x}\;+\;\sin\frac{2\pi x}{a}\,\sin\frac{\pi y}{x}\right)^2\\
&=\; |E_0|^2 \int\limits_0^a dx\,\sin^2\frac{\pi x}{a}\times\int\limits_0^a dy\,\sin^2\frac{2\pi y}{a}\\
&\quad+\; |E_0|^2 \int\limits_0^a dx\,\sin^2\frac{2\pi x}{a}\times\int\limits_0^a dy\,\sin^2\frac{\pi y}{a}\\
&\quad+\; 2|E_0|^2 \int\limits_0^a dx\,\sin\frac{\pi x}{a}\,\sin\frac{2\pi x}{a}\times\int\limits_0^a dy\,\sin\frac{2\pi y}{a}\,\sin\frac{\pi y}{a}\\
&=\; |E_0|^2\times\frac{a}{2}\times\frac{a}{2}\;+\;|E_0|^2\times\frac{a}{2}\times\frac{a}{2}\;+\;2|E_0|^2\times 0\times 0\\
&=\; |E_0|^2\times\frac{a^2}{2}
\end{aligned} \tag{S.56}$$

and therefore

$$\iint\limits_{\text{triangle}} \frac{|E_z|^2}{Z_0^2}\,dx\,dy = \frac{1}{2Z_0^2}\iint\limits_{\text{square}} |E_z|^2\,dx\,dy \;=\; \frac{|E_0|^2a^2}{4Z_0^2}\,. \tag{S.57}$$

Next, the perimeter integral. Along the vertical side of the triangle at $x=0$, we have

$$H_x \;=\; 0,\quad H_y \;=\; iH_1\left(\sin\frac{2\pi y}{a}\;+\;2\sin\frac{\pi y}{a}\right), \tag{S.58}$$

hence

$$\begin{aligned}
\int\limits_0^a dy\,|\mathbf{H}|^2 \;&=\; |H_1|^2\int\limits_0^a dy\,\left(\sin\frac{2\pi y}{a}\;+\;2\sin\frac{\pi y}{a}\right)^2\\
&=\; |H_1|^2\int\limits_0^a dy\,\left(\sin^2\frac{2\pi y}{a}\;+\;4\sin^2\frac{\pi y}{a}\;-\;4\sin\frac{2\pi y}{a}\times\sin\frac{\pi y}{a}\right)\\
&=\; |H_1|^2\left(\frac{a}{2}\;+\;4\times\frac{a}{2}\;-\;4\times 0\right)\;=\;\frac{5}{2}\,a|H_1|^2\,.
\end{aligned} \tag{S.59}$$

By symmetry, the integral over the horizontal side of the triangle at $y=0$ yields exactly the same result,

$$\int_0^a dx\,|\mathbf{H}|^2 \;=\; \frac{5}{2}\,a|H_1|^2\,. \tag{S.60}$$

As to the diagonal side of the triangle, we have

$$y \;=\; a \;-\; x, \qquad d\ell \;=\; \sqrt{2}\,dx, \tag{S.61}$$

while the magnetic field components are

$$H_x \;=\; -H_y \;=\; -iH_1\left(2\sin\frac{\pi x}{a}\,\cos\frac{2\pi x}{a} \;-\; \sin\frac{2\pi x}{a}\,\cos\frac{\pi x}{a}\right) \;=\; 2iH_1\,\sin^3\frac{\pi x}{a}\,, \tag{S.62}$$

hence

$$\begin{aligned}\int|\mathbf{H}|^2\,d\ell \;&=\; 8|H_1|^2\times\sqrt{2}\int_0^a dx\,\sin^6\frac{\pi x}{a}\\ &=\; 8\sqrt{2}\,|H_1|^2\times\frac{5a}{16}\\ &=\; \frac{5\sqrt{2}}{2}\,a|H_1|^2.\end{aligned} \tag{S.63}$$

Altogether, over the whole perimeter

$$\oint|\mathbf{H}|^2\,d\ell \;=\; \frac{5(2+\sqrt{2})}{2}\times a|H_1|^2, \tag{S.64}$$

or in terms of the electric amplitude $E_0$,

$$\oint|\mathbf{H}|^2\,d\ell \;=\; \frac{2+\sqrt{2}}{2}\,\frac{|E_0|^2a}{Z_0^2}\times\frac{\omega^2}{\Omega^2}\,. \tag{S.65}$$

Finally, plugging this perimeter integral (S.65) and the area integral (S.57) into eq. (S.31) for the attenuation rate, we get

$$\begin{aligned}\alpha \;&=\; \frac{R_s}{Z_0}\times\frac{\Omega^2}{\omega\sqrt{\omega^2-\Omega^2}}\times\frac{\oint|\mathbf{H}^2|\,d\ell}{(1/Z_0^2)\iint|E_z|^2\,dx\,dy}\\ &=\; \frac{R_s}{Z_0}\times\frac{\Omega^2}{\omega\sqrt{\omega^2-\Omega^2}}\times\frac{2+\sqrt{2}}{2}\,\frac{|E_0|^2a}{Z_0^2}\,\frac{\omega^2}{\Omega^2}\Bigg/\frac{|E_0|^2a^2}{4Z_0^2}\\ &=\; 2\sqrt{2}(\sqrt{2}+1)\,\frac{R_s}{aZ_0}\times\frac{\omega}{\sqrt{\omega^2-\Omega^2}}\end{aligned} \tag{S.66}$$

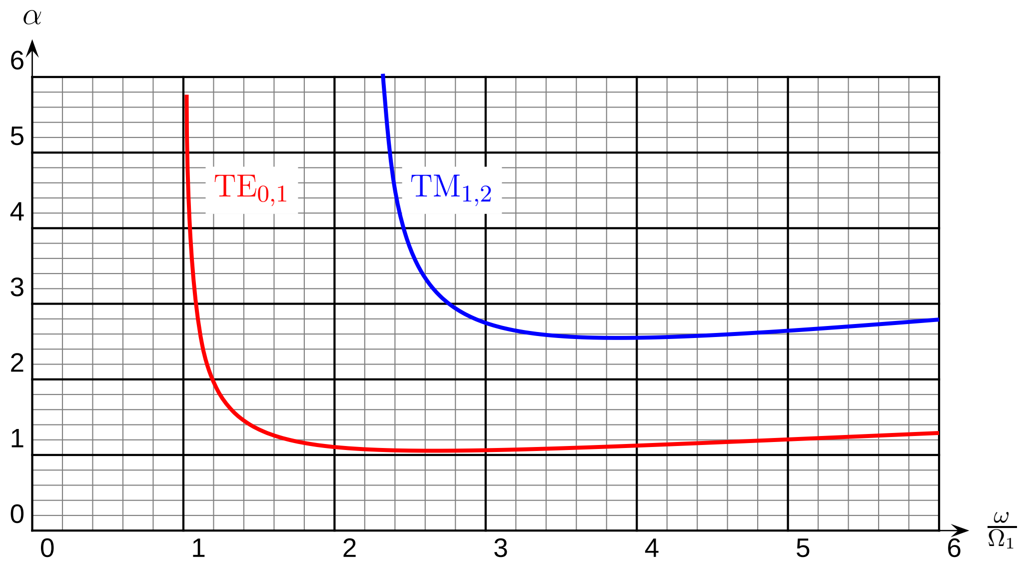
Problem **3**, postscript:

When comparing the attenuation rates (S.49) and (S.66) for the two modes of the the same waveguide, keep in mind that these modes have different cutoff frequencies, $\Omega_1 = (\pi c/a)$ for the $\text{TE}_{0,1}$ mode vs. $\Omega_2 = \sqrt{5}(\pi c/a)$ for the $\text{TM}_{1,2}$ mode. When both attenuation rates are expressed in terms of $\omega/\Omega_1$ — and also the surface resistivity at the same frequency $\Omega_1$, — we get

$$\alpha(\text{TE}_{0,1}) = 2(\sqrt{2}+1)\,\frac{R_s(\Omega_1)}{aZ_0} \times \frac{\omega^2 + \sqrt{2}\Omega_1}{\sqrt{\omega\Omega_1(\omega^2 - \Omega_1^2)}}\,, \tag{S.67}$$

$$\alpha(\text{TM}_{1,2}) = 2(\sqrt{2}+1)\,\frac{R_s(\Omega_1)}{aZ_0} \times \frac{\sqrt[4]{20}\,\omega^2}{\sqrt{\omega\Omega_1(\omega^2 - 5\Omega_1^2)}}\,. \tag{S.68}$$

Graphically,




Problem **4**:

The $\text{TE}_{1,1,1}$ mode of the cylindrical cavity has

$$\Gamma = \frac{j'_{1,1} \approx 1.84}{R}\,, \qquad \omega = c\sqrt{\Gamma^2 + \frac{\pi^2}{d^2}}\,, \tag{S.69}$$

and magnetic fields

$$\begin{aligned} H_z &= H_0\, J_1(\Gamma\rho)\, \cos\phi\, \sin\frac{\pi z}{d}\,, && \text{(S.70)}\\ H_\rho &= \frac{\pi H_0}{\Gamma d}\, J_1'(\Gamma\rho)\, \cos\phi\, \cos\frac{\pi z}{d}\,. && \text{(S.71)}\\ H_\phi &= -\frac{\pi H_0}{\Gamma d}\,\frac{J_1(\Gamma\rho)}{\Gamma\rho}\, \sin(\phi)\, \cos\frac{\pi x}{d}\,. && \text{(S.72)} \end{aligned}$$

Integrating $|\mathbf{H}|^2$ over the cavity's volume, we get

$$\begin{aligned} \iiint |H_z|^2\, d^3\mathbf{x} &= |H_0|^2 \times \int_0^d dz\, \sin^2\frac{\pi z}{d} \times \int_0^{2\pi} d\phi\, \cos^2\phi \times \int_0^R d\rho\, \rho\, J_1^2(\Gamma\rho)\\ &= |H_0|^2 \times \frac{d}{2} \times \pi \times \frac{1}{\Gamma^2} \int_0^{j'_{1,1}} dx\, x\, J_1^2(x). && \text{(S.73)} \end{aligned}$$

$$\begin{aligned} \iiint |H_x|^2\, d^3\mathbf{x} &= \left(\frac{\pi|H_0|}{\Gamma d}\right)^2 \times \int_0^d dz\, \cos^2\frac{\pi z}{d} \times \int_0^{2\pi} d\phi\, \cos^2\phi \times \int_0^R d\rho\, \rho\, \bigl(J_1'(\Gamma\rho)\bigr)^2\\ &= \frac{\pi^2|H_0|^2}{\Gamma^2 d^2} \times \frac{d}{2} \times \pi \times \frac{1}{\Gamma^2} \int_0^{j'_{1,1}} dx\, x\, \bigl(J_1'(x)\bigr)^2, && \text{(S.74)} \end{aligned}$$

$$\begin{aligned} \iiint |H_y|^2\, d^3\mathbf{x} &= \left(\frac{\pi|H_0|}{\Gamma d}\right)^2 \times \int_0^d dz\, \cos^2\frac{\pi z}{d} \times \int_0^{2\pi} d\phi\, \cos^2\phi \times \int_0^R d\rho\, \rho\, \left(\frac{J_1(\Gamma\rho)}{\Gamma\rho}\right)^2\\ &= \frac{\pi^2|H_0|^2}{\Gamma^2 d^2} \times \frac{d}{2} \times \pi \times \frac{1}{\Gamma^2} \int_0^{j'_{1,1}} dx\, x\, \frac{J_1^2(x)}{x^2}\,. && \text{(S.75)} \end{aligned}$$

Combining the last two integrals here, we get

$$\iiint |\mathbf{H}_t|^2\, d^3\mathbf{x} = \frac{\pi^2|H_0|^2}{\Gamma^2 d^2} \times \frac{\pi d}{2} \times \frac{1}{\Gamma^2} \int_0^{j'_{1,1}} dx\, x\, \left(\bigl(J_1'(x)\bigr)^2 + \frac{J_1^2(x)}{x^2}\right) \qquad \text{(S.76)}$$

where

$$\int_0^{j'_{1,1}} dx\, x \left( \left(J'_1(x)\right)^2 \,+\, \frac{J_1^2(x)}{x^2} \right) \;=$$

$$\langle\!\langle \text{ integrating by parts } \rangle\!\rangle$$

$$= \left( x J_1(x) J'_1(x) \right)\Big|_0^{j'_{1,1}} \,+\, \int_0^{j'_{1,1}} dx \left( -J_1(x) \Big( x J''_1(x) \,+\, J'_1(x) \Big) \,+\, \frac{J_1^2(x)}{x} \right)$$

$$= \;0 \quad \langle\!\langle \text{ because } J'_1(j'_{1,1}) = 0 \,\rangle\!\rangle \tag{S.77}$$

$$+\, \int_0^{j'_{1,1}} dx\, x\, J_1(x) \left( -J''_1(x) \,-\, \frac{J'_1(x)}{x} \,+\, \frac{J_1(x)}{x^2} \right)$$

$$\langle\!\langle \text{ by the Bessel equation } \rangle\!\rangle$$

$$= \int_0^{j'_{1,1}} dx\, x\, J_1(x) \times J_1(x),$$

the same integral as in eq. (S.73). Numerically,

$$C \;\stackrel{\text{def}}{=}\; \int_0^{j'_{1,1}} dx\, x\, J_1^2(x) \;\approx\; 0.4046. \tag{S.78}$$

Altogether,

$$\begin{aligned}
\iiint |\mathbf{H}|^2\, d^3\mathbf{x} \;&=\; |H_0|^2 \times \frac{\pi d}{2} \times \frac{C}{\Gamma^2} \;+\; \frac{\pi^2 |H_0|^2}{\Gamma^2 d^2} \times \frac{\pi d}{2} \times \frac{C}{\Gamma^2} \\
&=\; |H_0|^2 \times \frac{\pi C}{2\Gamma^2} \left( 1 \,+\, \frac{\pi^2}{d^2 \Gamma^2} \right) \\
&=\; |H_0|^2 \times dR^2 \times \frac{\pi C}{2 (j'_{1,1})^2} \left( 1 \,+\, \left( \frac{\pi R}{j'_{1.1} d} \right)^2 \right).
\end{aligned} \tag{S.79}$$

Next, let's integrate the $|\mathbf{H}|^2$ over the surface of the cavity. At each endcup disk (at

$z = 0$ and at $z = d$) we have $H_z = 0$ while

$$|H_\rho|^2 \;+\; |H_\phi|^2 \;=\; \left(\frac{\pi|H_0|}{\Gamma d}\right)^2 \times \left[\big(J_1'(\Gamma\rho)\big)^2\,\cos^2\phi \,+\, \frac{J_1^2(\Gamma\rho)}{(\Gamma\rho)^2}\,\sin^2\phi\right], \tag{S.80}$$

hence

$$\begin{aligned}
\iint\limits_{\text{endcup}} |\mathbf{H}|^2\,d^2\mathbf{x} \;&=\; \left(\frac{\pi|H_0|}{\Gamma d}\right)^2 \times \int\limits_0^R d\rho\,\rho \int\limits_0^{2\pi} d\phi\,\left[\big(J_1'(\Gamma\rho)\big)^2\,\cos^2\phi \,+\, \frac{J_1^2(\Gamma\rho)}{(\Gamma\rho)^2}\,\sin^2\phi\right]\\
&=\; \left(\frac{\pi|H_0|}{\Gamma d}\right)^2 \times \frac{\pi}{\Gamma^2} \int\limits_0^{j_{1,1}'} dx\,x\,\left[\big(J_1'(x)\big)^2 \,+\, \frac{J_1^2(x)}{x^2}\right]\\
&=\; \left(\frac{\pi|H_0|}{\Gamma d}\right)^2 \times \frac{\pi}{\Gamma^2} \times C\\
&=\; \frac{\pi^3 C}{(j_{1,1}')^4}\,\frac{R^4|H_0|^2}{d^2}\,.
\end{aligned} \tag{S.81}$$

On the other hand, at the sidewall at $\rho = R$

$$H_z \;=\; H_0\,J_1(j_{1,1}') \times \cos\phi\,\sin\frac{\pi z}{d}\,, \tag{S.82}$$

$$H_\rho \;\equiv\; 0, \tag{S.83}$$

$$H_\phi \;=\; -\frac{\pi H_0}{\Gamma d} \times \frac{J_1(j_{1,1}')}{j_{1,1}'} \times \sin\phi\,\cos\frac{\pi z}{d}\,, \tag{S.84}$$

hence

$$|\mathbf{H}|^2 \;=\; |H_0|^2\big(J_1(j_{1,1}')\big)^2 \times \left(\cos^2\phi\,\sin^2\frac{\pi z}{d} \,+\, \frac{\pi^2 R^2}{(j_{1,1}')^4 d^2} \times \sin^2\phi\,\cos^2\frac{\pi z}{d}\right), \tag{S.85}$$

which integrates to

$$\begin{aligned}
\iint\limits_{\text{sidewall}} |\mathbf{H}|^2\,d^2\mathbf{x} \;&=\; |H_0|^2\big(J_1(j_{1,1}')\big)^2 \times \left(\frac{2\pi Rd}{4} \,+\, \frac{\pi^2 R^2}{(j_{1,1}')^4 d^2} \times \frac{2\pi Rd}{4}\right)\\
&=\; \frac{A\pi}{2}\left(1 \,+\, \frac{\pi^2 R^2}{(j_{1,1}')^4 d^2}\right) \times Rd|H_0|^2
\end{aligned} \tag{S.86}$$

where

$$A \stackrel{\text{def}}{=} \left(J_1(j'_{1,1})\right)^2 \approx 0.3386. \tag{S.87}$$

Altogether,

$$\iint\limits_{\substack{\text{whole}\\\text{surface}}} |\mathbf{H}|^2\, d^2\mathbf{x} \;=\; |H_0|^2 \times \left( \frac{A\pi}{2}\, Rd \;+\; \frac{A\pi^3}{2(j'_{1,1})^4}\, \frac{R^3}{d} \;+\; \frac{2\pi^3 C}{(j'_{1,1})^4}\, \frac{R^4}{d^2} \right). \tag{S.88}$$

Taking the ratio of the volume integral (S.79) to the surface integral (S.88), we find (after a bit of algebra)

$$\begin{aligned} \frac{\iiint |\mathbf{H}|^2\, d^3\mathbf{x}}{\iint |\mathbf{H}|^2\, d^2\mathbf{x}} \;&=\; \frac{\frac{\pi C}{2(j'_{1,1})^2} R^2 d \;+\; \frac{\pi^3 C}{2(j'_{1,1})^4}\frac{R^4}{d}}{\frac{A\pi}{2} Rd \;+\; \frac{A\pi^3}{2(j'_{1,1})^4}\frac{R^3}{d} \;+\; \frac{2\pi^3 C}{(j'_{1,1})^4}\frac{R^4}{d^2}} \\ &=\; \frac{C}{A(j'_{1,1})^2}\, R \times \frac{1 \;+\; \frac{\pi^2}{(j'_{1,1})^2}(R/d)^2}{1 \;+\; \frac{\pi^2}{(j'_{1,1})^4}(R/d)^2 \;+\; \frac{4\pi^2 C}{A(j'_{1,1})^4}(R/d)^3} \\ &= \approx\; 0.3525\, R \times \frac{1 \;+\; 2.911(R/d)^2}{1 \;+\; 0.8588\,(R/d)^2 \;+\; 4.105\,(R/d)^3}\,. \end{aligned} \tag{S.89}$$

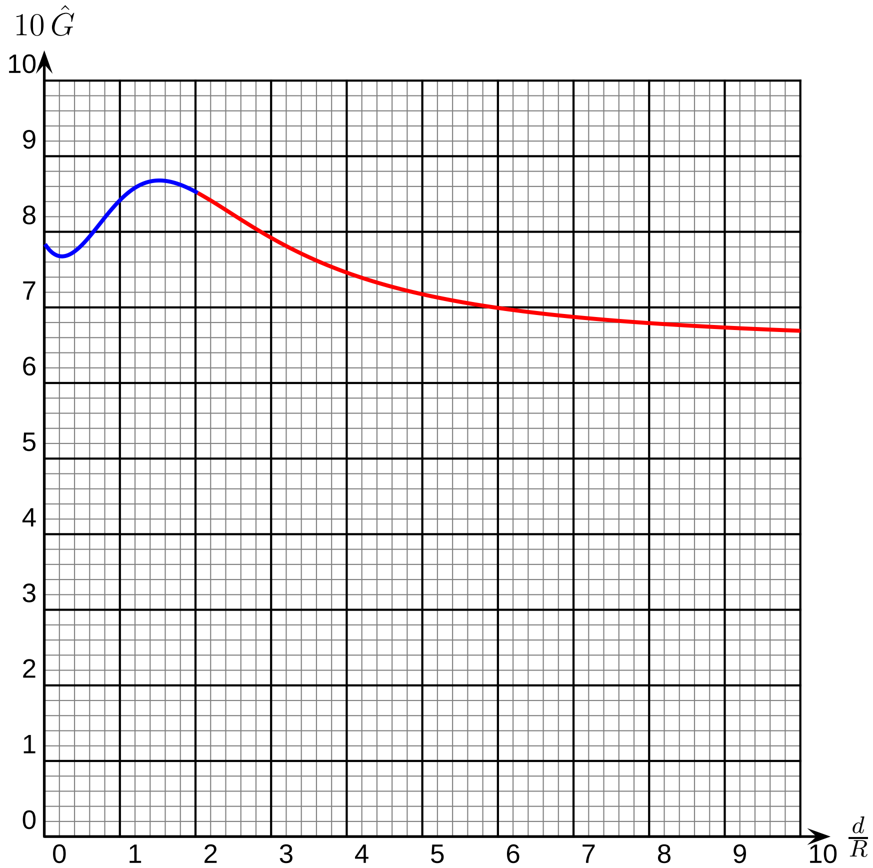
At the same time, for the $\mathrm{TE}_{1,1,1}$ mode

$$\frac{\omega_0}{c} \;=\; \sqrt{\frac{(j'_{1,1})^2}{R^2} \;+\; \frac{\pi^2}{d^2}} \;=\; \frac{j'_{1,1}}{R} \times \sqrt{1 \;+\; (\pi/j'_{1,1})^2 (R/d)^2} \tag{S.90}$$

hence

$$\begin{aligned} \hat{G} \;&=\; \frac{\omega_0}{c} \times \frac{\iiint |\mathbf{H}|^2\, d^3\mathbf{x}}{\iint |\mathbf{H}|^2\, d^2\mathbf{x}} \\ &\approx\; 0.649 \times \frac{(1 \;+\; 2.911\,(R/d)^2)^{3/2}}{1 \;+\; 0.8588\,(R/d)^2 \;+\; 4.105\,(R/d)^3}\,. \end{aligned} \tag{S.91}$$

**PS:** FYI, here is the plot of this geometric factor $\hat{G}$ as a function of the $d/R$ ratio:




Note: the plot line is colored blue for $(d/R) < 2.03$ and red for $(d/R) > 2.03$; the $TE_{1,1,1}$ mode has the lowest frequency only for the red part of the line. Over the red part, the geometric factor varies in a fairly narrow range between 0.85 and 0.65.