Problem **1**(a):

For a harmonically oscillating dipole moment $\mathbf{p}(t) = \mathbf{p}_0 e^{-i\omega t}$, the current density (1) becomes

$$\mathbf{J}(\mathbf{y},t) \;=\; -i\omega\mathbf{p}_0 e^{-i\omega t}\delta^{(3)}(\mathbf{y}), \tag{S.1}$$

hence the vector potential

$$\mathbf{A}(\mathbf{x},t) \;=\; e^{-i\omega t}\mathbf{A}(\mathbf{x}) \tag{S.2}$$

for

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\iiint d^3\mathbf{y}\,\mathbf{J}(y)\,\frac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|} \;=\; \frac{\mu_0}{4\pi}\,(-i\omega\mathbf{p}_0)\,\frac{e^{ikr}}{r}\,, \tag{S.3}$$

exactly as in eq. (2). Please note that this spherical wave does not depends on the direction $\mathbf{n} = \mathbf{x}/r$, so it's an exact solution of the wave equation for both far and intermediate zones of the problem (which in the zero-dipole-size limit means for all $r > 0$), and there are no subleading terms.

However, when we take the curl of the vector potential (2), we do get a subleading (WRT $1/r$) term in the magnetic field. (3): it obtains from taking the gradient of $1/r$ factor instead of the e^{ikr} factor. Indeed,

$$\frac{d}{dr}\,\frac{e^{ikr}}{r} \;=\; \frac{ike^{ikr}}{r} \;-\; \frac{e^{ikr}}{r^2} \;=\; ik\,\frac{e^{ikr}}{r}\left(1 \,+\, \frac{i}{kr}\right), \tag{S.4}$$

hence altogether

$$\mathbf{H}(\mathbf{x}) \;=\; \frac{1}{\mu_0}\nabla\times\mathbf{A} \;=\; \frac{-i\omega}{4\pi}\,\nabla\left(\frac{e^{ikr}}{r}\right)\times\mathbf{p}_0 \;=\; \frac{-i\omega}{4\pi}\,ik\,\frac{e^{ikr}}{r}\left(1 \,+\, \frac{i}{kr}\right)\mathbf{n}\times\mathbf{p}_0, \tag{S.5}$$

in perfect agreement with eq. (3) for the magnetic field.

As to the electric field $\mathbf{E}(\mathbf{x})$, it obtains from the Maxwell–Ampere Law as

$$\mathbf{E} \;=\; \frac{iZ_0}{k}\,\nabla\times\mathbf{H}. \tag{S.6}$$

For the magnetic fields as in eq. (3), this gives us

$$\begin{aligned}\mathbf{E} \;&=\; \frac{iZ_0\omega}{4\pi}\,\nabla\times\left(\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\mathbf{n}\times\mathbf{p}_0\right)\\ &=\; \frac{iZ_0\omega}{4\pi}\left[\frac{d}{dr}\left(\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\right)\mathbf{n}\times(\mathbf{n}\times\mathbf{p}_0)\;+\;\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\nabla\times(\mathbf{n}\times\mathbf{p}_0)\right]\end{aligned} \tag{S.7}$$

where

$$\frac{d}{dr}\left(\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\right) \;=\; ik\,\frac{e^{ikr}}{r}\left(1+\frac{2i}{kr}-\frac{2}{(kr)^2}\right) \tag{S.8}$$

while

$$\begin{aligned}\big[\nabla\times(\mathbf{n}\times\mathbf{p}_0)\big]_j \;&=\; \epsilon_{ijk}\epsilon_{k\ell m}(\nabla_j n_\ell)p_{0m}\\ &=\; (\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell})\,\frac{\delta_{j\ell}-n_jn_\ell}{r}\,p_{0m}\\ &=\; (\delta_{im}-3\delta_{im}-n_in_m+\delta_{im})\,\frac{p_{0m}}{r} \;=\; -(\delta_{im}+n_in_m)\frac{p_{0m}}{r}\,.\end{aligned} \tag{S.9}$$

Altogether,

$$\begin{aligned}\nabla\times\left(\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\mathbf{n}\times\mathbf{p}_0\right) \;&=\\ &=\; ik\,\frac{e^{ikr}}{r}\left(1+\frac{2i}{kr}-\frac{2}{(kr)^2}\right)\mathbf{n}\times(\mathbf{n}\times\mathbf{p}_0)\;+\;\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\frac{-\mathbf{p}_0-(\mathbf{n}\cdot\mathbf{p}_0)\mathbf{n}}{r}\\ &=\; ik\,\frac{e^{ikr}}{r}\left[\left(1+\frac{2i}{kr}-\frac{2}{(kr)^2}\right)\mathbf{n}\times(\mathbf{n}\times\mathbf{p}_0)\;+\;\frac{i}{kr}\left(1+\frac{i}{kr}\right)\big((\mathbf{n}\cdot\mathbf{p}_0)\mathbf{n}+\mathbf{p}_0\big)\right]\\ &=\; ik\,\frac{e^{ikr}}{r}\left[\mathbf{n}\times(\mathbf{n}\times\mathbf{p}_0)\;+\;\frac{i}{kr}\left(1+\frac{i}{kr}\right)\Big(2\mathbf{n}\times(\mathbf{n}\times\mathbf{p}_0)+(\mathbf{n}\cdot\mathbf{p}_0)\mathbf{n}+\mathbf{p}_0\Big)\right]\\ &=\; ik\,\frac{e^{ikr}}{r}\left[\mathbf{n}\times(\mathbf{n}\times\mathbf{p}_0)\;+\;\frac{i}{kr}\left(1+\frac{i}{kr}\right)\big(3(\mathbf{n}\cdot\mathbf{p}_0)\mathbf{n}-\mathbf{p}_0\big)\right],\end{aligned} \tag{S.10}$$

hence

$$\mathbf{E} \;=\; -\frac{Z_0\omega k}{4\pi}\,\frac{e^{ikr}}{r}\left[\mathbf{n}\times(\mathbf{n}\times\mathbf{p}_0)\;+\;\frac{i}{kr}\left(1+\frac{i}{kr}\right)\big(3(\mathbf{n}\cdot\mathbf{p}_0)\mathbf{n}-\mathbf{p}_0\big)\right], \tag{S.11}$$

in perfect agreement with eq. (4) for the electric field. *Quod erat demonstrandum.*

Problem **1**(b):

Eqs. (3–4) for the magnetic and the electric fields apply for all distances r from the dipole — short, medium, and long — as long as the dipole itself may be approximated as point-like. So let's take a closer look at their long-distance and short-distance limits, where the distances are viewed as long or short by comparison with the wavelength $\lambda = 2\pi/k$.

In the long distance regime $r \gg \lambda$, we may neglect all the negative powers of kr in eqs. (3–4), which leaves us with

$$\mathbf{H}(\mathbf{x},t) \approx \frac{k\omega}{4\pi}\frac{e^{ikr-i\omega t}}{r}(\mathbf{n}\times\mathbf{p}_0), \tag{S.12}$$

$$\mathbf{E}(\mathbf{x},t) \approx -\frac{Z_0 k\omega}{4\pi}\frac{e^{ikr-i\omega t}}{r}(\mathbf{n}\times(\mathbf{n}\times\mathbf{p}_0)). \tag{S.13}$$

These are precisely the radiation fields of a harmonic dipole we have discussed in class. Note that they diminish with distance as $1/r$, so that the radiation power density spreads out as $1/r^2$.

On the other hand, in the short distance regime $r \ll \lambda$, we focus on the highest negative powers of kr in eqs. (2–3), and we may also approximate $\exp(ikr) \approx 1$. Consequently, the short-distance limit of the electric field is

$$\begin{aligned}\mathbf{E}(\mathbf{x},t) &\approx -\frac{Z_0 k\omega}{4\pi}\frac{e^{-i\omega t}}{r}\frac{-\mathbf{p}_0 + 3(\mathbf{n}\cdot\mathbf{p}_0)\mathbf{n}}{(kr)^2}\\ &= \left(\frac{Z_0\omega}{4\pi k} = \frac{Z_0 c}{4\pi} = \frac{1}{4\pi\epsilon_0}\right)\frac{-\mathbf{p}_0 + 3(\mathbf{n}\cdot\mathbf{p}_0)\mathbf{n}}{r^3}\,e^{-i\omega t}.\end{aligned} \tag{S.14}$$

This is a quasistatic Coulomb field of the electric dipole $\mathbf{p}(t) = \mathbf{p}_0 e^{-i\omega t}$. That is, at any given instance of time t, the field (S.13) is the Coulomb field of the dipole moment we happen to have at that time. As any good dipole field, it scales with distance as $1/r^3$.

As to the magnetic field in the short-distance regime, the leading term in eq. (2) is

$$\mathbf{H}(\mathbf{x},t) = \frac{i\omega}{4\pi}\frac{\mathbf{n}\times\mathbf{p}_0}{r^2}\,e^{-i\omega t}. \tag{S.15}$$

Unlike the electric field (S.14), the short-distance magnetic field scales with distance as $1/r^2$, slower that any quasistatic magnetic multipole, but faster than the $1/r$ radiation-zone fields.

Also, the magnetic field (S.13) is not a quasistatic field, since it vanishes for $\omega \to 0$. Instead, this magnetic field is induced by the displacement current due to the time-dependent dipole field (S.14) in the short-distance zone. Indeed,

$$\begin{aligned}\nabla\times\mathbf{H}(\text{from eq. (S.15)}) &= -\frac{i\omega}{4\pi}\nabla\times\left(\frac{\mathbf{n}\times\mathbf{p}}{r^2}\right)e^{-i\omega t}\\ &= -\frac{i\omega}{4\pi}\frac{\mathbf{p}-3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})}{r^3}e^{-i\omega t}\\ &= \frac{\partial}{\partial t}\Big(\mathbf{D}=\epsilon_0\mathbf{E}(\text{from eq. (S.14)})\Big).\end{aligned} \tag{S.16}$$

Problem **1**(c):

The time-averaged Poynting vector of a harmonic wave is

$$\langle\mathbf{S}\rangle = \tfrac{1}{2}\,\mathrm{Re}\big(\mathbf{E}\times\mathbf{H}^*\big). \tag{S.17}$$

For the dipole wave (2–3), we have

$$\mathbf{E}\times\mathbf{H}^* = -\frac{Z_0k^2\omega^2}{16\pi^2r^2}\left[\begin{aligned}&\left(1-\frac{i}{kr}\right)\big(\mathbf{n}\times(\mathbf{n}\times\mathbf{p}_0)\big)\times(\mathbf{n}\times\mathbf{p}_0)^*\\ +\frac{i}{kr}&\left(1+\frac{1}{k^2r^2}\right)\big(3(\mathbf{n}\cdot\mathbf{p}_0)\mathbf{n}-\mathbf{p}_0\big)\times(\mathbf{n}\times\mathbf{p}_0)^*\end{aligned}\right] \tag{S.18}$$

where

$$\begin{aligned}\big(\mathbf{n}\times(\mathbf{n}\times\mathbf{p}_0)\big)\times(\mathbf{n}\times\mathbf{p}_0)^* &= (\mathbf{n}\times\mathbf{p}_0)^*\times\big((\mathbf{n}\times\mathbf{p}_0)\times\mathbf{n}\big)\\ &= (\mathbf{n}\times\mathbf{p}_0)\Big(\mathbf{n}\cdot(\mathbf{n}\times\mathbf{p}_0)^* = 0\Big)\\ &\quad -\mathbf{n}\Big((\mathbf{n}\times\mathbf{p}_0)^*\cdot(\mathbf{n}\times\mathbf{p}_0) = \|\mathbf{n}\times\mathbf{p}_0\|^2\Big)\\ &= -\|\mathbf{n}\times\mathbf{p}_0\|^2\,\mathbf{n}\end{aligned} \tag{S.19}$$

while

$$
\begin{aligned}
\big(3(\mathbf{n}\cdot\mathbf{p}_0)\mathbf{n}-\mathbf{p}_0\big)\times(\mathbf{n}\times\mathbf{p}_0)^* \;&=\; 3(\mathbf{n}\cdot\mathbf{p}_0)\Big(\mathbf{n}(\mathbf{n}\cdot\mathbf{p}_0)^* \;-\; \mathbf{p}_0^*\Big) \\
&\quad -\; \Big(\mathbf{n}(\mathbf{p}_0\cdot\mathbf{p}_0^*) \;-\; \mathbf{p}_0^*(\mathbf{n}\cdot\mathbf{p}_0)\Big) \\
&=\; \Big(3|(\mathbf{n}\cdot\mathbf{p}_0)|^2 \;-\; \|\mathbf{p}_0\|^2\Big)\mathbf{n} \;-\; 2(\mathbf{n}\cdot\mathbf{p}_0)\mathbf{p}_0^*.
\end{aligned} \tag{S.20}
$$

Altogether,

$$
\mathbf{E}\times\mathbf{H}^* \;=\; \frac{Z_0k^2\omega^2}{16\pi^2r^2}\left[\begin{array}{c}
\left(1\;-\;\dfrac{i}{kr}\right)\|\mathbf{n}\times\mathbf{p}_0\|^2\,\mathbf{n} \\
-\dfrac{i}{kr}\left(1\;+\;\dfrac{1}{k^2r^2}\right)\big(3|\mathbf{n}\cdot\mathbf{p}_0|^2-\|\mathbf{p}_0\|^2\big)\mathbf{n} \\
+\dfrac{2i}{kr}\left(1\;+\;\dfrac{1}{k^2r^2}\right)(\mathbf{n}\cdot\mathbf{p}_0)\mathbf{p}_0^*
\end{array}\right]. \tag{S.21}
$$

Next, we take the real part of this cross product, thus

$$
\mathrm{Re}\left[\left(1\;-\;\frac{i}{kr}\right)\|\mathbf{n}\times\mathbf{p}_0\|^2\,\mathbf{n}\right] \;=\; \|\mathbf{n}\times\mathbf{p}_0\|^2\,\mathbf{n}, \tag{S.22}
$$

$$
\mathrm{Re}\left[\frac{-i}{kr}\left(1\;+\;\frac{1}{k^2r^2}\right)\big(3|\mathbf{n}\cdot\mathbf{p}_0|^2-\|\mathbf{p}_0\|^2\big)\mathbf{n}\right] \;=\; 0, \tag{S.23}
$$

$$
\mathrm{Re}\left[\frac{2i}{kr}\left(1\;+\;\frac{1}{k^2r^2}\right)(\mathbf{n}\cdot\mathbf{p})\mathbf{p}^*\right] \;=\; -\frac{2}{kr}\left(1\;+\;\frac{1}{k^2r^2}\right)\mathrm{Im}\big((\mathbf{n}\cdot\mathbf{p})\mathbf{p}^*\big), \tag{S.24}
$$

where

$$
\begin{aligned}
2\,\mathrm{Im}\big((\mathbf{n}\cdot\mathbf{p})\mathbf{p}^*\big) \;&=\; -i(\mathbf{n}\cdot\mathbf{p}_0)\mathbf{p}_0^* \;+\; i(\mathbf{n}\cdot\mathbf{p}_0^*)\mathbf{p} \\
&=\; -i\mathbf{n}\times(\mathbf{p}_0^*\times\mathbf{p}_0) \\
&=\; \mathbf{n}\times\mathrm{Im}(\mathbf{p}_0^*\times\mathbf{p}_0),
\end{aligned} \tag{S.25}
$$

and therefore, the time-averaged Poynting vector

$$
\langle\mathbf{S}\rangle \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(\mathbf{E}\times\mathbf{H}^*\big) \;=\; \frac{Z_0k^2\omega^2}{32\pi^2r^2}\left[\|\mathbf{n}\times\mathbf{p}_0\|^2\,\mathbf{n}\;-\;\frac{1}{kr}\left(1\;+\;\frac{1}{k^2r^2}\right)\mathbf{n}\times\mathrm{Im}(\mathbf{p}_0^*\times\mathbf{p}_0)\right]. \tag{S.26}
$$

Altogether, we find that the radiation power flow has two components: the radial power flow

$$\langle\mathbf{S}\rangle_{\rm rad} \;=\; \frac{Z_0 k^2\omega^2}{32\pi^2 r^2}\,\|\mathbf{n}\times\mathbf{p}_0\|^2\,\mathbf{n} \tag{S.27}$$

which diminishes with distance as $1/r^2$, thus distance-independent power per solid angle

$$\frac{dP}{d\Omega} \;=\; \frac{Z_0 k^2\omega^2}{32\pi^2}\,\|\mathbf{n}\times\mathbf{p}_0\|^2, \tag{S.28}$$

and the lateral power flow

$$\langle\mathbf{S}\rangle_{\rm lat} \;=\; \frac{Z_0 k\omega^2}{32\pi^2 r^3}\left(1+\frac{1}{k^2r^2}\right)\Big(-\mathbf{n}\times{\rm Im}(\mathbf{p}_0^*\times\mathbf{p}_0)\Big) \tag{S.29}$$

which diminishes with distance at a faster rate $1/r^3$.

Note: for a linear dipole — whose components (p_x,p_y,p_z) oscillate with the same phase, — the complex amplitude vector $\mathbf{p}_0$ is parallel to its complex conjugate $\mathbf{p}_0^*$, thus $\mathbf{p}_0^*\times\mathbf{p}_0=0$, and hence no lateral power flow, $\langle\mathbf{S}\rangle_{\rm lat}=0$. On the other hand, a non-linear dipole — for which the 3 components (p_x,p_y,p_z) oscillate with different phases — has complex amplitude vector $\mathbf{p}_0$ that is *not* parallel to its complex conjugate $\mathbf{p}_0^*$. For such a non-linear dipole $\mathbf{p}_0^*\times\mathbf{p}_0\neq 0$, and that gives rise to a non-trivial lateral power flow (S.29).

Problem **1**(d):

The linear momentum density of the EM fields

$$\mathbf{g} \;=\; \frac{1}{c^2}\mathbf{S} \tag{S.30}$$

gives rise to the angular momentum density

$$\vec{\mathcal{L}} \stackrel{\rm def}{=} \frac{d\mathbf{L}}{d\,\mathrm{volume}} \;=\; \mathbf{x}\times\mathbf{g} \;=\; \frac{\mathbf{x}\times\mathbf{S}}{c^2}\,. \tag{S.31}$$

For a purely radial Poynting vector, this angular momentum density would vanish. But as we saw in part (c), the radiation of a non-linear dipole has a lateral component to its

Poynting vector, thus non-zero angular momentum density

$$\begin{aligned}
\vec{\mathcal{L}} &= \frac{1}{c^2}\, r\mathbf{n}\times\langle\mathbf{S}\rangle_{\text{lat}} \\
&= \frac{r}{c^2}\,\frac{Z_0 k\omega^2}{32\pi^2 r^3}\left(1+\frac{1}{k^2r^2}\right)\mathbf{n}\times\Big(-\mathbf{n}\times\operatorname{Im}(\mathbf{p}_0^*\times\mathbf{p}_0)\Big) \\
&= \frac{Z_0\omega^3}{32\pi^2c^3r^2}\left(1+\frac{1}{k^2r^2}\right)\mathbf{n}\times\Big(-(\mathbf{n}\times\operatorname{Im}(\mathbf{p}_0^*\times\mathbf{p}_0)\Big) \\
&\xrightarrow[kr\gg1]{} \frac{Z_0\omega^3}{32\pi^2c^3r^2}\,\mathbf{n}\times\Big(-(\mathbf{n}\times\operatorname{Im}(\mathbf{p}_0^*\times\mathbf{p}_0)\Big).
\end{aligned} \tag{S.32}$$

This angular momentum density flows outward with the radiation itself. In the far zone of $kr \gg 1$, the radiation flows out radially with speed c, so the angular momentum flow density is simply

$$\mathcal{M}_{ij} \approx c\mathcal{L}_i n_j\,. \tag{S.33}$$

Consequently, the net rate at which the radiation carries away the angular momentum is simply

$$\vec{\tau}_{\text{net}} \stackrel{\text{def}}{=} \frac{d\mathbf{L}_{\text{EM}}}{dt} = \oiint\limits_{\substack{\text{large}\\\text{sphere}}} c\vec{\mathcal{L}}\,d^2\text{area} = \lim_{r\to\infty}\oiint cr^2\vec{\mathcal{L}}\,d^2\Omega. \tag{S.34}$$

In out case,

$$\lim_{r\to\infty}(cr^2\vec{\mathcal{L}}) = \frac{Z_0\omega^3}{32\pi^2c^2}\,\mathbf{n}\times\Big(-(\mathbf{n}\times\operatorname{Im}(\mathbf{p}_0^*\times\mathbf{p}_0)\Big), \tag{S.35}$$

hence

$$\tau_i^{\text{net}} = \frac{Z_0\omega^3}{32\pi^2c^2}\oiint d^2\Omega(\mathbf{n})\Big(-\epsilon_{ijk}n_j\epsilon_{k\ell m}n_\ell\operatorname{Im}(\mathbf{p}_0^*\times\mathbf{p}_0)_m\Big) \tag{S.36}$$

where

$$\oiint d^2\Omega(\mathbf{n})\Big(-\epsilon_{ijk}n_j\epsilon_{k\ell m}n_\ell\Big) = \oiint d^2\Omega(\mathbf{n})\,(-n_jm_\ell+\delta_{j\ell}) = +\frac{8\pi}{3}\,\delta_{im}\,, \tag{S.37}$$

and therefore

$$\vec{\tau}_{\text{net}} = +\frac{Z_0\omega^3}{12\pi c^2}\operatorname{Im}(\mathbf{p}_0^*\times\mathbf{p}_0). \tag{S.38}$$

Note: Physically, a steady increase of the EM radiation's angular momentum at the rate (S.38) means that the non-linear dipole creating this radiation supplies it with not only power but also *torque*. Specifically, it effectively applies that torque (S.38) at the EM radiation! And by the angular version of the Newton's third law, the radiation acts with an opposite torque $-\vec{\tau}_{\rm net}$ on the non-linear oscillator.

Problem **1**(e):

For the sake of definiteness, let the electron in the classical Rutherford atom rotate counter-clockwise in the (x, y) plane, thus

$$\mathbf{x} \;=\; (r\cos\omega t, r\sin\omega t, 0) \;=\; \mathrm{Re}\Big((r, ir, 0)e^{-i\omega t}\Big), \tag{S.39}$$

while the angular velocity and the angular momentum of the atom point in the $\hat{\mathbf{z}}$ direction,

$$\vec{\omega} \;=\; \omega\hat{\mathbf{z}}, \quad \mathbf{L} \;=\; m\omega r^2\hat{\mathbf{z}} \;=\; mr^2\vec{\omega}. \tag{S.40}$$

The rotating dipole moment

$$\mathbf{p}(t) \;=\; -er(i, i, 0)e^{-i\omega t}, \tag{S.41}$$

has complex amplitude vector $\mathbf{p}_0 = -er(1, i, 0)$ that's not parallel to its complex conjugate $\mathbf{p}_0^* = -er(1, -i, 0)$, so there is a non-zero cross product

$$\mathbf{p}_0^* \times \mathbf{p}_0 \;=\; e^2r^2(1, -i, 0) \times (1, +i, 0) \;=\; e^2r^2(0, 0, 2i), \tag{S.42}$$

$$\mathrm{Im}(\mathbf{p}_0^* \times \mathbf{p}_0) \;=\; 2e^2r^2\hat{\mathbf{z}}, \tag{S.43}$$

which gives rise to the radiation torque

$$\vec{\tau}_{\rm net} \;=\; +\frac{Z_0e^2}{6\pi c^2}\,\omega^3r^2\,\hat{\mathbf{z}}. \tag{S.44}$$

Note: this is the torque the atom supplies to the radiation it emits. The torque by the radiation on the atom has the opposite direction, thus

$$\frac{d\mathbf{L}_{\rm atom}}{dt} \;=\; -\vec{\tau}_{\rm net} \;=\; -\frac{Z_0e^2}{12\pi c^3}\,\omega^3r^2\,\hat{\mathbf{z}}. \tag{S.45}$$

Note the direction of this torque is precisely opposite to the direction $+\hat{\mathbf{z}}$ of the atom's own angular momentum (S.40).

At the same time, the net EM power emitted by the rotating oscillator is

$$P_{\text{net}} \;=\; \frac{Z_0\omega^2}{12\pi c^2}\,\|\mathbf{p}_0\|^2 \;=\; \frac{Z_0\omega^2}{12\pi c^2}\times 2e^2r^2 \;=\; \frac{Z_0e^2}{6\pi c^2}\times\omega^4r^2. \tag{S.46}$$

Similar to the angular momentum, this power comes at the expense of the atom's own energy U, so it's lost at the rate

$$\frac{dU}{dt} \;=\; -P_{\text{net}} \;=\; -\frac{Z_0e^2}{6\pi c^2}\times\omega^4r^2. \tag{S.47}$$

Comparing this formula to eq. (S.45) for the rate of the angular momentum loss, we immediately see that

$$\frac{dU}{dt} \;=\; -\omega\left|\frac{d\mathbf{L}}{dt}\right|, \tag{S.48}$$

and further more

$$\frac{dU}{dt} \;=\; +\vec{\omega}\cdot\frac{d\mathbf{L}}{dt} \tag{S.49}$$

since the two vectors on the RHS have precisely opposite directions.

Problem **1**(f):

A classical particle moving in a Coulomb field has several integrals of motion, including the net energy

$$U \;=\; \frac{m\mathbf{v}^2}{2} \;-\; \frac{\alpha}{r} \tag{S.50}$$

where $\alpha = e^2/4\pi\epsilon_0$ for the hydrogen atom, the angular momentum

$$\mathbf{L} \;=\; \mathbf{x}\times m\mathbf{v}\,, \tag{S.51}$$

and the Runge–Lenz vector

$$\mathbf{K} \;=\; \mathbf{v}\times\mathbf{L} \;-\; \alpha\mathbf{n}. \tag{S.52}$$

These integrals of motion are not completely independent; instead, the Runge–Lenz vector is always $\perp$ to the angular momentum, while their magnitudes are related to the energy as

$$\mathbf{L}^2 U \;=\; -\frac{m}{2}\big(\alpha^2 - \mathbf{K}^2\big)^2. \tag{S.53}$$

For an elliptic orbit, the direction of the angular momentum is $\perp$ to the orbit's plane while the direction of the Runge–Lenz vector points towards the perihelion. Also, $|\mathbf{K}| = \alpha \times$ excentricity, so for a circular orbit — and only for a circular orbit — $\mathbf{K} = 0$. Consequently, eq. (S.53) gives us a criterion of a circular orbit in terms of its energy and angular momentum:

$$\text{an orbit is circular if and only if} \quad \mathbf{L}^2 U \;=\; -\frac{m\alpha^2}{2}\,. \tag{S.54}$$

One can easily verify eq. (S.54) for a circular orbit without using the Runge–Lenz vector, although it would not prove that any orbit obeying this criterion must be circular. Using first-year Newtonian mechanics, we have

$$m\omega^2 r \;=\; \frac{\alpha}{r^2} \quad\Longrightarrow\quad \omega^2 \times r^3 \;=\; \frac{\alpha}{m}\,, \tag{S.55}$$

hence

$$L \;=\; m\omega r^2 \;=\; \sqrt{\alpha m} \times \sqrt{r}, \tag{S.56}$$

$$U \;=\; \frac{m\omega^2 r^2}{2} \;-\; \frac{\alpha}{r} \;=\; -\frac{\alpha}{2r}\,, \tag{S.57}$$

$$UL^2 \;=\; -\frac{\alpha^2 m}{2}\,. \tag{S.58}$$

With all this in mind, let us now address the results of part (e) for the Rutherford atom. The fact that $d\mathbf{L}/dt$ has precisely opposite direction from the remaining angular momentum $\mathbf{L}$ of the atom means that the direction of $\mathbf{L}$ remains fixed while its magnitude slowly diminishes. In other words, *the plane of the electron's orbit remains fixed* while the orbital radius slowly shrinks to zero.

Next, suppose the orbit is initially circular, then eq. (6) implies that

$$\frac{d}{dt}(\mathbf{L}^2 U) \;=\; 2U\mathbf{L}\cdot\frac{d\mathbf{L}}{dt} \;+\; \mathbf{L}^2\left(\frac{dU}{dt} = \vec{\omega}\cdot\frac{d\mathbf{L}}{dt}\right) \;=\; \frac{d\mathbf{L}}{dt}\cdot\Big(2U\mathbf{L} \,+\, \mathbf{L}^2\vec{\omega}\Big) \;=\; 0$$

because

$$2U\mathbf{L} \;+\; \mathbf{L}^2\vec{\omega} \;=\; \left(2U = \frac{-\alpha}{r}\right)(\mathbf{L} = mr^2\vec{\omega}) \;+\; (\mathbf{L}^2 = \alpha mr)\vec{\omega} \;=\; -\alpha mr\vec{\omega} \;+\; \alpha mr\vec{\omega} \;=\; 0. \tag{S.59}$$

Consequently, if the criterion (S.54) initially holds true for a circular orbit, then it continues to hold true, so the orbit stays circular. *Quod erat demonstrandum.*

Problem **2**(a):

The Efimenko equations for the electric and magnetic fields of given charge and current densities follow from the retarded Green's function of the wave equation. I have explained that issue in class back in early October, and the Efimenko equations themselves appear on the last page of my notes on Maxwell equations. In the notations of the present homework, the Efimenko equations become:

$$\mathbf{H}(\mathbf{x},t) \;=\; \frac{-1}{4\pi}\iiint d^3\mathbf{y}\begin{pmatrix} \dfrac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\times\mathbf{J}(\mathbf{y},t-\frac{|\mathbf{x}-\mathbf{y}|}{c}) \\ +\,\dfrac{(\mathbf{x}-\mathbf{y})}{c|\mathbf{x}-\mathbf{y}|^2}\times\dot{\mathbf{J}}(\mathbf{y},t-\frac{|\mathbf{x}-\mathbf{y}|}{c}) \end{pmatrix}, \tag{S.60}$$

$$\mathbf{E}(\mathbf{x},t) \;=\; \frac{1}{4\pi\epsilon_0}\iiint d^4\mathbf{y}\begin{pmatrix} \dfrac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,\rho(\mathbf{y},t-\frac{|\mathbf{x}-\mathbf{y}|}{c}) \\ +\,\dfrac{(\mathbf{x}-\mathbf{y})}{c|\mathbf{x}-\mathbf{y}|^2}\,\dot{\rho}(\mathbf{y},t-\frac{|\mathbf{x}-\mathbf{y}|}{c}) \\ -\,\dfrac{1}{c^2|\mathbf{x}-\mathbf{y}|}\,\dot{\mathbf{J}}(\mathbf{y},t-\frac{|\mathbf{x}-\mathbf{y}|}{c}) \end{pmatrix}. \tag{S.61}$$

Applying eq. (S.60) to the current (1) of the point-like dipole, we immediately obtain

$$\begin{aligned}
\mathbf{H}(\mathbf{x},t) &= \frac{-1}{4\pi}\iiint d^3\mathbf{y}\left(\begin{array}{c} \dfrac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\times\dot{\mathbf{p}}(t-\frac{|\mathbf{x}-\mathbf{y}|}{c})\delta^{(3)}(\mathbf{y}) \\ +\,\dfrac{(\mathbf{x}-\mathbf{y})}{c|\mathbf{x}-\mathbf{y}|^2}\times\ddot{\mathbf{p}}(t-\frac{|\mathbf{x}-\mathbf{y}|}{c})\delta^{(3)}(\mathbf{y}) \end{array}\right) \\
&= \frac{-1}{4\pi}\left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\times\dot{\mathbf{p}}(t-\tfrac{|\mathbf{x}|}{c})\;+\;\frac{\mathbf{x}}{c|\mathbf{x}|^2}\times\ddot{\mathbf{p}}(t-\tfrac{|\mathbf{x}|}{c})\right) \\
&= \frac{-1}{4\pi}\left(\frac{\mathbf{n}}{r^2}\times\dot{\mathbf{p}}(t_{\text{ret}})\;+\;\frac{\mathbf{n}}{cr}\times\ddot{\mathbf{p}}(t_{\text{ret}})\right),
\end{aligned}\tag{S.62}$$

exactly as in eq. (8).

Eq. (9) for the electric field takes a bit more work. Plugging ρ and $\mathbf{J}$ from eq. (1) into the Efimenko equation (S.61) for the electric field, we get

$$\mathbf{E}(\mathbf{x},t) = \frac{-1}{4\pi\epsilon_0}\iiint d^3\mathbf{y}\left(\begin{array}{c} \dfrac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,(\mathbf{p}(t-\frac{|\mathbf{x}-\mathbf{y}|}{c})\cdot\nabla_y)\delta^{(3)}(\mathbf{y}) \\ +\,\dfrac{(\mathbf{x}-\mathbf{y})}{c|\mathbf{x}-\mathbf{y}|^2}\,(\dot{\mathbf{p}}(t-\frac{|\mathbf{x}-\mathbf{y}|}{c})\cdot\nabla_y)\delta^{(3)}(\mathbf{y}) \\ +\,\dfrac{1}{c^2|\mathbf{x}-\mathbf{y}|}\,\ddot{\mathbf{p}}(t-\frac{|\mathbf{x}-\mathbf{y}|}{c})\delta^{(3)}(\mathbf{y}) \end{array}\right). \tag{S.63}$$

Integrating the third term here is completely straightforward,

$$\iiint d^3\mathbf{y}\,\frac{1}{c^2|\mathbf{x}-\mathbf{y}|}\,\ddot{\mathbf{p}}(t-\tfrac{|\mathbf{x}-\mathbf{y}|}{c})\delta^{(3)}(\mathbf{y}) \;=\; \frac{1}{c^2|\mathbf{x}|}\,\ddot{\mathbf{p}}(t-\tfrac{|\mathbf{x}|}{c}) \;=\; \frac{1}{c^2r}\,\ddot{\mathbf{p}}(t_{\text{ret}}), \tag{S.64}$$

but the first two terms in (S.63) need more care due to the derivative of the δ-function.

Integrating the first term, we get

$$
\begin{aligned}
\iiint d^4\mathbf{y}\,\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,(\mathbf{p}(t-\tfrac{|\mathbf{x}-\mathbf{y}|}{c})\cdot\nabla_y)\delta^{(3)}(\mathbf{y}) \;=&\; \\
&\langle\!\langle\,\text{integrating by parts}\,\rangle\!\rangle \\
&= -\iiint d^4\mathbf{y}\,\delta^{(3)}(\mathbf{y})\,\frac{\partial}{\partial y_j}\left(\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,p_j(t-\tfrac{|\mathbf{x}-\mathbf{y}|}{c})\right) \\
&= +\iiint d^4\mathbf{y}\,\delta^{(3)}(\mathbf{y})\,\frac{\partial}{\partial x_j}\left(\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,p_j(t-\tfrac{|\mathbf{x}-\mathbf{y}|}{c})\right) \\
&= \frac{\partial}{\partial x_j}\left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\,p_j(t-\tfrac{|\mathbf{x}|}{c})\right) \\
&= \nabla_j\left(\frac{\mathbf{n}}{r^2}\,p_j(t_{\mathrm{ret}})\right).
\end{aligned}
\tag{S.65}
$$

Likewise, for the second term we get

$$
\iiint d^4\mathbf{y}\,\frac{(\mathbf{x}-\mathbf{y})}{c|\mathbf{x}-\mathbf{y}|^2}\,(\dot{\mathbf{p}}(t-\tfrac{|\mathbf{x}-\mathbf{y}|}{c})\cdot\nabla_y)\delta^{(3)}(\mathbf{y}) \;=\; \nabla_j\left(\frac{\mathbf{n}}{rc}\,\dot{p}_j(t_{\mathrm{ret}})\right). \tag{S.66}
$$

Moreover, due to $\mathbf{x}$-dependence of the retarded time (9), we have non-zero gradients of functions of t_{ret},

$$
\nabla f(t_{\mathrm{ret}}) \;=\; (\nabla t_{\mathrm{ret}})\,\dot{f}(t_{\mathrm{ret}}) \;=\; -\frac{\mathbf{n}}{c}\,\dot{f}(t_{\mathrm{ret}}). \tag{S.67}
$$

Consequently,

$$
\begin{aligned}
\text{first term}_i \;&=\; \nabla_j\left(\frac{n_i}{r^2}\,p_j(t_{\mathrm{ret}})\right) \\
&=\; \nabla_j\left(\frac{n_i}{r^2}\right)\,p_j(t_{\mathrm{ret}}) \;+\; \frac{n_i}{r^2}\,\left(\nabla_j p_j(t_{\mathrm{ret}}) \;=\; -\frac{n_j}{c}\,\dot{f}(t_{\mathrm{ret}})\right) \\
&=\; \frac{\delta_{ij}-3n_in_j}{r^3}\,p_j(t_{\mathrm{ret}}) \;-\; \frac{n_in_j}{cr^2}\,\dot{p}_j(t_{\mathrm{ret}}),
\end{aligned}
\tag{S.68}
$$

and likewise

$$
\begin{aligned}
\text{second term}_i \;&=\; \nabla_j\left(\frac{n_i}{cr}\,\dot{p}_j(t_{\mathrm{ret}})\right) \\
&=\; \nabla_j\left(\frac{n_i}{cr}\right)\,\dot{p}_j(t_{\mathrm{ret}}) \;+\; \frac{n_i}{cr}\,\left(\nabla_j \dot{p}_j(t_{\mathrm{ret}}) \;=\; -\frac{n_j}{c}\,\ddot{f}(t_{\mathrm{ret}})\right) \\
&=\; \frac{\delta_{ij}-2n_in_j}{cr^2}\,\dot{p}_j(t_{\mathrm{ret}}) \;-\; \frac{n_in_j}{c^2r}\,\ddot{p}_j(t_{\mathrm{ret}}).
\end{aligned}
\tag{S.69}
$$

Altogether, eq. (S.63) for the electric field evaluates to

$$\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= \frac{-1}{4\pi\epsilon_0}\left(\text{first term} \;+\; \text{second term} \;+\; \text{third term}\right) \\
&= \frac{-1}{4\pi\epsilon_0}\begin{pmatrix} \dfrac{\mathbf{p}-3(\mathbf{n}\cdot\mathbf{p})\mathbf{n}}{r^3} - \dfrac{(\mathbf{n}\cdot\dot{\mathbf{p}})\mathbf{n}}{cr^2} \\ +\dfrac{\dot{\mathbf{p}}-2(\mathbf{n}\cdot\dot{\mathbf{p}})\mathbf{n}}{cr^2} - \dfrac{(\mathbf{n}\cdot\ddot{\mathbf{p}})\mathbf{n}}{c^2 r} \\ +\dfrac{\ddot{\mathbf{p}}}{c^2 r} \end{pmatrix} \\
&= \frac{-1}{4\pi\epsilon_0}\left(\frac{\mathbf{p}-3(\mathbf{n}\cdot\mathbf{p})\mathbf{n}}{r^3} + \frac{\dot{\mathbf{p}}-3(\mathbf{n}\cdot\dot{\mathbf{p}})\mathbf{n}}{cr^2} + \frac{\ddot{\mathbf{p}}-(\mathbf{n}\cdot\ddot{\mathbf{p}})\mathbf{n}}{c^2 r}\right)
\end{aligned} \tag{S.70}$$

where $\mathbf{p}$, $\dot{\mathbf{p}}$, and $\ddot{\mathbf{p}}$ are all evaluated at the retarded time (10). By inspection, the last line of eq. (S.70) is in perfect agreement with eq. (9). *Quod erat demonstrandum.*

Problem **2**(b):

In the long-distance limit, the EM field (7–8) are dominated by the terms with decrease with distance as $1/r$ rather than $1/r^2$ or $1/r^3$, thus

$$\mathbf{H}(\mathbf{x},t) \approx -\frac{\mathbf{n}\times\ddot{\mathbf{p}}(t_{\rm ret})}{4\pi cr}, \tag{S.71}$$

$$\mathbf{E}(\mathbf{x},t) \approx \frac{(\mathbf{n}\cdot\ddot{\mathbf{p}}(t_{\rm ret}))\mathbf{n} - \ddot{\mathbf{p}}(t_{\rm ret})}{4\pi\epsilon_0 c^2 r} = Z_0\,\frac{\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{p}}(t_{\rm ret}))}{4\pi cr}. \tag{S.72}$$

In this limit, the Poynting vector becomes

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = -\frac{Z_0}{16\pi^2c^2r^2}\left(\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{p}})\right)\times(\mathbf{n}\times\ddot{\mathbf{n}}) = +\frac{Z_0}{16\pi^2c^2r^2}\left\|\mathbf{n}\times\ddot{\mathbf{p}}(t_{\rm ret})\right\|^2\mathbf{n} \tag{S.73}$$

hence the power emitted into a solid angle $d\Omega$ is

$$\frac{dP}{d\Omega} = \frac{Z_0}{16\pi^2c^2}\left\|\mathbf{n}\times\ddot{\mathbf{p}}(t_{\rm ret})\right\|^2, \tag{S.74}$$

and the net power radiated by the dipole is

$$P_{\text{net}}(t) = \frac{Z_0}{16\pi^2 c^2} \oiint d^2\Omega(\mathbf{n}) \left\| \mathbf{n} \times \ddot{\mathbf{p}}(t_{\text{ret}}) \right\|^2 = \frac{Z_0}{6\pi c^2} \left\| \ddot{\mathbf{p}}(t_{\text{ret}}) \right\|^2, \tag{S.75}$$

exactly as in eq. (11).

BTW, the retarded time $t_{\text{ret}} = t - \frac{r}{c}$ is retarded relative to the time t at which we detect this radiation at long distance r from the dipole. By the clock of the dipole itself, the energy loss happens at the same time as the $\ddot{\mathbf{p}}$, thus

$$\frac{dU_{\text{dipole}}(t')}{dt'} = -\frac{Z_0}{6\pi c^2} \left\| \ddot{\mathbf{p}}(t') \right\|^2. \tag{S.76}$$

Problem **2**(c):

The parallel-plate capacitor in question has capacitance

$$C = \frac{\epsilon_0 A}{b}. \tag{S.77}$$

When it's charged to initial charge Q_0 and then allowed to discharge via resistor R, it's charge decreases exponentially as

$$Q(t) = Q_0 \times \exp(-t/\tau) \quad \text{for } \tau = RC. \tag{S.78}$$

The dipole moment of this capacitor is

$$p(t) = bQ(t) = bQ_0 \exp(-t/\tau), \tag{S.79}$$

hence

$$\ddot{p} = \frac{bQ_0}{\tau^2} \exp(-t/\tau), \tag{S.80}$$

which causes EM radiation at net power

$$P = \frac{Z_0}{6\pi c^2} \frac{b^2 Q_0^2}{\tau^4} \times \exp(-2t/\tau). \tag{S.81}$$

Integrating this power over the discharge time, we find the net energy carried by the EM radiation to be

$$\Delta U_{\rm EM} = \int_0^\infty dt\, P(t) = \frac{Z_0}{6\pi c^2}\,\frac{b^2 Q_0^2}{\tau^4} \times \int_0^\infty dt\, e^{-2t/\tau} = \frac{Z_0}{6\pi c^2}\,\frac{b^2 Q_0^2}{\tau^4} \times \frac{\tau}{2}\,. \qquad \text{(S.82)}$$

Compared to the initial energy stored in the capacitor

$$U_0 = \frac{Q_0^2}{2C} = \frac{Q_0^2 b}{2\epsilon_0 A}\,, \qquad \text{(S.83)}$$

the fraction of this energy carried by the EM radiation is

$$\frac{\Delta U_{\rm EM}}{U_0} = \frac{Z_0\epsilon_0}{6\pi c^2} \times \frac{Ab}{\tau^3} = \frac{1}{6\pi} \times \frac{Ab}{(c\tau)^3} \qquad \text{(S.84)}$$

where the second equality follows from $Z_0\epsilon_0 c = 1$.

Problem **2**(d):

For the specific example of $A = 100\ \text{cm}^2 = 0.01\ \text{m}^2$, $b = 1\ \text{mm} = 10^{-3}$ m and $R = 10\ \Omega$, we have

$$C = \frac{\epsilon_0 A}{b} = 88.5\ \text{pF}, \quad \tau = RC = 0.885\ \text{ns}, \quad c\tau = 0.265\ m, \qquad \text{(S.85)}$$

and hence

$$\frac{\Delta U_{\rm EM}}{U_0} = \frac{1}{6\pi} \times \frac{Ab}{(c\tau)^3} = \frac{10^{-5}\ \text{m}^3}{6\pi(0.265\ \text{m})^3} = 2.85\ \times 10^{-5}. \qquad \text{(S.86)}$$

Problem **3**(a):

The quadrupole moment tensor of a system of point charges is

$$Q_{ij} = \sum_n q_n\bigl(\tfrac{3}{2}x_{n,i}x_{n,j} - \tfrac{1}{2}r_n^2\bigr). \qquad \text{(S.87)}$$

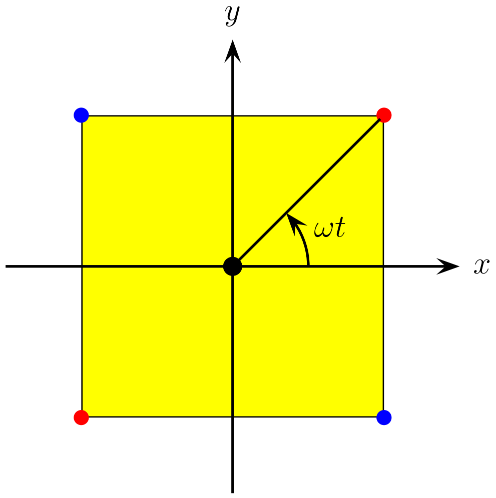
The 4 charges in question are all in the same plane — which we take to be the (x, y) plane, — hence $Q_{xz} = Q_{yz} = 0$. Also, all 4 charges lie at the same distance $r = a/\sqrt{2}$ from the origin and the net charge $\sum_n q_n$ vanishes, hence $\sum_n q_n r_n^2 = 0$ and therefore

$$Q_{zz} = 0 \quad \text{and} \quad Q_{xx} + Q_{yy} = 0. \tag{S.88}$$

The remaining independent components of the quadrupole tensor form a complex combination

$$\mathcal{Q} = Q_{xx} - Q_{yy} + 2iQ_{xy} = \frac{3}{2}\sum_n q_n(x_n + iy_n)^2. \tag{S.89}$$

For the charges at the corners of a rotating square



$$\begin{aligned} \text{charge } q_n &= (-1)^n q \\ \text{at } x_n + iy_n &= i^n \times e^{i\omega t} \times \frac{a}{\sqrt{2}} \\ \text{for } n &= 0, 1, 2, 3. \end{aligned} \tag{S.90}$$

we have

$$\forall n: \quad q_n(x_n + iy_n)^2 = +\frac{qa^2}{2} \times e^{2i\omega t} \tag{S.91}$$

and hence

$$\mathcal{Q} = 3qa^2 \times e^{2i\omega t}. \tag{S.92}$$

In terms of the quadrupole tensor components, this means

$$Q_{xx} = -Q_{yy} = \tfrac{1}{2}\,\mathrm{Re}(\mathcal{Q}) = \tfrac{3}{2}qa^2 \times \cos(2\omega t), \qquad Q_{xy} = \tfrac{1}{2}\,\mathrm{Im}(\mathcal{Q}) = \tfrac{3}{2}qa^2 \times \sin(2\omega t), \tag{S.93}$$

or in matrix notations

$$Q_{ij}(t) \;=\; \frac{3qa^2}{2}\begin{pmatrix} +\cos(2\omega t) & +\sin(2\omega t) & 0 \\ +\sin(2\omega t) & -\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{S.94}$$

Note that this quadrupole tensor oscillates with frequency 2ω, *i.e.*, twice the rotation frequency of the charges. As to the complex amplitude of the quadrupole oscillation,

$$Q_{ij}(t) \;=\; \frac{3qa^2}{2}\,\mathrm{Re}\left[e^{-2i\omega t}\begin{pmatrix} +1 & +i & 0 \\ +i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right], \tag{S.95}$$

hence

$$\text{amplitude } Q_{ij} \;=\; \frac{3qa^2}{2}\begin{pmatrix} +1 & +i & 0 \\ +i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{S.96}$$

Problem **3**(b–c):

As explained in class, the EM power radiated in a particular direction $\mathbf{n}$ is

$$\frac{dP}{d\Omega} \;=\; \frac{Z_0\omega_{\mathrm{osc}}^2}{2c^2}\times\left(|\mathbf{f}(\mathbf{n})|^2\,-\,|\mathbf{n}\cdot\mathbf{f}(\mathbf{n})|^2\right) \tag{S.97}$$

where

$$\mathbf{f}(\mathbf{n}) \;=\; \frac{1}{4\pi}\iiint d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\,\exp(-ik\mathbf{n}\cdot\mathbf{y}). \tag{S.98}$$

In the long wavelength approximation, the leading contribution to the $\mathbf{f}$ comes from the lowest oscillating multipole moment, electric or magnetic. For the system at hand, the lowest oscillating moment is the electric quadrupole; as we saw in part (a), it has frequency

$\omega_{\rm osc} = 2\omega$ and amplitude (S.96). For a general electric quadrupole,

$$f_j(\mathbf{n}) \;=\; \frac{\omega_{\rm osc}^2}{12\pi c}\, Q_{jk} n_k\,, \tag{S.99}$$

so for the quadrupole in question

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \;=\; \frac{\omega_{\rm osc}^2 q a^2}{8\pi c} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} (n_x + i n_y), \tag{S.100}$$

or in spherical coordinates

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \;=\; \frac{\omega_{\rm osc}^2 q a^2}{8\pi c} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \sin\theta\, e^{i\phi}. \tag{S.101}$$

Consequently,

$$\mathbf{f}^* \cdot \mathbf{f} \;=\; \frac{\omega_{\rm osc}^4 q^2 a^4}{64\pi^2 c^2} \times 2\sin^2\theta, \tag{S.102}$$

$$\mathbf{n} \cdot \mathbf{f} \;=\; \frac{\omega_{\rm osc}^4 q^2 a^4}{64\pi^2 c^2} \times \big(\sin\theta\, e^{i\phi}\big)^2, \tag{S.103}$$

hence

$$\big(|\mathbf{f}(\mathbf{n})|^2 \;-\; |\mathbf{n}\cdot\mathbf{f}(\mathbf{n})|^2\big) \;=\; \frac{\omega_{\rm osc}^4 q^2 a^4}{64\pi^2 c^2} \times \big(2\sin^2\theta \;-\; \sin^4\theta\big), \tag{S.104}$$

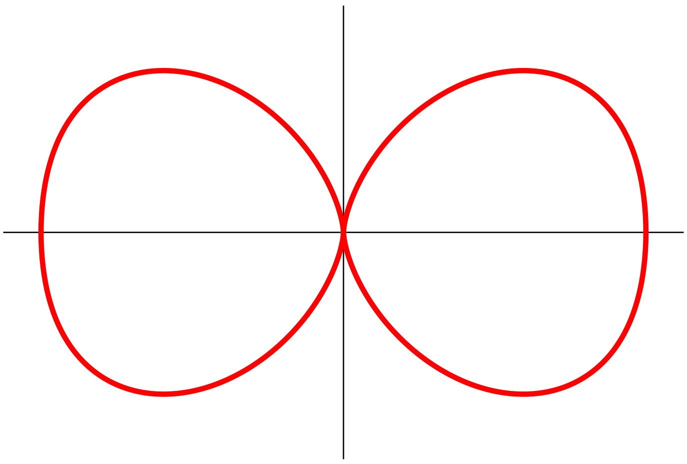
and therefore

$$\frac{dP}{d\Omega} \;=\; \frac{Z_0 q^2 a^4 \omega_{\rm osc}^6}{128\pi^2 c^4} \times \sin^2\theta(2 - \sin^2\theta). \tag{S.105}$$

In particular, the angular dependence of the radiated power has form

$$\frac{dP}{d\Omega} \;\propto\; \sin^2\theta(2 - \sin^2\theta) \;=\; 1 \;-\; \cos^4\theta. \tag{S.106}$$

Graphically,



As to the total power radiated by the rotating quadrupole,

$$P_{\text{net}} \;=\; \frac{Z_0 q^2 a^4 \omega_{\text{osc}}^6}{128\pi^2 c^4} \times \oiint d^2\Omega\,(1-\cos^4\theta) \tag{S.107}$$

where $\omega_{\text{osc}} = 2\omega$ and

$$\oiint d^2\Omega\,(1-\cos^4\theta) \;=\; 2\pi\int_{-1}^{+1} d\cos\theta\,(1-\cos^4\theta) \;=\; 4\pi\times\left(1-\frac{1}{5}\right) \;=\; \frac{16\pi}{5}\,. \tag{S.108}$$

Thus altogether,

$$P_{\text{net}} \;=\; \frac{8Z_0 q^2 q^4\omega^6}{5\pi c^4} \;=\; \frac{8q^2}{5\pi\epsilon_0}\times\frac{a^4\omega^6}{c^5}\,. \tag{S.109}$$